LIMITS OF PERFORMANCE IN REFERENCE-TRACKING AND PATH-FOLLOWING FOR NONLINEAR SYSTEMS

Technical Report

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Abstract: We investigate limits of performance in reference-tracking and path-following and highlight an essential difference between them. For a class of nonlinear systems, we show that in reference-tracking, the smallest achievable $\mathcal{L}_2$-norm of the tracking error is equal to the least amount of control energy needed to stabilize the zero-dynamics of the error system. We then show that this fundamental performance limitation does not exist when the control objective is to force the output to follow a geometric *path* without a timing law assigned to it. This is true even when an additional desired speed assignment is required to be satisfied asymptotically or in finite time.

Keywords: Limits of performance, non-minimum phase nonlinear systems, path-following, reference-tracking, cheap-control.

1. INTRODUCTION

Obstacles to achieving perfect tracking with linear feedback systems have been quantified with classical Bode integrals and as the limits of cheap optimal control performance [Kwakernaak and Sivan, 1972; Middleton, 1991; Qiu and Davison, 1993; Seron *et al.*, 1999; Chen *et al.*, 2000]. While in the absence of unstable zero dynamics (*non-minimum phase zeros*) perfect tracking of any reference signal is possible, this is not the case with unstable zero dynamics. A formula derived by Qiu and Davison [1993], see also [Su *et al.*, 2003], shows that the tracking error increases as the signal frequencies approach those of the unstable zeros.

Seron *et al.* [1999] re-interpreted the Qiu-Davison formula and generalized it to a class of nonlinear systems. They showed that the best attainable value of the $\mathcal{L}_2$-norm of the tracking error – J_T – is equal to the lowest control effort, measured by the $\mathcal{L}_2$-norm, needed to stabilize the zero dynamics driven by the system output $y(t)$. It is its role as a stabilizing control input that prevents the output $y(t)$ from perfect tracking. Extensions to non-right-invertible systems are given in [Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002].

In the first part of this paper we present an internal model analog of the results in [Seron *et al.*, 1999; Braslavsky *et al.*, 2002]. In this more general case, the zero dynamics system to be stabilized is driven by the tracking error. As before, the best attainable value of J_T is the control needed for stabilization.

In the second part of the paper we present our main result for the path-following problem. When, for a class of nonlinear systems, the control objective is to force the output to follow a geometric *path* without a timing law assigned to it, we show that the fundamental performance limitations imposed on reference-tracking by unstable zero dynamics do not apply. Furthermore, the same is

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true for the speed-assigned path-following problem, where an additional desired speed is required to be satisfied asymptotically or in finite time.

Recently, in [Aguiar *et al.*, 2004*c*] we present conditions under which for non-minimum phase linear systems these path following problems can be solved with arbitrarily small $\mathcal{L}_2$-norm of the path following error.

In Section 2 we formulate the reference-tracking and path-following problems. Performance limitations for linear systems for reference-tracking and recent results for path-following are briefly revised in Section 3. Section 4 presents the main results of the paper and concluding remarks are given in Section 5.

2. REFERENCE-TRACKING AND PATH-FOLLOWING PROBLEMS

2.1 Reference-tracking

For linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

Davison [1976] and Francis [1977] consider the reference signal $r(t)$ generated by a known exosystem

$$\dot{w} = Sw, \quad r = Qw, \tag{2}$$

and show that the so-called *servomechanism or regulator problem* is solvable if and only if (A, B) is stabilizable, (C, A) is detectable, the number of inputs is at least as large as the number of outputs ($m \geq q$), and the zeros of (A, B, C, D) do not coincide with the eigenvalues of S. The internal model approach of [Francis and Wonham, 1976; Francis, 1977] designs the reference-tracking controller

$$u(t) = Kx(t) + (\Gamma - K\Pi)w(t),$$

where K is such that $(A + BK)$ is Hurwitz, and Π and Γ satisfy

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma, \\ 0 &= C\Pi + D\Gamma - Q. \end{aligned}$$

Then, the *tracking error* $e_T(t) := y(t) - r(t)$ converges to zero, and the transients $\tilde{x} := x - \Pi w$ and $\tilde{u} := u - \Gamma w$ are governed by $\dot{\tilde{x}} = (A + BK)\tilde{x}$, $\tilde{u} = K\tilde{x}$.

For the nonlinear regulator problem

$$\dot{x} = f(x, u), \qquad y = h(x, u), \tag{3}$$

$$\dot{w} = s(w), \qquad r = q(w), \tag{4}$$

where $f(0, 0) = 0$, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control, $y \in \mathbb{R}^q$ the output, $w \in \mathbb{R}^p$ the internal state of the exosystem, and $r \in \mathbb{R}^q$ the reference signal, Isidori and Byrnes [1990] shows that it is solvable if and only if there exist smooth mappings $\Pi(w)$, and $c(w)$, satisfying $\Pi(0) = 0$, $c(0) = 0$, and

$$\begin{aligned} &\frac{\partial \Pi}{\partial w} s(w) = f(\Pi(w), c(w)), \\ &h(\Pi(w), c(w)) - q(w) = 0. \end{aligned} \tag{5}$$

Krener [1992] presented necessary and sufficient conditions for local solvability of (5) when either the exosystem has a semisimple pole structure or the plant has a semisimple zero structure.

2.2 Path-following

Path-following problems are primarily concerned with the design of control laws that drive an object (robot arm, mobile robot, ship, aircraft, etc.) to reach and follow a geometric *path*. A secondary goal is to force the object moving along the path to satisfy some additional dynamic specification. A common approach to the path-following problem is to parameterize the geometric path y_d by a *path variable* θ and then select a *timing law* for θ, [Hauser and Hindman, 1995; Al-Hiddabi and McClamroch, 2002; Skjetne *et al.*, 2004; Aguiar *et al.*, 2004*a*]. Extending the approach of [Hauser and Hindman, 1995], a path-following controller was proposed in [Skjetne *et al.*, 2004] for a class of uncertain strict feedback nonlinear systems. A framework for path-following as a method to avoid some limitations in reference-tracking was described in [Aguiar *et al.*, 2004*a*]. The key idea is to use θ as an additional control input to stabilize the unstable zero-dynamics while the original control variables keep the system on the path.
In path-following, the output $y(t)$ is required to reach and follow a geometric path $y_d(\theta)$ generated by the exosystem

$$\begin{aligned} \frac{d}{d\theta} w(\theta) &= s(w(\theta)), \quad w(\theta_0) = w_0 \\ y_d(\theta) &= q(w(\theta)), \end{aligned} \tag{6}$$

where θ is the scalar path parameter, and $w \in \mathbb{R}^p$ is the exogenous state. For a given *timing law* $\theta(t)$, the *path-following error* is defined as

$$e_P(t) := y(t) - y_d(\theta(t)). \tag{7}$$

The two path-following problems considered in the paper are:
Geometric path-following: For a desired path $y_d(\theta)$, design a controller that achieves:

i) *boundedness:* the state $x(t)$ is uniformly bounded for all $t \geq 0$ and every initial condition $(x(0), w(\theta(0))) = (x_0, w_0)$, in some neighborhood of $(0, 0)$,
ii) *error convergence:* the path-following error $e_P(t)$ converges to zero as $t \to \infty$, and
iii) *forward motion:* $\dot{\theta}(t) > c$ for all $t \geq 0$, where c is a positive constant.

Speed-assigned path-following: In addition to geometric path-following, given a desired constant speed $v_d > 0$, it is required that either $\dot{\theta}(t) \to v_d$ as $t \to \infty$, or $\dot{\theta}(t) = v_d$ $\forall t \geq T$ and some $T \geq 0$.

As illustrated by Skjetne *et al.* [2004], these path-following problems provide natural settings for many engineering applications.

From a theoretical standpoint our main interest is to determine whether the freedom to select a timing law $\theta(t)$ can be used to achieve an arbitrarily small $\mathcal{L}_2$-norm of the path-following error, that is, whether $\delta^\star > 0$ in

$$\int_0^\infty \|e_P(t)\|^2 \, dt \le \delta^\star \tag{8}$$

can be made arbitrarily small, while keeping the closed-loop stable.

3. LIMITS OF PERFORMANCE FOR LINEAR SYSTEMS

3.1 Reference-tracking

An important issue in reference-tracking problems is whether the $\mathcal{L}_2$-norm of the tracking error can be made arbitrarily small, that is, whether

$$\int_0^\infty \|e_T(t)\|^2 \, dt \le \delta^\star, \tag{9}$$

can be satisfied for an arbitrary $\delta^\star > 0$, while keeping the closed-loop stable. For this to be the case, the zeros of (A, B, C, D) must lie in the open left half-plane $\mathbb{C}^-$.

The non-minimum phase zeros, that is the zeros in $\mathbb{C}^+$, impose a fundamental limitation on the attainable tracking performance (9). This is revealed by analyzing the cheap control and showing that the limit as $\epsilon \to 0$ of the optimal value of the cost functional

$$J_\epsilon := \min_{\tilde{u}} \int_0^\infty \left[\|e_T(t)\|^2 + \epsilon^2 \|\tilde{u}(t)\|^2\right] dt \tag{10}$$

with $\tilde{u}$ defined in Section 2.1, is strictly positive, [Kwakernaak and Sivan, 1972]. Qiu and Davison [1993] proved the following result:

Theorem 1. Let $x(0) = 0$, $r(t) := \eta_1 \sin \omega t + \eta_2 \cos \omega t$, and assume that (1) is right-invertible and has no zero at $j\omega$. The best attainable performance $J_T := \lim_{\epsilon \to 0} J_\epsilon$ is given by $J_T = \eta' M \eta$, $\eta = \mathrm{col}(\eta_1, \eta_2)$ for some positive semi-definite M and

$$\mathrm{trace}\, M = \sum_{i=1}^{p} \left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega} \right),$$

where $z_1, z_2, \ldots, z_p$ are the non-minimum phase zeros of (A, B, C, D).

For more general reference signals, [Su *et al.*, 2003] give explicit formulas which show the dependence of J_T on the non-minimum phase zeros and their frequency-dependent directional information.

3.2 Path-following

In contrast to reference-tracking, the attainable performance for path-following is not limited by non-minimum phase zeros, [Aguiar *et al.*, 2004*c*]. Let

$$y_d(\theta) := \sum_{k=1}^{n_d} \left[a_k e^{j\omega_k \theta} + a_k^\star e^{-j\omega_k \theta}\right], \tag{11}$$

where the $\omega_k > 0$ are real numbers and the a_k are non-zero complex vectors, which can be generated by an exosystem of the form (6). As in [Qiu and Davison, 1993; Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002; Su *et al.*, 2003], we assume that initially the system is at rest, i.e., $x(0) = 0$.

Theorem 2. For the geometric path-following problem, if (A, B) is stabilizable, then for any given positive constant $\delta^\star$ there exist constant matrices K and L, and a timing law $\theta(t)$ such that the feedback law

$$u(t) = Kx(t) + Lw(\theta(t)) \tag{12}$$

achieves (8).

Proof. See [Aguiar *et al.*, 2004*c*]. □

We stress that the stabilizability of (A, B) is the *only* condition (necessary and sufficient) for the solvability of the geometric path-following problem using (12).

Next we show that an arbitrarily small $\mathcal{L}_2$-norm of the path-following error is attainable even when the speed assignment v_d is specified beforehand.

Theorem 3. For the speed-assigned path-following problem, let v_d be specified so that the eigenvalues of $v_d S$ do not coincide with the zeros of (3), and assume that (A, B) is stabilizable. Then, (8) can be satisfied for any $\delta^\star > 0$ with a timing law $\theta(t)$ and a controller of the form (12) but with time-varying piecewise-constant matrices K and L.

Proof. See [Aguiar *et al.*, 2004*c*]. □

4. LIMITS OF PERFORMANCE FOR NONLINEAR SYSTEMS

To obtain analogous results for nonlinear systems we first extend the results of [Seron *et al.*, 1999; Braslavsky *et al.*, 2002] to the case when the reference signal for reference-tracking is generated by an exosystem (4). We then show that, in contrast to reference-tracking, the path-following problems can be solved with arbitrarily small $\mathcal{L}_2$−norm of the path-following error.

4.1 Reference-tracking

We consider the class of nonlinear systems which are locally diffeomorphic to systems in strict-feedback form [Krstić *et al.*, 1995, Appendix G] [3]:

$$\dot{z} = f_0(z) + g_0(z)\xi_1, \tag{13a}$$
$$\dot{\xi}_1 = f_1(z, \xi_1) + g_1(z, \xi_1)\xi_2,$$
$$\vdots$$
$$\dot{\xi}_r = f_r(z, \xi_1, \ldots, \xi_r) + g_r(z, \xi_1, \ldots, \xi_r)u, \tag{13b}$$
$$y = \xi_1, \tag{13c}$$

where $z \in \mathbb{R}^{n_z}$, $\xi := \mathrm{col}(\xi_1, \ldots, \xi_r)$, $\xi_i \in \mathbb{R}^m$, $\forall i \in \{1, \ldots, r\}$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$. $f_i(\cdot)$ and $g_i(\cdot)$ are $\mathcal{C}^k$ functions of their arguments (for

[3] When convenient we use the compact form (3) for (13). In that case, $f(\cdot)$ denotes the vector field described by the right-hand-side of (13a)–(13b), $h(\cdot)$ the output map described by (13c), and $x = \mathrm{col}(z, \xi_1, \ldots, \xi_r)$.

some large k), $f_i(0,\ldots,0)=0$, and the matrices $g_i(\cdot)$, $i=0,\ldots,r$ are always nonsingular. We assume that initially the system is at rest, i.e., $(z,\xi)=(0,0)$.

When the reference-tracking problem is solvable, there exist maps $\Pi=\mathrm{col}(\Pi_0,\ldots,\Pi_r)$, $\Pi_0:\mathbb{R}^p\to\mathbb{R}^{n_z}$, $\Pi_i:\mathbb{R}^p\to\mathbb{R}^m$, $\forall i\in\{1,\ldots,r\}$, and $c:\mathbb{R}^p\to\mathbb{R}^m$ that satisfies (5). The following locally diffeomorphic change of coordinates

$$\tilde{z}=z-\Pi_0(w), \tag{14}$$
$$\tilde{\xi}:=\mathrm{col}(\tilde{\xi}_1,\ldots,\tilde{\xi}_r), \tag{15}$$
$$\tilde{\xi}_i=\xi_i-\Pi_i(w),\quad i=1,\ldots,r \tag{16}$$
$$\tilde{u}=u-c(w), \tag{17}$$

transforms the system (13) into the *error system*

$$\dot{\tilde{z}}=\tilde{f}_0(\tilde{z},w)+\tilde{g}_0(\tilde{z},w)e_T, \tag{18a}$$
$$\begin{aligned}\dot{\tilde{\xi}}_1&=\tilde{f}_1(\tilde{z},\tilde{\xi}_1,w)+\tilde{g}_1(\tilde{z},\tilde{\xi}_1,w)\tilde{\xi}_2,\\ &\vdots \\ \dot{\tilde{\xi}}_r&=\tilde{f}_r(\tilde{z},\tilde{\xi}_1,\ldots,\tilde{\xi}_r,w)+\tilde{g}_r(\tilde{z},\tilde{\xi}_1,\ldots,\tilde{\xi}_r,w)\tilde{u},\end{aligned} \tag{18b}$$
$$e_T=\tilde{\xi}_1, \tag{18c}$$

where

$$\begin{aligned}\tilde{f}_0&:=f_0(\tilde{z}+\Pi_0(w))-f_0(\Pi_0(w))\\ &\quad+\Big[g_0(\tilde{z}+\Pi_0(w))-g_0(\Pi_0(w))\Big]q(w),\\ \tilde{g}_0&:=g_0(\tilde{z}+\Pi_0(w)),\end{aligned}$$

$\tilde{f}_0(0,w)=0$, $\tilde{g}_0(\tilde{z},0)=g_0(\tilde{z})$, and $\tilde{f}_i(\cdot)$, $\tilde{g}_i(\cdot)$, $i=1,\ldots,r$ are appropriate defined functions that satisfy $\tilde{f}_i(0,\ldots,0,w)=0$ and $\tilde{g}_i(\tilde{z},\ldots,\tilde{\xi}_i,0)=g_i(\tilde{z},\ldots,\tilde{\xi}_i)$. For our analysis we consider the following two optimal control problems.

Cheap control problem: For the system consisting of the error system (18) and the exosystem (4) with initial condition $\big(\tilde{z}(0),\tilde{\xi}(0),w(0)\big)=\big(\tilde{z}_0,\tilde{\xi}_0,w_0\big)$, find the optimal feedback law $\tilde{u}=\alpha^{cc}_{\delta,\epsilon}(\tilde{z},\tilde{\xi},w)$ that minimizes the cost functional

$$\frac{1}{2}\int_0^\infty\big(\|e_T(t)\|^2+\delta\|\tilde{z}(t)\|^2+\epsilon^{2r}\|\tilde{u}(t)\|^2\big)\,dt \tag{19}$$

for some small $\delta>0$, $\epsilon>0$. We denote by $J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)$ the corresponding optimal value. The best-attainable cheap control performance for trajectory tracking is then defined by $J_T:=\lim_{(\delta,\epsilon)\to 0}J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)$. As shown in [Krener, 2001], $J^{cc}_{\delta,\epsilon}(\tilde{z},\tilde{\xi},w)$ is $\mathcal{C}^{k-2}$ in some neighborhood of $(0,0,0)$ for every $\delta>0$, $\epsilon>0$ under the following assumption:

Assumption 1. The local linearization around $(z,\xi)=(0,0)$ of system (13) is stabilizable and detectable, and the local linearization around $w=0$ of the exosystem (4) is stable.

Minimum-energy problem: For the system

$$\dot{\tilde{z}}=\tilde{f}_0(\tilde{z},w)+\tilde{g}_0(\tilde{z},w)e_T, \tag{20a}$$
$$\dot{w}=s(w), \tag{20b}$$

with e_T viewed as the input, and initial condition $\big(\tilde{z}(0),w(0)\big)=\big(\tilde{z}_0,w_0\big)$, find the optimal feedback law $e_T=\alpha^{me}_\delta(\tilde{z},w)$ that minimizes the cost

$$\frac{1}{2}\int_0^\infty\big(\delta\|\tilde{z}(t)\|^2+\|e_T(t)\|^2\big)\,dt, \tag{21}$$

for some small $\delta>0$. We denote by $J^{me}_\delta(\tilde{z}_0,w_0)$ the corresponding optimal value. Under Assumption 1, $J^{me}_\delta(\tilde{z},w)$ is $\mathcal{C}^{k-2}$ in some neighborhood of $(0,0)$.

Our analysis reveals that the best attainable value of J_T is equal to the least control effort needed to stabilize the corresponding zero dynamics system (20) driven by the tracking error e_T.

Theorem 4. Suppose that Assumption 1 holds and that (5) has a solution in some neighborhood of $w=0$. Then, for any initial condition $(\tilde{z}(0),\tilde{\xi}(0),w(0))=(\tilde{z}_0,\tilde{\xi}_0,w_0)$ in some neighborhood of $(0,0,0)$ there exists a solution to the cheap control problem and

$$J_T=\lim_{\delta\to 0}J^{me}_\delta \tag{22}$$

Proof (outline). Under Assumption 1 and from the formulations of the cheap control and minimum-energy problems, we conclude that for every $\delta>0$, $\epsilon>0$, and every initial condition $(\tilde{z}_0,\tilde{\xi}_0,w_0)$ in some neighborhood of $(0,0,0)$, the value functions $J^{me}_\delta(\tilde{z},w)$ and $J^{cc}_{\delta,\epsilon}(\tilde{z},\tilde{\xi},w)$ exist and satisfy

$$J^{me}_\delta(\tilde{z}_0,w_0)\le J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0).$$

Therefore,

$$\lim_{\delta\to 0}J^{me}_\delta\le\lim_{(\epsilon,\delta)\to 0}J^{cc}_{\delta,\epsilon}:=J_T. \tag{23}$$

On the other hand, from Lemma 1 in Appendix we have

$$J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)\le J^{me}_\delta(\tilde{z}_0,w_0)+O(\epsilon). \tag{24}$$

Therefore, the result (22) follows from (23)–(24), and the fact that δ, ϵ can be made arbitrarily small. □

4.2 Path-following

One can also define a cheap control problem for path-following by replacing e_T by e_P in (19). The following result shows that, in contrast to reference-tracking, the path-following problem can be solved with arbitrarily small $\mathcal{L}_2$-norm for the error.

We assume that the vector field $s(w)$ and the output map $q(w)$ of the exosystem (6) are linear, i.e., that $s(w)=Sw$, $q(w)=Qw$, and that all eigenvalues of $S\in\mathbb{R}^{p\times p}$ are non-zero and semisimple.

Theorem 5. Assume that (5) has a solution for $s(w)=v_dSw$, for v_d almost everywhere on $(0,\infty)$. There exists a neighborhood around $w=0$ such that for all initial conditions $w(\theta(0))=w_0$ in that neighborhood and any $\delta^\star>0$, we can find a timing law for $\theta(t)$ and a feedback law

$$u=c(w)+\alpha_{\delta,\epsilon}(z,\xi,w) \tag{25}$$

that solves the geometric path-following and satisfies (8).

Proof (outline). Choosing the timing law

$$\dot{\theta}(t) = v_d, \qquad \theta(0) = 0, \tag{26}$$

where $v_d > 0$ is a constant to be selected latter, the path-following problem can be viewed as the reference-tracking of the signal $r(t)$ generated by

$$\dot{w}(t) = v_d S w(t), \quad r(t) = Qw(t). \tag{27}$$

The substitution in (5) yields

$$\begin{aligned} \frac{\partial \Pi}{\partial w} v_d S w &= f(\Pi(w), c(w)), \\ h(\Pi(w), c(w)) - Qw &= 0. \end{aligned} \tag{28}$$

The feedback law (25) is determined by the function $c(w)$ that solves (28) and $\alpha_{\delta,\epsilon}(z, \xi, w)$ which minimizes (19) for the error system (18) together with the exosystem (27) and for some small $\delta > 0$, $\epsilon > 0$. From the choice of the timing law (26) and applying Theorem 4, we can conclude that in the limit as $(\epsilon, \delta) \to 0$, $J_P = J_T = \lim_{\delta \to 0} J_\delta^{me}$.

Now, we need to show that J_δ^{me} can be made arbitrarily small be selecting a sufficiently large v_d. To this end, we prove in Lemma 2 in Appendix that for the minimum-energy problem and every initial condition in some neighborhood of $(\tilde{z}, w) = (0, 0)$, there exist a sufficiently small $\delta > 0$ in (21) and a feedback law $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$ for which $J_\delta^{me}(\tilde{z}_0, w_0)$ can be bounded by

$$J_\delta^{me}(\tilde{z}_0, w_0) \le \frac{1}{2} \tilde{z}_0' P_0 \tilde{z}_0,$$

for some $P_0 > 0$ that does not depend on v_d. Observe also that $\tilde{z}_0 = \Pi_0(w_0)$, since $z(0) = 0$. The proof is completed using Lemma 3 in Appendix. □

Next we show that an arbitrarily small $\mathcal{L}_2$-norm of the path-following error is attainable even when the speed assignment v_d is specified beforehand.

Theorem 6. Let v_d be specified so that (28) has a solution in some neighborhood of $w = 0$. For the speed-assigned path-following problem, (8) can be satisfied for any $\delta^\star > 0$ with a suitable timing law $\theta(t)$ and a controller of the form (25) with time-varying piecewise-continuous mappings $c(w)$ and $\alpha(z, \xi, w)$.

Proof (outline). To construct a path-following controller that satisfies (8) we start with

$$u = c_\sigma(w) + \alpha_\sigma(z, \xi, w), \tag{29a}$$

$$\dot{\theta} = v_\sigma, \tag{29b}$$

where for each positive constant v_ℓ, $\ell \in \mathcal{I} := \{0, 1, 2, \ldots N\}$, the mappings $\Pi_\ell := col(\Pi_{\ell_0}, \Pi_{\ell_\xi})$, $\Pi_{\ell_\xi} := col(\Pi_{\ell_{\xi_1}}, \ldots, \Pi_{\ell_{\xi_r}})$, $\Pi_{\ell_0} : \mathbb{R}^p \to R^{n_z}$, $\Pi_{\ell_i} : \mathbb{R}^p \to R^m$, $i = 1, \ldots, r$, and $c_\ell : \mathbb{R}^p \to \mathbb{R}^m$ satisfy

$$\begin{aligned} \frac{\partial \Pi_\ell}{\partial w} v_\ell S w &= f(\Pi_\ell(w), c_\ell(w)), \\ h(\Pi_\ell(w), c_\ell(w)) - Qw &= 0, \end{aligned} \tag{30}$$

and $\sigma(t) : [t_0 := 0, \infty) \to \mathcal{I}$, is the piecewise constant switching signal

$$\sigma(t) = \begin{cases} i, & t_i \le t < t_{i+1}, \; i = 0, \ldots, N-1 \\ N, & t \ge t_N \end{cases}$$

Each $\alpha_\ell(z, \xi, w)$ is the optimal feedback-law that minimizes

$$\int_0^\infty \left(\|e_P\|^2 + \delta \|z - \Pi_{\ell_0}(w)\|^2 + \epsilon^{2r} \|u - c_\ell(w)\|^2 \right) dt,$$

for some $\delta > 0$, $\epsilon > 0$ small. We observe that (29) is a speed-assignment path-following controller for which $\dot{\theta}(t)$ converges to $v_N = v_d$ in finite time.

We now prove that for any given $\delta^\star$, (8) can be satisfied by appropriate selection of the finite sequence $t_0, t_1, \ldots, t_N$ together with $(v_0, \Pi_0, \alpha_0, c_0)$, $(v_1, \Pi_1, \alpha_1, c_1), \ldots, (v_N, \Pi_N, \alpha_N, c_N)$ used in the feedback controller (29). To this end, we show in Lemma 4 in Appendix that J_P is bounded by

$$\begin{aligned} J_P \le{}& \frac{1}{2} \tilde{z}_0' P_0 \tilde{z}_0 + \frac{\lambda_{max}(P_0)}{2} \gamma \sum_{\ell=1}^N (v_{\ell-1} - v_\ell)^2 \\ &+ \lambda_{max}(P_0) \sum_{\ell=1}^N \tilde{z}_{\ell-1}(t_\ell)' \left[\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell) \right] \\ &+ \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^N \|\tilde{z}_{\ell-1}(t_\ell)\|^2, \end{aligned} \tag{31}$$

where $P_0 > 0$, γ is a positive constant, $\tilde{z}_0 = \tilde{z}(0)$, $\bar{z}_\ell := \Pi_{\ell_0}(w)$, and the transient errors $\tilde{z}_\ell := z - \Pi_{\ell_0}(w)$ converge to zero as $t \to \infty$.

We show that each term of (31) can be upper-bounded by $\frac{\delta^\star}{4}$ so that $J_P \le \delta^\star$. Applying the same arguments used to prove Theorem 5, the first term in (31) can be bounded by $\frac{\delta^\star}{4}$ by choosing a sufficiently large v_0. To prove that the second term in (31) is smaller than $\frac{\delta^\star}{4}$, we select the parameters v_ℓ, $\ell \in \mathcal{I}$ to satisfy

$$v_{\ell-1} - v_\ell = \mu, \quad v_N = v_d, \quad \ell = 1, 2, \ldots, N \tag{32}$$

where $\mu := \frac{\delta^\star}{4\gamma(v_0 - v_N)}$, and $N := \frac{v_0 - v_N}{\mu}$. Then, it follows that

$$\gamma \sum_{\ell=1}^N (v_{\ell-1} - v_\ell)^2 \le \gamma N \mu^2 = \gamma(v_0 - v_N)\mu = \frac{\delta^\star}{4}.$$

The above selection for the v_ℓ, $\ell \in \mathcal{I}$, is made under the constraint that the reference-tracking of the signal $r(t)$ generated by (27) (with v_d replaced by v_ℓ) be solvable. This can always be satisfied by appropriately adjusting v_0. Finally, for a given finite N, each of the last two terms in (31) can be made smaller than $\frac{\delta}{4}$ by choosing t_ℓ, $\ell = 1, 2, \ldots, N$ sufficiently large. □

5. CONCLUSIONS

We have first presented an analogy of the results in [Seron *et al.*, 1999; Braslavsky *et al.*, 2002] for reference-tracking using the internal model approach. We have shown that the best attainable value of J_T is the least amount of energy required to stabilize the corresponding zero-dynamics of the error system. In main part of the paper we have demonstrated that the performance limitations can be avoided by reformulating the problem as path-following, where the path variable θ is treated as an additional control variable. This conceptual result may be of practical significance, because the path-following formulation is convenient for many applications. Design of path-following controllers for non-minimum phase systems is a topic of current research.

APPENDIX

Lemma 1. Suppose that Assumption 1 holds. For every $\delta > 0$ and every initial condition $(\tilde{z}_0, \tilde{\xi}_0, w_0)$ for the error system (18) and the exosystem (4) in some neighborhood of $(0,0,0)$, there exist a sufficiently small $\epsilon > 0$ and feedback law $\tilde{u} = \hat{\alpha}^{cc}_{\delta,\epsilon}(\tilde{z}, \tilde{\xi}, w)$ for which (19) is smaller than or equal to

$$J^{me}_{\delta}(\tilde{z}_0, w_0) + O(\epsilon). \tag{33}$$

Proof. This proof is inspired from [Braslavsky *et al.*, 2002]. Let $\alpha^{me}_{\delta}(\tilde{z}, w)$ be the optimal control law for the minimum-energy problem, that is,

$$\alpha^{me}_{\delta}(\tilde{z}, w) = -\tilde{g}_0' \frac{\partial' J^{me}_{\delta}}{\partial \tilde{z}}, \tag{34}$$

which exists under Assumption 1. Consider the change of coordinates

$$\eta_1 = \tilde{\xi}_1 - \alpha^{me}_{\delta}, \quad \eta_2 = \dot{\tilde{\xi}}_1 - \dot{\alpha}^{me}_{\delta}, \dots \quad \eta_r = \tilde{\xi}^{(r-1)}_1 - \alpha^{(r-1)}_{\delta} \tag{35}$$

that takes (18) to the form

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)\alpha^{me}_{\delta}(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)C\eta, \\ \dot{\eta} &= A\eta + B\tilde{f}_{\eta}(\tilde{z}, \eta, w) + Bv, \\ \eta_1 &= C\eta, \end{aligned} \tag{36}$$

where $\eta := \mathrm{col}(\eta_1, \eta_2, \dots, \eta_r)$, the input $v := \tilde{g}\tilde{u}$, $\tilde{g} := \Pi^r_{i=1}\tilde{g}_i$, $\tilde{f}_{\eta}(\cdot)$ is a suitable defined function, and

$$A := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ I \end{bmatrix}, \quad C := \begin{bmatrix} I & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{37}$$

The proof is organized in two parts: we first prove that for any $\delta > 0$ and any initial condition $(\tilde{z}_0, \eta_0, w_0)$ in some neighborhood of $(0,0,0)$ for system (36) together with the exosystem (4), there exist a sufficient small $\epsilon > 0$ and a feedback law $v = \hat{\alpha}_{\delta,\epsilon}(\tilde{z}, \eta, w)$ for which

$$V_{\delta,\epsilon}(\tilde{z}_0, \eta_0, w_0) := \frac{1}{2}\int_0^{\infty} \left(\|\eta_1 + \alpha^{me}_{\delta}(\tilde{z}, w)\|^2 + \delta\|\tilde{z}\|^2 + \epsilon^{2r}\|v\|^2\right) dt$$

is smaller than or equal to (33) along trajectories to (4), (36). Then, in the second part, we derive a feedback law for $\tilde{u}$ that is obtained from $\hat{\alpha}_{\delta,\epsilon}$ and show that (19) is smaller than or equal to (33).

Let $P \in \mathbb{R}^{mr \times mr}$ be the unique positive-definite solution of the algebraic Riccati equation

$$PA + A'P - PBB'P = -C'C \tag{38}$$

where A, B, C are given by (37), and consider the feedback law

$$v = -\frac{1}{\epsilon^r}B'PE\eta, \tag{39}$$

where

$$E = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \epsilon I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \epsilon^{r-1}I \end{bmatrix}.$$

Substituting (39) into (36), we obtain the closed-loop system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)\alpha^{me}_{\delta}(\tilde{z}, w) \\ &\quad + \tilde{g}_0(\tilde{z}, w)CE\eta, \\ \epsilon E\dot{\eta} &= (A - BB'P)E\eta + \epsilon^r B\tilde{f}_{\eta}(\tilde{z}, \eta, w). \end{aligned} \tag{40}$$

We now show that (39) achieves asymptotic stability of the closed-loop system (40). Consider the candidate Lyapunov function

$$W = J^{me}_{\delta}(\tilde{z}, w) + \frac{\epsilon}{2}\eta' EQE\eta,$$

where $Q \in \mathbb{R}^{mr \times mr}$ is the unique positive definite solution of the algebraic Riccati equation

$$Q(A - BB'P) + (A' - PBB')Q = -C'C - I$$

which exists because $(A - BB'P)$ is Hurwitz. Computing $\dot{W}$ along (40), we obtain

$$\begin{aligned} \dot{W} &= \frac{\partial J^{me}_{\delta}}{\partial \tilde{z}}(\tilde{f}_0 + \tilde{g}_0\alpha^{me}_{\delta} + \tilde{g}_0 CE\eta) + \frac{\partial J^{me}_{\delta}}{\partial w}s(w) \\ &\quad + \frac{1}{2}\eta' E[Q(A - BB'P) + (A' - PBB')Q]E\eta \\ &\quad + \epsilon^r \eta' EQB\tilde{f}_{\eta} \\ &= -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|\alpha^{me}_{\delta}\|^2 - \alpha^{me\,\prime}_{\delta} C\eta - \frac{1}{2}\|CE\eta\|^2 \\ &\quad - \frac{1}{2}\|E\eta\|^2 + \epsilon^r\eta' EQB\tilde{f}_{\eta}, \end{aligned}$$

where we have used the fact that $J^{me}_{\delta}(\tilde{z}, w)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial J^{me}_{\delta}}{\partial \tilde{z}}\tilde{f}_0(\tilde{z}, w) + \frac{\partial J^{me}_{\delta}}{\partial w}s(w) + \frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\left\|\tilde{g}_0' \frac{\partial' J^{me}_{\delta}}{\partial \tilde{z}}\right\|^2 = 0. \tag{41}$$

By noting that $\tilde{f}_{\eta}(\cdot)$ satisfies

$$\|\tilde{f}_{\eta}(\tilde{z}, \eta, w)\| \le \gamma_{\tilde{z}}\|\tilde{z}\| + \gamma_{\eta}\|\eta\|,$$

for any positive constants $\rho_{\tilde{z}}$, ρ_{η}, ρ_w so that $\|\tilde{z}\| \le \rho_{\tilde{z}}$, $\|\eta\| \le \rho_{\eta}$, $\|w\| \le \rho_w$, and sufficient large positive constants $\gamma_{\tilde{z}}$, γ_{η}, it follows that

$$\begin{aligned} \dot{W} &\le -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|C\eta + \alpha^{me}_{\delta}\|^2 - \frac{1}{2}\|E\eta\|^2 \\ &\quad + \epsilon^r\left(\gamma_1\|\tilde{z}\| + \gamma_2\|\eta\|\right)\|E\eta\| \\ &\le -\frac{1}{2}(\delta - \gamma_1\epsilon^r)\|\tilde{z}\|^2 - \frac{1}{2}(1 - \gamma_1\epsilon^r - 2\gamma_2\epsilon)\|E\eta\|^2 \end{aligned}$$

where $\gamma_1 := \gamma_{\tilde{z}}\|QB\|$, $\gamma_2 := \gamma_{\eta}\|QB\|$, and we have used the fact that $\epsilon^r\|\eta\| \le \epsilon\|E\eta\|$ for $\epsilon < 1$.

Consequently, for any $\delta > 0$, we can always pick a sufficiently small $\epsilon > 0$ such that $\dot{W} \leq 0$. Asymptotic stability of $(\tilde{z}, \eta) = (0, 0)$ thus follows from LaSalle's theorem. Note also that W converges to zero as $t \to \infty$ because $J_\delta^{me}(0, w) = 0$.

To prove that

$$V_{\delta,\epsilon}(\tilde{z}_0, \eta_0, w_0) \leq J_\delta^{me}(\tilde{z}_0, w_0) + O(\epsilon), \tag{42}$$

we define the positive-definite function

$$\hat{V} := J_\delta^{me}(\tilde{z}, w) + \epsilon \hat{V}_\epsilon(\eta),$$

where

$$\hat{V}_\epsilon := \frac{1}{2}\eta' E P E \eta.$$

The time-derivatives of J_δ^{me} and $\hat{V}_\epsilon$ along (40) satisfy

$$\dot{V}_\delta^{me} = -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|\alpha_\delta^{me}\|^2 - \alpha_\delta^{me\prime} C\eta, \tag{43}$$

$$\dot{\hat{V}}_\epsilon = -\frac{1}{2\epsilon}\|CE\eta\|^2 - \frac{1}{2\epsilon}\|B'PE\eta\|^2 + \epsilon^{r-1}\eta' EPB\tilde{f}_\eta. \tag{44}$$

and

$$\dot{\hat{V}} = -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|C\eta + \alpha_\eta\|^2 - \frac{1}{2}\|B'PE\eta\|^2 + \epsilon^r \eta' EPB\tilde{f}_\eta. \tag{45}$$

From the definitions of $\hat{V}$ and W, and the fact that W converges to zero as $t \to \infty$, we can conclude that $\hat{V}$ also converges to zero as $t \to \infty$. Thus, integrating (45) and by noticing that

$$V_{\delta,\epsilon} = \frac{1}{2}\int_0^\infty \left(\|C\eta + \alpha_0\|^2 + \delta\|\tilde{z}\|^2 + \|B'PE\eta\|^2\right) dt,$$

we get

$$V_{\delta,\epsilon} = \hat{V}(\tilde{z}_0, \eta_0, w_0) + \int_0^\infty \epsilon^r \eta' EPB\tilde{f}_\eta \, dt.$$

Integrating (44), the term $\int_0^\infty \epsilon^{r-1}\eta' EPB\tilde{f}_\eta \, dt$ can be upper-bounded by $\|\int_0^\infty \epsilon^{r-1}\eta' EPB\tilde{f}_\eta \, dt\| \leq V_\epsilon(\eta_0)$, and therefore we conclude (42).

The second part of the proof consists of showing that for every $\delta > 0$ and every initial condition in some neighborhood of $(z, \xi, w) = (0, 0, 0)$, there exists a $\epsilon^\star > 0$ for which (19) with the feedback law

$$\tilde{u} = \tilde{g}^{-1}\hat{\alpha}_{\delta,\epsilon^\star}(\tilde{z}, \eta, w) = -\frac{1}{\epsilon^{\star r}}\tilde{g}^{-1}B'PE\eta$$

is smaller than or equal to (33).

Let γ^r be the lowest value of the smallest singular value of $\tilde{g}$ in a compact set containing the trajectories generated by u. Note that $\gamma > 0$ because $\tilde{g}_i^{-1}$, $i = 1, \ldots, r$ are nonsingular. Let $\epsilon^\star = \frac{\epsilon}{\gamma}$, then,

$$\frac{1}{2}\int_0^\infty \left(\|e_T\|^2 + \delta\|\tilde{z}\|^2 + \epsilon^{2r}\|\tilde{u}\|^2\right) dt \leq \frac{1}{2}\int_0^\infty \left(\|e_T\|^2 + \delta\|\tilde{z}\|^2 + \frac{\epsilon^{2r}}{\gamma^{2r}}\|\tilde{g}\tilde{u}\|^2\right) dt = V_{\delta,\epsilon^\star} \leq J_\delta^{me}(\tilde{z}_0, w_0) + O(\epsilon^\star).$$

□

Lemma 2. Consider the minimum-energy problem formulated in Section 4. For every initial condition $(\tilde{z}(0), w(0)) = (\tilde{z}_0, w_0)$ for (20) in some neighborhood of $(0, 0)$, there exist a sufficiently small $\delta > 0$ and a feedback law $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$ for which (21) is smaller than or equal to

$$\frac{1}{2}\tilde{z}_0' P_0 \tilde{z}_0.$$

for some $P_0 > 0$ that does not depend on v_d.

Proof. Let

$$\dot{z} = F_0 z + G_0 \xi_1$$

be the local linearization around $z = 0$ of (13a). The $\tilde{z}$ dynamics (18a) can be written as

$$\dot{\tilde{z}} = F_0\tilde{z} + G_0 e + h_{\tilde{f}_0}(\tilde{z}, w) + h_{\tilde{g}_0}(\tilde{z}, w)e_T, \tag{46}$$

where $h_{\tilde{f}_0}(\tilde{z}, w) := \tilde{f}_0(\tilde{z}, w) - F_0\tilde{z}$, $h_{\tilde{g}_0}(\tilde{z}, w) := \tilde{g}_0(\tilde{z}, w) - G_0$ and satisfy for all $\|\tilde{z}\| \leq \rho_{\tilde{z}}$, $\|w\| \leq \rho_w$

$$\|h_{\tilde{f}_0}(\tilde{z}, w)\| \leq \gamma_1\|\tilde{z}\|^2 + \gamma_2\|\tilde{z}\| \, \|\Pi_0(w)\|,$$
$$\|h_{\tilde{g}_0}(\tilde{z}, w)\| \leq \gamma_3\|\tilde{z}\|^2 + \gamma_4\|\Pi_0(w)\|,$$

for some positive constants γ_i, $i = 1, \ldots, 4$. Consider (46) in closed-loop with the feedback law

$$e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w) := -G_0' P_0 \tilde{z}, \tag{47}$$

where $P_0 > 0$ satisfies

$$F_0' P_0 + P_0 F_0 + I - P_0 G_0 G_0' P_0 = 0. \tag{48}$$

Let

$$\hat{V}_\delta^{me} := \frac{1}{2}\tilde{z}' P_0 \tilde{z}.$$

Computing its time-derivative along the trajectories of (46), (47), yields

$$\begin{aligned}\dot{\hat{V}}_\delta^{me} &= \frac{1}{2}\tilde{z}'[P_0F_0 + F_0'P_0 - 2P_0G_0G_0'P_0]\tilde{z} \\ &\quad + \frac{1}{2}\tilde{z}'[Ph_{\tilde{f}_0} + h_{\tilde{f}_0}'P - 2P_0h_{\tilde{g}_0}G_0'P_0]\tilde{z} \\ &\leq -\frac{1}{2}\|e_T\|^2 - \frac{1}{2}\Big[1 - 2\lambda_{max}(P_0)\big(\gamma_1\|\tilde{z}\|^2 \\ &\quad + \gamma_2\|\tilde{z}\| \, \|\Pi_0(w)\| - \lambda_{max}(P_0)(\gamma_3\|\tilde{z}\| \\ &\quad + \gamma_4\|\Pi_0(w)\|)\big)\Big]\|\tilde{z}\|^2.\end{aligned}$$

Therefore, there exist sufficiently small positive constants $\rho_{\tilde{z}}$, ρ_w and a $\delta > 0$ such that for all $\|\tilde{z}\| \leq \rho_{\tilde{z}}$, $\|w\| \leq \rho_w$, $\dot{\hat{V}}_\delta^{me}$ satisfies

$$\dot{\hat{V}}_\delta^{me} \leq -\frac{1}{2}\|e_T\|^2 - \frac{\delta}{2}\|\tilde{z}\|^2. \tag{49}$$

Thus, $\hat{V}_\delta^{me}$ converges to zero as $t \to \infty$ and integrating (49) we obtain

$$\hat{V}_\delta^{me}(\tilde{z}_0) \geq \frac{1}{2}\int_0^\infty (\|e_T\|^2 + \delta\|\tilde{z}\|^2) \, dt.$$

□

Lemma 3. Consider the reference-tracking problem for the nonlinear system (13) with the vector field $s(w)$ and the output map $q(w)$ of the exosystem (4) given by $s(w) = v_d S w$, $q(w) = Qw$.

Suppose that all eigenvalues of $S \in \mathbb{R}^{p\times p}$ are non-zero and semisimple, and that (28) has a solution in some neighborhood of $w = 0$ for some $v_d > 0$. Then, for any $\rho > 0$, there exists a sufficiently large $v_d > 0$ such that the reference-tracking problem is solvable. Moreover, the mapping $\Pi_0 : \mathbb{R}^p \to \mathbb{R}^{n_z}$ that satisfies

$$\frac{\partial \Pi_0(w)}{\partial w} Sw = \mu\big[f_0(\Pi_0(w)) + g_0(\Pi_0(w))Qw\big] \tag{50}$$

$\mu := \frac{1}{v_d}$, is bounded by

$$\|\Pi_0(w)\| \le \rho. \tag{51}$$

Proof. Without loss of generality, we assume that S is a diagonal matrix. For a given neighborhood $\{w \in \mathbb{C}^p : \|w\| \le \epsilon\}$ for which the solution exists and a given $\rho > 0$, we expand $\Pi_0(w)$ in Taylor series

$$\Pi_0(w) = \Pi_0^{[1]}(w) + \frac{1}{2}\Pi_0^{[2]}(w) + \cdots + \frac{1}{k!}\Pi_0^{[k]}(w) + O(w)^{k+1} \tag{52}$$

where the superscript $[d]$ denotes terms composed of homogeneous polynomials of degree d, i.e.,

$$\Pi_0^{[k]}(w) = \sum_{1\le i_1\le\cdots\le i_k\le p} \Pi_{0_{i_1\ldots i_k}} w_{i_1}\ldots w_{i_k}, \tag{53}$$

$\Pi_{0_{i_1\ldots i_k}} \in \mathbb{C}^{n_z}$, and pick a sufficiently large N such that

$$\|\Pi_0(w) - \sum_{k=1}^{N}\frac{1}{k!}\Pi_0^{[k]}(w)\| \le \frac{\rho}{2}, \quad \forall_{\|w\|\le\epsilon}. \tag{54}$$

We prove (51) if there exists $\mu > 0$ for which all terms $k = 1, 2, \ldots, N$ of $\Pi_{0_{i_1\ldots i_k}}$ can be bounded by

$$\|\Pi_{0_{i_1\ldots i_k}}\| \le \delta, \quad \delta := \frac{\rho}{2(e^{p\epsilon}-1)}. \tag{55}$$

Note that

$$\begin{aligned}\|\Pi_0(w)\| &\le \|\Pi_0(w) - \sum_{k=1}^{N}\frac{1}{k!}\Pi_0^{[k]}(w)\| \\ &\quad + \|\sum_{k=1}^{N}\frac{1}{k!}\Pi_0^{[k]}(w)\| \\ &\le \frac{\rho}{2} + \sum_{k=1}^{\infty}\frac{1}{k!}\|\Pi_0^{[k]}(w)\|.\end{aligned}$$

Since $\|\Pi_0^{[k]}(w)\| \le \delta p^k \|w\|^k$, it follows that

$$\|\Pi_0(w)\| \le \frac{\rho}{2} + \delta\sum_{k=1}^{\infty}\frac{1}{k!}(p\epsilon)^k = \frac{\rho}{2} + \delta(e^{p\epsilon}-1) = \rho.$$

Expanding $f_0(z)$, $g_0(z)$ in Taylor series

$$f_0(z) = F_0 z + \frac{1}{2}f_0^{[2]}(z) + \cdots + \frac{1}{k!}f_0^{[k]}(z) + O(z)^{k+1},$$

$$g_0(z) = G_0 + g_0^{[1]}(z) + \cdots + \frac{1}{k!}g_0^{[k-1]}(z) + O(z)^k,$$

and inserting these series expansions together with the expansion of $\Pi_0(w)$ into (50) and matching the first order terms, we obtain

$$\frac{\partial \Pi_0^{[1]}}{\partial w}(w)Sw = \mu\big[F_0\Pi_0^{[1]}(w) + G_0 Qw\big]. \tag{56}$$

Substituting $\Pi_0^{[1]}(w) = \sum_{i=1}^{p}\Pi_i w_i$, $\Pi_i \in \mathbb{R}^{n_z}$ in (56) and matching the term in w_i, yields

$$\lambda_i \Pi_i w_i = \mu\big[F_0\Pi_i + G_0 Q\mathrm{e}_i\big]w_i, \quad \forall_i \tag{57}$$

where the λ_i denote the eigenvalues of S, and e_i the i^{th} unit vector. Taking norms we get

$$\|\Pi_i\| \le \mu\frac{\bar{f}_0\|\Pi_i\| + a_{1_i}}{|\lambda_i|},$$

where $\bar{f}_0 := \|F_0\|$ and $a_{1_i} := \|G_0 Q\mathrm{e}_i\|$. Therefore, we can conclude that $\|\Pi_i\| \le \delta$ provided that

$$\mu \le \min_{1\le i\le p}\frac{\delta|\lambda_i|}{\bar{f}_0\delta + a_{1_i}} \text{ and } \mu < \frac{|\lambda_i|}{\bar{f}_0}.$$

To guarantee that (57) has a non trivial solution for all Π_i, (suppose for example that Π_i is an eigenvector of F_0), the constant μ should also satisfies the non-resonance constraint [Krener, 1992]

$$\mu z_i \ne \lambda_j, \quad i = 1, \ldots, n_z;\ j = 1, \ldots, p$$

where z_i are the eigenvalues of F_0, which turns out to be the zeros of the local linearization around the origin of (13).

Consider now the degree two part of (50) that satisfies

$$\frac{\partial \Pi_0^{[2]}}{\partial w}(w)Sw = \mu\big[F_0\Pi_0^{[2]}(w) + f_0^{[2]}(\Pi_0^{[1]}(w)) + \frac{1}{2}g_0^{[1]}(\Pi_0^{[1]}(w))Qw\big]. \tag{58}$$

We can expand $\Pi_0^{[2]}$, $f_0^{[2]}$, and $g_0^{[1]}Qw$ in terms of the ω_i as

$$\begin{aligned}\Pi_0^{[2]}(w) &= \sum \Pi_{0_{ij}} w_i w_j,\\ f_0^{[2]}(\Pi_0^{[1]}(w)) &= \sum f_{0_{ij}} w_i w_j,\\ g_0^{[1]}(\Pi_0^{[1]}(w))Qw &= \sum g_{0_{ij}} w_i w_j,\end{aligned}$$

where $\Pi_{0_{ij}} \in \mathbb{C}^{n_z}$, $f_{0_{ij}} \in \mathbb{C}^{n_z}$, $g_{0_{ij}} \in \mathbb{C}^{n_z}$, and the sums range over $1 \le i \le j \le p$. Substituting these equations into (58), matching the terms $w_i w_j$ and noticing that

$$\frac{\partial \Pi_0^{[2]}}{\partial w}(w)Sw = \sum \Pi_{0_{ij}}(\lambda_i + \lambda_j)w_i w_j, \quad \forall_{i,j}$$

we obtain

$$(\lambda_i + \lambda_j)\Pi_{0_{ij}} w_i w_j = \mu\big[F_0\Pi_{0_{ij}} + f_{0_{ij}} + \frac{1}{2}g_{0_{ij}}\big]w_i w_j.$$

Taking norms, yields

$$\|\Pi_{0_{ij}}\| \le \mu\frac{\bar{f}_0\|\Pi_{0_{ij}}\| + a_{2_{ij}}}{|\lambda_i + \lambda_j|},$$

where $a_{2_{ij}} := \|f_{0_{ij}} + \frac{1}{2}g_{0_{ij}}\|$. Therefore, we can conclude that $\|\Pi_{0_{ij}}\| \le \delta$ provided that

$$\mu \le \min_{1\le i\le j\le p}\frac{\delta|\lambda_i + \lambda_j|}{\bar{f}_0\delta + a_{2_{ij}}} \text{ and } \mu < \frac{|\lambda_i + \lambda_j|}{\bar{f}_0}.$$

Also, for all z_i, μ should satisfies the second non-resonance constraint

$$\mu z_i \neq \lambda_{j_1} + \lambda_{j_2}, \quad 1 \leq j_1 \leq j_2 \leq p.$$

Generalizing to an arbitrary degree k, we have that the degree k equations of (50) satisfies

$$\frac{\partial \Pi_0^{[k]}}{\partial w}(w)Sw = \mu\big[F_0\Pi_0^{[k]}(w) + f_0^{[k]}(w) + \frac{1}{k}g_0^{[k-1]}(w)Qw\big], \quad (59)$$

where $f_0^{[k]}(w)$ and $g_0^{[k-1]}(w)$ are the degree k and $k-1$ parts of the composition of $f_0(z)$ and $g_0(z)$ with the expansion of $\Pi_0(w)$ up to degree $k-1$. As before, expanding $\Pi_0^{[k]}$, $f_0^{[k]}$, and $g_0^{[k-1]}Qw$ in terms of $w_{i_1}w_{i_2}\dots w_{i_k}$ with $i_1 \leq i_2 \leq \cdots \leq i_k \leq p$, and substituting these expansions in (59) yields

$$(\lambda_{i_1} + \cdots + \lambda_{i_k})\Pi_{0_{i_1\dots i_k}} w_{i_1}\dots w_{i_k} = \mu\big[F_0\Pi_{0_{i_1\dots i_k}} + f_{0_{i_1\dots i_k}} + \frac{1}{k}g_{0_{i_1\dots i_k}}\big]w_{i_1}\dots w_{i_k}.$$

Taking norms, we get

$$\|\Pi_{0_{i_1\dots i_k}}\| \leq \mu\frac{\bar f_0\|\Pi_{0_{i_1\dots i_k}}\| + a_{k_{i_1\dots i_k}}}{|\lambda_{i_1} + \cdots + \lambda_{i_k}|},$$

where $a_{k_{i_1\dots i_k}} := \|f_{0_{i_1\dots i_k}} + \frac{1}{k}g_{0_{i_1\dots i_k}}\|$. Thus, it follows that $\|\Pi_{0_{i_1\dots i_k}}\| \leq \delta$ provided that

$$\mu \leq \min_{1\leq i_1\leq\cdots\leq i_k\leq p} \frac{\delta|\lambda_{i_1} + \cdots + \lambda_{i_k}|}{\bar f_0\delta + a_{k_{i_1\dots i_k}}},$$

and

$$\mu < \frac{|\lambda_{i_1} + \cdots + \lambda_{i_k}|}{\bar f_0}.$$

Once more, the selection of μ should satisfies all the non-resonance constraints. The k^{th} non-resonance constraint is given by

$$\mu z_i \neq \lambda_{j_1} + \cdots + \lambda_{j_k}, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq p.$$

□

Lemma 4. Under the conditions of Theorem 6, there exists a sufficiently large constant $\gamma > 0$ such that J_P satisfies (31).

Proof. We first compute

$$J_\ell := \int_{t_\ell}^{\infty} \|e_P(t)\|^2\, dt, \quad \ell \in \mathcal{I}$$

with $\sigma(t) = \ell$ for all $t \geq t_\ell$ and note that

$$J_P \leq \sum_{\ell=0}^{N} J_\ell. \quad (60)$$

As in the derivation of proof of Theorem 5, we get

$$J_\ell \leq \frac{1}{2}\tilde z_\ell(t_\ell)' P_0 \tilde z_\ell(t_\ell), \quad (61)$$

where $P_0 > 0$ satisfies (48), $\tilde z_\ell := z - \bar z_\ell$ and $\bar z_\ell = \Pi_{\ell_0}(w)$, that is, $\bar z_\ell$ is the steady-state of z when $\sigma(t) = \ell$ for all $t \geq t_\ell$. For $\ell = 1, 2, \dots, N$ we substitute $z(t_\ell) = \tilde z_{\ell-1}(t_\ell) + \bar z_{\ell-1}(t_\ell)$ in (61) and get

$$\begin{aligned} J_\ell &\leq \frac{1}{2}\big(\tilde z_{\ell-1}(t_\ell) + \bar z_{\ell-1}(t_\ell) - \bar z_\ell(t_\ell)\big)' P_0 \big(\tilde z_{\ell-1}(t_\ell) + \bar z_{\ell-1}(t_\ell) - \bar z_\ell(t_\ell)\big) \\ &\leq \frac{\lambda_{max}(P_0)}{2}\Big(\|\bar z_{\ell-1}(t_\ell) - \bar z_\ell(t_\ell)\|^2 + 2\tilde z'_{\ell-1}(t_\ell)\big[\bar z_{\ell-1}(t_\ell) - \bar z_\ell(t_\ell)\big] + \|\tilde z_{\ell-1}(t_\ell)\|^2\Big). \end{aligned} \quad (62)$$

We now prove that $\tilde\Pi_{\ell_0}(w) := \bar z_{\ell-1} - \bar z_\ell = \Pi_{\ell-1_0}(w) - \Pi_{\ell_0}(w)$ can be written as

$$\tilde\Pi_{\ell_0}(w) = \alpha_\ell(w)\tilde\mu_l, \quad (63)$$

where $\tilde\mu := \mu_{\ell-1} - \mu_\ell$ and $\alpha_\ell(w)$ is a continuous functions that satisfies

$$\begin{aligned} \frac{\partial\alpha}{\partial w}(w)Sw &= f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw \\ &+ \mu_\ell\big[\beta_f(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\alpha(w) \\ &+ \beta_g(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\alpha(w)Qw\big]. \end{aligned} \quad (64)$$

To show (63), we consider (50) for each $\Pi_{\ell-1_0}(w)$, $\Pi_{\ell_0}(w)$, that is,

$$\frac{\partial\Pi_{\ell-1_0}(w)}{\partial w}Sw = \mu_{\ell-1}\big[f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw\big]$$

$$\frac{\partial\Pi_{\ell_0}(w)}{\partial w}Sw = \mu_\ell\big[f_0(\Pi_{\ell_0}(w)) + g_0(\Pi_{\ell_0}(w))Qw\big]$$

Subtracting the equations, yields

$$\begin{aligned} \frac{\partial\tilde\Pi_{\ell_0}(w)}{\partial w}Sw &= \tilde\mu_\ell\big[f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw\big] + \mu_\ell\big[f_0(\Pi_{\ell-1_0}(w)) \\ &- f_0(\Pi_{\ell_0}(w)) + \big(g_0(\Pi_{\ell-1_0}(w)) - g_0(\Pi_{\ell_0}(w))\big)Qw\big]. \end{aligned} \quad (65)$$

Applying the mean value theorem and replacing (63) in (65), it follows (64).

Assuming that $\Pi_{\ell-1_0}(w)$ and $\Pi_{\ell_0}(w)$ exist, then $\alpha(w)$ is bounded and there exists a sufficiently large constant $\gamma > 0$ such that $\|\tilde\Pi_{\ell_0}(w)\|^2 \leq \gamma\tilde\mu_\ell^2$. This fact, together with (60) and (62) follows (31).

□

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