Minimum-Energy State Estimation for Systems with Perspective Outputs

(Technical Report)

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Abstract

This paper addresses the state estimation of systems with perspective outputs. We derive a minimum-energy estimator which produces an estimate of the state that is "most compatible" with the dynamics, in the sense that it requires the least amount of noise energy to explain the measured outputs. Under suitable observability assumptions, the estimate converges globally asymptotically to the true value of the state in the absence of noise and disturbance. In the presence of noise, the estimate remains bounded away from the true value of the state. These results are also extended to solve the estimation problem when the plant outputs are transmitted through a network. In that case, we assume that the measurements arrive at discrete-time instants, are time-delayed, noisy, and may not be complete. We show that the re-designed minimum-energy estimator preserves the same convergence properties. We apply these results to the estimation of position and orientation for a mobile robot that uses a monocular charged-coupled-device (CCD) camera mounted on-board to observe the apparent motion of stationary points. In the context of our application, the estimator can deal directly with the usual problems associated with vision systems such as noise, latency and intermittency of observations. Experimental results are presented and discussed.

Index Terms

Visual servo control; Observers for nonlinear systems; Estimation; Robotics; Networked control systems

I. Introduction

The state estimation of nonlinear systems has received considerable attention in the literature during the last few decades. A particular branch of engineering where observers for nonlinear systems have been applied rather extensively is Computer Vision. In particular to solve the problems of pose estimation, shape tracking, 3D surface estimation, among others. In this paper, the practical motivation is the pose estimation problem for mobile robots using measurements from a charged-coupled-device (CCD) camera mounted on-board that observes the apparent motion of stationary points. The dynamics of these systems belongs to the class of systems with perspective outputs, which will be introduced in the next subsection. The reader is referred to [1]–[3] for several other examples of perspective systems in the context of motion and shape estimation.

We are interested in designing state-estimators for systems with perspective outputs in the presence of noise and disturbances. We also consider the case where measurements are transmitted through a network, arriving at discrete-time instants, are time-delayed, and may not be complete.

The remaining of this section introduces the problem of state estimation for systems with perspective outputs using a minimum-energy approach, presents previous related work, describes the main contributions, and provides a brief overview of pose estimation of autonomous vehicles using visual information.

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A. State estimation for systems with perspective outputs

Consider a continuous-time system described by

$$\dot{x} = A(u)x + b(u) + G(u)\mathbf{d}, \tag{1}$$

$$\alpha_j y_j = C_j(u)x + d_j(u) + \mathbf{n}_j, \qquad j \in \mathcal{I} := \{1, 2, \dots, N\} \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ its input, $y_j \in \mathbb{R}^{q_j}$ its jth perspective output, $\mathbf{d} \in \mathbb{R}^{n_d}$ an input disturbance that cannot be measured, and $\mathbf{n}_j \in \mathbb{R}^{q_j}$ measurement noise affecting the jth output. Each $\alpha_j \in \mathbb{R}$, $j \in \mathcal{I}$ denotes a scalar that is determined by a normalization constraint such as

$$\|y_j\| = 1 \qquad \text{or} \qquad v_j' y_j = 1, \tag{3}$$

where the $v_j \in \mathbb{R}^{q_j}$ denote constant vectors. We call (1)–(2) a *state-affine system with multiple perspective outputs*, or for short simply a *system with perspective outputs*. These type of systems are inspired by the (single output) perspective systems introduced by Ghosh et al. [1]. Notice that when the matrices A, b, and all the C_j, d_j are constant, and $\mathbf{d}$ and all the $\mathbf{n}_j$ are zero, we essentially have a *perspective linear system* in the sense of [1].

In the last few years, the observability of perspective linear systems was been systematically studied in the literature and Dayawansa et al. [4] provide an elegant algebraic observability test. It should be noted that for perspective linear systems without inputs it is never possible to recover the norm of the state because the system is homogeneous on the initial conditions. Therefore Dayawansa et al. [4] only consider state indistinguishability up a homogeneous scaling of the state. However, as shown in [5], in the presence of inputs it is in principle possible to recover the whole state from perspective outputs.

Motivated by the above considerations, one of the main contributions of this paper is the design of a state-estimator for (1)–(2). In Section II we propose a minimum-energy estimator that produces an estimate for the state of the perspective system that is "most compatible" with the system's dynamics and measured outputs. In particular, the optimal state estimate $\hat{x}$ at time t is defined to be the value for the state that is compatible with the observations collected up to time t and the dynamics of the system for the "smallest" possible measurement noise $\mathbf{n}_j$ and disturbances $\mathbf{d}$, with "smallest" understood in an integral-square sense. This formulation is purely deterministic but leads to a state-estimator that resembles a Kalman-Bucy filter. In fact, if the same approach towards state-estimation was applied to a linear system with linear outputs, one would arrive precisely at the Kalman-Bucy filter that would be obtained in a stochastic setting [6].

Minimum-energy estimators were first proposed by Mortensen [7] and further refined by Hijab [8]. Game theoretical versions of these estimators were proposed by McEneaney [9]. It was recently shown by Krener [10] that this type of estimators is globally convergent when the system is observable for *every* input. Under less restrictive observability assumptions, in Section II-B we show that for perspective systems with multiple inputs, the state-estimator proposed has the desirable property that the state-estimate converges asymptotically to the true value of the state in the absence of noise and disturbance. In the presence of noise, the estimate remains bounded away from the true value of the state. We can therefore use this state-estimator to design output-feedback controllers by using the estimated state to drive state-feedback controllers.

Another problem that is tackled in this paper is state-estimation for systems with perspective outputs when the measurements are transmitted through a network. Over the past few years there has been a considerable research effort in the area of networked control systems. The reader is refereed to [11] for a survey on this topic. In Section III we assume that the measurements are acquired only at discrete times t_i', $i = 1, 2, \dots, k$, with $t_1' < t_2' < \dots < t_k'$, and that we only have access to them after a time-delay τ_i. The sequence of measurements is therefore given by

$$\mathbf{y}_j(t_i) := y_j(t_i') = y_j(t_i - \tau_i),$$

where $\mathbf{y}$ denotes the time-delay observed variable, and $t_i := t_i' + \tau_i$. We also suppose that the measurements may not be complete, that is, at time t_i' only the outputs y_j with $j \in \mathcal{I}_i$ are available, where $\mathcal{I}_i \subseteq \mathcal{I}$ and the inclusion may be strict when some measurements are missing.

The problem under consideration is then to design an observer which estimates the continuous-time state vector $x(t)$ governed by equation (1), given the discrete time-delay measurements $\mathbf{y}(t_i)$ expressed by the output equation

$$\boldsymbol{\alpha}_{j\,i}\mathbf{y}_j(t_i) = C_j(u(t_i - \tau_i))x(t_i - \tau_i) + d_j(u(t_i - \tau_i)) + \mathbf{n}_j(t_i - \tau_i), \quad j \in \mathcal{I}_i, \tag{4}$$

where $\boldsymbol{\alpha}_{j_i}$ is a normalization constraint such that (3) holds for $\mathbf{y}_j(t_i)$.

Convergence properties and observability conditions under which the estimate state $\hat{x}$ converges to the process state x are investigated in Section III-B.

B. Pose estimation of autonomous vehicles using visual information

A fundamental problem in mobile robotics is the determination of position and orientation with respect to an inertial coordinate system using a camera mounted on a robot that observes the apparent motion on the image of stationary points. The linear and angular velocities of the camera can be assumed known in its own coordinate system (possibly with errors due to noise) but not in the inertial coordinate system. This is quite reasonable in mobile robotics where the motion of the camera is determined by the applied control signals. The problem of estimating the position and orientation of a camera mounted on a rigid body from the apparent motion of point features has a long tradition in the computer vision literature (cf., e.g., [12]–[17] and references therein). Filtering-like or iterative algorithms that continuously improve the estimates as more data (i.e., images) are acquired and that are robust with respect to measurement noise are especially desirable. Soatto et al. [14] formulated the visual motion estimation problem in terms of identification of nonlinear implicit systems with parameters on a topological manifold and propose a dynamic solution either in the local coordinates or in the embedding space of the parameter manifold. In [17], rigid-body pose estimation using inertial sensors and a monocular camera is considered. A local convergent observer where the states evolve on $SO(3)$ is proposed (the rotation estimation is decoupled from the position estimation). In the area of wheeled mobile robots, Ma et al. [18] address the problem of tracking an arbitrarily shaped continuous ground curve by formulating it as controlling the shape of the curve in the image plane. Observability of the curve dynamics is studied and an extended Kalman filter is proposed to dynamically estimate the image quantities needed for feedback control from the actual noisy images. An application for landing an unmanned air vehicle using vision in the control loop is described in [19]. In [15], the autonomous aircraft landing problem based on measurements provided by airborne vision and inertial sensors is addressed. The authors cast the problem in a linear parametrically varying framework and solve it using tools that borrows from the theory of linear matrix inequalities. These results are extended in [20] to deal with the so-called out-of-frame events.

In Section IV we formulate the problem of estimating the position and orientation of a controlled rigid body using measurements from an on-board CCD camera as a state-estimation problem of a perspective system. The problem is then solved by using the minimum-energy estimators derived in the previous sections. One of the main contributions is that—opposite what happens with most previous algorithms—the ones proposed here are globally convergent provided that suitable observability assumptions are satisfied. These assumptions are independent of the initialization of the estimator and depend solely on the motion of the camera. Global convergence to the correct position and orientation is achieved in the absence of noise. When there is noise, the magnitude of the estimation error is essentially proportional to the amount of noise. Another difference with respect to several other algorithms is that we also estimate scale. This can be achieved either through known (scaling) information about the points observed and/or through the knowledge of the camera's linear velocity. We also consider singular configurations for the points under observation, e.g., all points coplanar.

Another novelty of this paper is that we explicitly address the fact that the noisy measurements arrive at discrete-time instants, are time-delayed, and may not be complete. In this way, we can deal with usual problems associated to vision systems such as noise, latency, and occlusions.

The theoretical results were experimentally validated by applying them to estimate the position and orientation of a mobile robot using measurements from an on-board CCD camera. We then use these estimates to close the feedback loop and control the robot to a desired position, defined with respect to visual landmarks. The results obtained are discussed in Section IV-C.

This paper builds upon and extends previous results by the authors [21]–[23].

II. State Estimation and Control for Systems with Perspective Outputs

Consider the perspective output system (1)–(2). For appropriate noise and disturbance signals, essentially every value for the state x at a time $t \in \mathbb{R}$ is compatible with any outputs y_j observed on the interval $[0, t)$. However, we will favor estimates for the state that can be made compatible with the measured outputs utilizing "small" noise and disturbance signals. In fact, we formulate state estimation as a deterministic optimization problem in which the estimate $\hat{x}(t)$ of the state at time $t \geq 0$ is the value for which the measured outputs can be made compatible with the system dynamics (1)–(2) for the "smallest" possible noise $\mathbf{n}_j$ and disturbance $\mathbf{d}$. More specific, we address the following problem:

Problem 1: Given an input u and measured outputs y_j, $j \in \mathcal{I}$, defined on an interval $[0,t)$, compute the estimate $\hat{x}(t)$ of the state at time t defined by

$$\hat{x}(t) := \arg\min_{z\in\mathbb{R}^n} J(z,t), \tag{5}$$

where

$$J(z;t) := \min_{\mathbf{d},\mathbf{n}_j,\alpha_j} \Big\{ (x(0)-\hat{x}_0)' P_0 (x(0)-\hat{x}_0) + \int_0^t \Big(\|\mathbf{d}\|^2 + \sum_{j=1}^{N} \|\mathbf{n}_j\|^2 \Big) d\tau : \\ x(t)=z,\ \dot{x} = A(u)x + b(u) + G(u)\mathbf{d},\ \alpha_j y_j = C_j(u)x + d_j(u) + \mathbf{n}_j \Big\}, \tag{6}$$

and $P_0 > 0$, $\hat{x}_0$ encode a-priori information about the state.

Remark 1: The approach just described towards state estimation can be viewed as the computation of a "generalized pseudo-inverse" that attempts to recover the current value of the state $x(t)$ from the measured output $y(\tau)$, $\tau \in [0,t]$. To understand what is meant by this let U_t denote the triple consisting of the quantities that cannot be measured but affect the value of the state at time t. Namely, the initial state, past noise and past disturbances:

$$U_t := \{x(0); \mathbf{n}(\tau), \tau \in [0,t]; \mathbf{d}(\tau), \tau\in[0,t]\}.$$

We denote by $\mathcal{U}_t$ the space of such triples. The system dynamics (1)–(2) define the following two operators

$$X_t : \mathcal{U}_t \to \mathbb{R}^n \qquad\qquad O_t : \mathcal{U}_t \to \mathcal{Y}_t$$
$$U_t \mapsto x(t) \qquad\qquad U_t \mapsto Y_t := \{y_j(\tau), \tau\in[0,t], j = 1,2,\dots,N\}$$

where $\mathcal{Y}_t$ denotes the appropriate output space. One can then view state estimation as solving the following system of equations

$$\hat{x} = X_t(\hat{U}_t), \qquad\qquad Y_t = O_t(\hat{U}_t), \tag{7}$$

for the unknowns $\hat{x}$ and $\hat{U}_t$. If the observation operator O_t had a left inverse O_t^{-1} (e.g., in the absence of noise and if the system was observable) the solution to (7) would be unique and given by $\hat{x} = X_t\big(O_t^{-1}(Y_t)\big)$. However, in general this is not the case and the approach we propose is to replace the left inverse of O_t by its "pseudo-inverse." In particular, we define the estimate to be $\hat{x} = X_t\big(O_t^{\perp}(Y_t)\big)$, where $O_t^{\perp}(Y_t)$ denotes the min-norm solution to $Y_t = O_t(U_t)$, i.e., $O_t^{\perp}(Y_t) = \arg\min_{\substack{U_t\in\mathcal{U}_t:\\ Y_t = O_t(U_t)}} \|U_t\|_{\mathcal{U}_t}$. The norm $\|\cdot\|_{\mathcal{U}_t}$ (or more precisely its square) is specified by the cost (6). □

A. The observer equations

We now present the observer equations that can be derived using dynamic programming. The following result solves Problem 1.

Theorem 1: The solution to the state-estimation problem defined by (5) and (6) is given by

$$\dot{Q} = A(u)Q + QA(u)' + G(u)G(u)' - QWQ, \qquad Q(0) = P_0^{-1}, \tag{8}$$

$$\dot{\hat{x}} = (A(u) - QW)\hat{x} + b(u) - Qw, \qquad \hat{x}(0) = \hat{x}_0, \tag{9}$$

where

$$W(t) := \sum_{j=1}^{N} C_j'(u)\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)C_j(u), \qquad w(t) := \sum_{j=1}^{N} C_j'(u)\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)d_j(u), \qquad t \geq 0.$$

Furthermore, the cost function $J(z;t)$ defined in (6) is quadratic and can be written as

$$J(z;t) = \big(z - \hat{x}(t)\big)' P(t) \big(z - \hat{x}(t)\big) + c(t), \tag{10}$$

where $c(t)$ satisfies

$$\dot{c} = \hat{x}' W \hat{x} + \sum_{j=1}^{N} d_j' \Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big) d_j, \qquad c(0) = 0. \tag{11}$$

Before proving Theorem 1 note that we can rewrite the state-estimation equation (9) as

$$\dot{\hat{x}} = A(u)\hat{x} + b(u) + Q\sum_{j=1}^{N} C_j(u)'\Big(\hat{\alpha}_j y_j - C_j(u)\hat{x} - d_j(u)\Big), \qquad \hat{\alpha}_j = \frac{y_j'\big(C_j(u)\hat{x} + d_j(u)\big)}{\|y_j\|^2}, \tag{12}$$

which emphasizes the parallel between (9) and a Kalman-Bucy filter for linear systems.

Proof: [Theorem 1] The function $J(z;t)$, $z \in \mathbb{R}^n$, $t \geq 0$ can be viewed as a cost-to-go. To derive the dynamic programming operator we can consider an elementary time interval dt and write[1]

$$\begin{aligned}
J(z;t) = &\min_{\mathbf{d},\alpha_j} \Big\{ (x(0)-\hat{x}_0)'P_0(x(0)-\hat{x}_0) + \Big(\|\mathbf{d}\|^2 + \sum_{j=1}^{N} \|\alpha_j y_j - C_j z - d_j\|^2\Big)dt \\
&+ \int_0^{t-dt} \Big(\|\mathbf{d}\|^2 + \sum_{j=1}^{N} \|\alpha_j y_j - C_j x - d_j\|^2\Big)d\tau : \\
&x(t-dt) = z - (Az + b + G\mathbf{d})dt,\ \dot{x} = Ax + b + G\mathbf{d}\Big\} \\
= &\min_{\mathbf{d},\alpha_j} \Big(\|\mathbf{d}\|^2 + \sum_{j=1}^{N} \|\alpha_j y_j - C_j z - d_j\|^2\Big)dt + J(z - (Az + b + G\mathbf{d})dt; t - dt).
\end{aligned}$$

Subtracting $J(z;t-dt)$ from both sides of the above equation, dividing by dt, and taking the limit as $dt \to 0$, leads to

$$\begin{aligned}
J_t(z;t) = &\min_{\mathbf{d},\alpha_j} \|\mathbf{d}\|^2 + \sum_{j=1}^{N} \|\alpha_j y_j - C_j z - d_j\|^2 - J_z(z;t)(Az + b + G\mathbf{d}) \\
= &\min_{\mathbf{d},\alpha_j} \big\|\mathbf{d} - \frac{1}{2}G'J_z(z;t)'\big\|^2 - \frac{1}{4}\|G'J_z(z;t)'\|^2 - J_z(z;t)(Az+b) \\
&+ \sum_{j=1}^{N} \|y_j\|^2\alpha_j^2 - 2y_j'(C_j z + d_j)\,\alpha_j + \|C_j z + d_j\|^2 \\
= &-\frac{1}{4}\|G'J_z(z;t)'\|^2 - J_z(z;t)(Az+b) \\
&+ \min_{\alpha_j} \sum_{j=1}^{N} \|y_j\|^2\Big(\alpha_j - \frac{y_j'(C_j z + d_j)}{\|y_j\|^2}\Big)^2 + (C_j z + d_j)'\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)(C_j z + d_j) \\
= &-\frac{1}{4}\|G'J_z(z;t)'\|^2 - J_z(z;t)(Az+b) + \sum_{j=1}^{N}(C_j z + d_j)'\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)(C_j z + d_j),
\end{aligned} \tag{13}$$

where J_t and J_z denote the partial derivatives of J with respect to t and z, respectively. The value of $J(z;t)$ can then be determined from the linear partial differential equation (13) with initial condition

$$J(z;0) = (z - \hat{x}_0)'P_0(z - \hat{x}_0), \qquad z \in \mathbb{R}^n. \tag{14}$$

It turns out that the solution to (13)–(14) can be written as (10) for appropriately defined signals $\hat{x}(t)$ and $c(t)$. The signal $\hat{x}$ is then precisely the estimate for the state x of the perspective linear system. Moreover, matching (14) with (10) we conclude that $P(0) = P_0$, $\hat{x}(0) = \hat{x}_0$, and $c(0) = 0$. To verify that the solution to (13)–(14) can indeed be written as (10), we replace this equation in (13) and obtain

$$\begin{aligned}
&-2(z-\hat{x})'P\dot{\hat{x}} + (z-\hat{x})'\dot{P}(z-\hat{x}) + \dot{c} \\
&\qquad = -\|G'P(z-\hat{x})\|^2 - 2(z-\hat{x})'P(Az+b) + \sum_{j=1}^{N}(C_j z + d_j)'\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)(C_j z + d_j),
\end{aligned}$$

or equivalently

$$\begin{aligned}
&z\big(\dot{P} + PA + A'P + PGG'P - W\big)z + 2z'\big(-P\dot{\hat{x}} - \dot{P}\hat{x} - PGG'P\hat{x} - A'P\hat{x} + Pb - w\big) \\
&\qquad + \dot{c} + 2\hat{x}'P\dot{\hat{x}} + \hat{x}'\dot{P}\hat{x} + \hat{x}'PGG'P\hat{x} - 2\hat{x}'Pb - \sum_{j=1}^{N} d_j'\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)d_j = 0.
\end{aligned}$$

[1]For simplicity of notation, we will drop the dependence on u of the matrices A, b, G, C_j and d_j.

This equation holds provided that

$$\dot{P} + PA + A'P + PGG'P - W = 0 \tag{15}$$

$$-P\dot{\hat{x}} - \dot{P}\hat{x} - PGG'P\hat{x} + Pb - A'P\hat{x} - w = 0 \tag{16}$$

$$\dot{c} + 2\hat{x}'P\dot{\hat{x}} + \hat{x}'\dot{P}\hat{x} + \hat{x}'PGG'P\hat{x} - 2\hat{x}'Pb - \sum_{j=1}^{N} d_j'\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)d_j = 0. \tag{17}$$

Replacing (15) in (16) and these two equations in (17), we obtain (11) and

$$-\dot{P} = PA + A'P + PGG'P - W \tag{18}$$

$$P\dot{\hat{x}} = PA\hat{x} + Pb - W\hat{x} - w \tag{19}$$

$$\dot{c} = \hat{x}'W\hat{x} + \sum_{j=1}^{N} d_j'\Big(I - \frac{y_j y_j'}{\|y_j\|^2}\Big)d_j. \tag{20}$$

It turns out that $P(t)$ remains positive definite for all time (see Claim 1 in Appendix). Therefore (19) is actually equivalent to (9) with $Q = P^{-1}$. Using the fact that $\dot{Q} = -Q\dot{P}Q$, it is straightforward to conclude that the matrix Q can be generated directly from (8). ■

B. Estimator convergence

We are now interesting in determining under what conditions does the state estimate $\hat{x}$ converges to the true state x of the perspective system. The following technical assumption is needed:

Assumption 1: There exist positive constants $\delta, \Delta \in (0, \infty)$ such that $\delta I \leq G(u)G'(u) \leq \Delta I$, $\forall u \in \mathbb{R}^m$.

This assumption essentially guarantees that $G(u)$ is bounded and full-row rank, "uniformly" over all possible inputs. The following result establishes the convergence of the state estimate.

Theorem 2: Assuming that the solution to the process (1)–(2) exists globally, the solution to state estimator (8)–(9) also exists globally. Moreover, when Assumption 1 holds and Q remains uniformly bounded, there exist positive constants $c, \lambda, \gamma_d, \gamma_1, \ldots, \gamma_N$ such that

$$\|\tilde{x}(t)\| \leq ce^{-\lambda t}\|\tilde{x}(0)\| + \gamma_d \sup_{\tau \in (0,t)} \|\mathbf{d}(\tau)\| + \sum_{j=1}^{N} \gamma_j \sup_{\tau \in (0,t)} \|\mathbf{n}_j(\tau)\|, \qquad t > 0, \tag{21}$$

where $\tilde{x}(t) := \hat{x}(t) - x(t)$ denotes the state estimation error.

Proof: See the Appendix. ■

Some assumption on the observability[2] of (1)–(2) would be expected to achieve convergence of the estimated state $\hat{x}$ to the process state x. In Theorem 2 this assumption appear in the form of the requirement that Q remains bounded. In the remaining of this section we investigate conditions under which this happens.

From (8) it is clear that Q remains bounded if $W(t) \geq \epsilon I > 0$, $\forall t \geq 0$ because in this case the term $-QWQ$ eventually dominates for very large Q. However, this case is not very interesting because, e.g., for the single output case ($N = 1$) the matrix W typically has rank equal to $(\operatorname{rank} C_1) - 1 \leq n - 1$. The following Lemma provides a significantly weaker condition for the boundedness of Q.

Lemma 1: The matrix Q remains bounded along trajectories of the system (1)–(2) and state-estimator (8)–(9), provided that there exist positive constants T, ϵ such that the following condition

$$\frac{1}{T}\int_0^T \Phi(t+\tau, t)'W(t+\tau)\Phi(t+\tau, t)d\tau \geq \epsilon I > 0, \qquad \forall t \geq 0, \tag{22}$$

holds, where $\Phi(t, \tau)$ denotes the state transition matrix of $\dot{z} = A(u)z$.

Proof: See the Appendix. ■

[2]In the present setup, the correct notion is actually constructability because we are attempting to reconstruct the state from past outputs [24, Section 3.3].

To get some intuition for the meaning of (22) note that for $\int_0^T \Phi(t+\tau,t)'W(t+\tau)\Phi(t+\tau,t)d\tau$ to be singular, there would have to be a vector x_0 such that

$$x_0'\Phi(t+\tau,t)'W(t+\tau)\Phi(t+\tau,t)x_0 = 0, \qquad \forall \tau \in (0,T),\ t\geq 0,$$

or equivalently, such that

$$\beta_j(t+\tau)y_j(t+\tau) = C_j\big(u(t+\tau)\big)\Phi(t+\tau,t)x_0, \qquad \forall \tau \in (0,T),\ t\geq 0, \quad j\in\{1,\dots,N\}, \tag{23}$$

for appropriate scalars $\beta_j(t)$. In essence this means that (22) fails when all the y_j evolve as if u, $\mathbf{d}$, and all the $\mathbf{n}_j$ were zero. In fact, we can view (22) as a persistence of excitation-like condition that requires x to evolve is some interesting way, other than just following the homogeneous dynamics of (1)–(2), along which scaling information could not be recovered.

It is interesting to note the parallel between the integral in (22) and the constructibility Gramian for linear system [24, Section 3.3]. In fact, if W were replaced by $\sum_{j=1}^N C_j'C_j$, the integral in (22) is precisely the constructibility Gramian for the system (1) with *linear* outputs $C_jx+d_j+\mathbf{n}_j$, $j\in\{1,2,\dots,N\}$.

Combining Theorem 2 and Lemma 1 we obtain the following:

Corollary 1: When Assumption 1 holds and there exist constants T, ϵ such that (22) holds, the state-estimate $\hat{x}$ converges exponentially fast to the state x in the absence of disturbance input and measurement noise. When the disturbance and noise are bounded but nonzero, $\hat{x}$ may not converge to x but remains bounded away from it.

Remark 2: Given the input-to-state stability (ISS) like result with respect to $(\mathbf{d},\mathbf{n}_1,\dots,\mathbf{n}_N)$ described in Theorem 2, we can use the state-estimator to design output-feedback controllers by using the estimated state $\hat{x}$ to drive state-feedback controllers. This is straightforward provided that the state-feedback controllers are robust (in ISS sense) with respect to $(\mathbf{d},\tilde{x})$ (cf., e.g., [25], [26], [27, Section 5.3]).

III. State Estimation from Discrete Noisy Time-Delayed Measurements

This section addresses the state estimation of continuous-time systems with perspective outputs, whose measurements arrive at discrete-time instants, are time-delayed, noisy, and may not be complete.

Before we formulate the optimization problem, observe from (1) that $x(t_i)$ satisfies

$$x(t_i) = \Phi(t_i,t_i-\tau_i)x(t_i-\tau_i) + \int_{t_i-\tau_i}^{t_i}\Phi(t_i,\sigma)\big[b(u(\sigma))+G(u(\sigma))\mathbf{d}(\sigma)\big]\,d\sigma,$$

where $\Phi(t,t_0)$ is the transition matrix of system (1) satisfying the differential equation $\dot{\Phi}=A(u)\Phi$. Therefore,

$$x(t_i-\tau_i) = \Phi^{-1}(t_i,t_i-\tau_i)x(t_i) - \Phi^{-1}(t_i,t_i-\tau_i)\int_{t_i-\tau_i}^{t_i}\Phi(t_i,\sigma)\big[b(u(\sigma))+G(u(\sigma))\mathbf{d}(\sigma)\big]\,d\sigma.$$

Replacing this equation in (4) we get

$$\boldsymbol{\alpha}_{j_i}\mathbf{y}_j(t_i) = \bar{C}_j(u)x(t_i) + \bar{d}_j(u) + \bar{\mathbf{n}}_j(t_i), \quad j\in\mathcal{I}_i, \tag{24}$$

where

$$\begin{aligned}
\bar{C}_j(u) &:= C_j(u(t_i-\tau_i))\Phi(t_i-\tau_i,t_i),\\
\bar{d}_j(u) &:= -\bar{C}_j(u)\int_{t_i-\tau_i}^{t_i}\Phi(t_i,\sigma)b(u(\sigma))\,d\sigma + d_j(u(t_i-\tau_i)),\\
\bar{\mathbf{n}}_j(t_i) &:= -\bar{C}_j(u)\int_{t_i-\tau_i}^{t_i}\Phi(t_i,\sigma)G(u(\sigma))\mathbf{d}(\sigma)\,d\sigma + \mathbf{n}_j(t_i-\tau_i).
\end{aligned}$$

The minimum-energy estimation problem can then be stated as follows:

Problem 2: Given an input u defined on an interval $[0,t)$, and measured outputs $\mathbf{y}_j(t_i)$, $j\in\mathcal{I}_i$ with $i=0,1,\dots k$, $t_0:=0\leq t_1\leq\cdots\leq t_k\leq t$, compute the estimate $\hat{x}(t)$ of the state at time t defined by

$$\hat{x}(t) := \arg\min_{z\in\mathbb{R}^n} J(z,t), \tag{25}$$

where

$$J(z;t) := \min_{\substack{\mathbf{d}:[0,t),\bar{\mathbf{n}}_j(t_i),\boldsymbol{\alpha}_{j\,i}\\ i=0,1,\ldots k}} \Big\{ (x(0)-\hat{x}_0)'P_0(x(0)-\hat{x}_0) + \int_0^t \|\mathbf{d}(\sigma)\|^2\, d\sigma + \sum_{i=0}^{k}\sum_{j\in\mathcal{I}_i} \|\bar{\mathbf{n}}_j(t_i)\|^2 :$$
$$x(t)=z,\ \dot{x}=A(u)x+b(u)+G(u)\mathbf{d},\quad \boldsymbol{\alpha}_{j\,i}\mathbf{y}_j(t_i)=\bar{C}_j(u)x(t_i)+\bar{d}_j(u)+\bar{\mathbf{n}}_j(t_i)\Big\}, \tag{26}$$

and $P_0 > 0$, $\hat{x}_0$ encode a-priori information about the state.

The estimate $\hat{x}(t)$ can be interpreted as the value for which the measured outputs can be made compatible with the system dynamics (1) and (24) for the "smallest" possible noise $\bar{\mathbf{n}}_j$ and disturbance $\mathbf{d}$. This formulation considers the case when all the measurements may not be available at each time t_i because $\mathcal{I}_i$ can be a strict subset of $\mathcal{I}$.

A. *The observer equations*

In what follows, given a signal x with a discontinuity at time t, we denote by $x(t^-)$ the limit from below of $x(\tau)$ as $\tau \uparrow t$, *i.e.*, $x(t^-) := \lim_{\tau\uparrow t} x(\tau)$. Without loss of generality we take all signals to be continuous from above at every point, *i.e.*, $x(t) = \lim_{\tau\downarrow t} x(\tau)$. The following result solves Problem 2.

Theorem 3: The estimate $\hat{x}(t)$ of the state at time $t \geq t_0 := 0$ defined by (25) and (26) can be computed as a solution to the *impulse system* defined by the following dynamic equations for $t_i \leq t < t_{i+1}$, $i = 0, 1, \ldots, k$

$$\dot{P}(t) = -P(t)A(u) - A(u)'P(t) - P(t)G(u)G(u)'P(t), \qquad P(t_i) = P_i \tag{27}$$
$$\dot{\hat{x}}(t) = A(u)\hat{x}(t) + b(u), \qquad \hat{x}(t_i) = \hat{x}_i \tag{28}$$

and the following impulse equations at $t = t_{i+1}$, $i = 0, 1, \ldots, k-1$

$$P(t_{i+1}) = P(t_{i+1}^-) + W(t_{i+1}), \tag{29}$$
$$\hat{x}(t_{i+1}) = \hat{x}(t_{i+1}^-) - P(t_{i+1})^{-1}\big[W(t_{i+1})\hat{x}(t_{i+1}^-) + w(t_{i+1})\big] \tag{30}$$

where

$$W(t_{i+1}) := \sum_{j\in\mathcal{I}_{i+1}} \bar{C}_j'(u)\Big(I - \frac{\mathbf{y}_j(t_{i+1})\mathbf{y}_j(t_{i+1})'}{\|\mathbf{y}_j(t_{i+1})\|^2}\Big)\bar{C}_j(u), \quad w(t_{i+1}) := \sum_{j\in\mathcal{I}_{i+1}} \bar{C}_j'(u)\Big(I - \frac{\mathbf{y}_j(t_{i+1})\mathbf{y}_j(t_{i+1})'}{\|\mathbf{y}_j(t_{i+1})\|^2}\Big)\bar{d}_j(u). \tag{31}$$

Furthermore, the cost function $J(z;t)$ defined in (26) is quadratic and can be written as

$$J(z;t) = \big(z - \hat{x}(t)\big)'P(t)\big(z - \hat{x}(t)\big) + c(t), \tag{32}$$

where $c(0) = 0$ and, for all $i = 0, 1, \ldots k-1$,

$$c(t) = c(t_i), \qquad t_i \leq t < t_{i+1} \tag{33}$$
$$c(t) = -(P(t^-)\hat{x}(t^-) + \hat{x}(t^-)'P(t^-)x(t^-) + c(t^-)$$
$$- w(t))'\big[P(t^-) + W(t)\big]^{-T}(P(t^-)\hat{x}(t^-) - w(t)) + \sum_{j\in\mathcal{I}_{i+1}} \bar{d}_j \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right)\bar{d}_j, \qquad t = t_{i+1} \tag{34}$$

Remark 3: From equation (28) we can conclude, as it was expected, that since in the time interval (t_k, t_{k+1}) there is no additional information, the dynamic equation of the observer for $\hat{x}(t)$ is a replica of the state dynamic equation of system (1). Notice also that for $t = t_{i+1}$, $i = 0, 1, \ldots, k-1$, the state-estimation equation (30) can be rewritten as

$$\hat{x}(t_{i+1}) = \hat{x}(t_{i+1}^-) + P(t_{i+1})^{-1}\sum_{j\in\mathcal{I}_{i+1}} \bar{C}_j'(u)\Big(\hat{\alpha}_j\mathbf{y}_j(t_{i+1}) - \bar{C}_j(u)\hat{x}(t_{i+1}^-) - \bar{d}_j\Big),$$
$$\hat{\alpha}_j = \frac{\mathbf{y}_j'(t_{i+1})\big(\bar{C}_j(u)\hat{x}(t_{i+1}^-) + \bar{d}_j\big)}{\|\mathbf{y}_j(t_{i+1})\|^2}$$

which emphasizes the parallel between (30) and a Kalman filter for linear systems. □

Proof: [Theorem 3] We start by proving (32). Take some $t \in (t_i, t_{i+1})$. Since $J(z;t)$ is a cost-to-go it must satisfy the dynamic programming equation

$$\begin{aligned} J(z;t) = &\min_{\substack{\mathbf{d}:[0,t),\alpha_j(t_i)\\ i=0,1,\dots k}} \Big\{ (x(0)-\hat{x}_0)'P_0(x(0)-\hat{x}_0) + \|\mathbf{d}\|^2 dt + \int_0^{t-dt} \|\mathbf{d}(\sigma)\|^2 \, d\sigma \\ &+ \sum_{i=0}^{k} \sum_{j\in\mathcal{I}_i} \|\alpha_j \mathbf{y}_j(t_i) - \bar{C}_j x(t_i) - \bar{d}_j\|^2 : x(t-dt) = z - (Az+b+G\mathbf{d})dt, \dot{x} = Ax+b+G\mathbf{d} \Big\} \\ = &\min_{\mathbf{d}} \Big\{ \|\mathbf{d}\|^2 dt + J(z-(Az+b+G\mathbf{d})dt; t-dt) \Big\}. \end{aligned}$$

Subtracting $J(z;t-dt)$ from both sides of the above equation, dividing by dt, and taking the limit as $dt \to 0$, leads to

$$\begin{aligned} J_t(z;t) &= \min_{\mathbf{d}} \Big\{ \|\mathbf{d}\|^2 - J_z(z;t)(Az+b+G\mathbf{d}) \Big\} \\ &= \min_{\mathbf{d}} \Big\{ \|\mathbf{d} - \frac{1}{2} G' J_z(z;t)'\|^2 - \frac{1}{4} \|G' J_z(z;t)'\|^2 - J_z(z;t)(Az+b) \Big\} \\ &= -\frac{1}{4} \|G' J_z(z;t)'\|^2 - J_z(z;t)(Az+b), \end{aligned} \tag{35}$$

where J_t and J_z denote the partial derivatives of J with respect to t and z, respectivel. For $k=0$, the value of $J(z;t)$ is determined from the linear partial differential equation (35) with initial condition

$$J(z;0) = (z-\hat{x}_0)'P_0(z-\hat{x}_0), \quad z \in \mathbb{R}^n \tag{36}$$

and can be written as (32) for appropriately defined signals $\hat{x}(t)$ and $c(t)$. The signal $\hat{x}$ is then precisely the estimate for the state x of the perspective system (1), (24). Moreover, matching (36) with (32) we conclude that $P(0) = P_0$, $\hat{x}(0) = \hat{x}_0$, $c(0) = 0$. To verify that the solution to (35)–(36) can be written as (32), we replace this equation in (35), and obtain

$$\begin{aligned} -2(z-\hat{x})'P\dot{\hat{x}} + (z-\hat{x})'\dot{P}(z-\hat{x}) + \dot{c} = \\ - \|G'P(z-\hat{x})\|^2 - 2(z-\hat{x})P(Az+b), \end{aligned}$$

or equivalently

$$\begin{aligned} z'(\dot{P} + PA + A'P + PGG'P)z + 2z'(-P\dot{\hat{x}} - \dot{P}\hat{x} - PGG'P\hat{x} - A'P\hat{x} + Pb) \\ + \dot{c} + 2\hat{x}'P\dot{\hat{x}} + \hat{x}'\dot{P}\hat{x} + \hat{x}'PGG'\hat{x} - 2\hat{x}'Pb = 0. \end{aligned}$$

This equation holds provided that

$$\dot{P} + PA + A'P + PGG'P = 0, \tag{37}$$

$$-P\dot{\hat{x}} - \dot{P}\hat{x} - PGG'P\hat{x} - A'P\hat{x} + Pb = 0, \tag{38}$$

$$\dot{c} + 2\hat{x}'P\dot{\hat{x}} + \hat{x}'\dot{P}\hat{x} + \hat{x}'PGG'\hat{x} - 2\hat{x}'Pb = 0. \tag{39}$$

Replacing (37) in (38) and these two equations in (39), we conclude that (27)–(28) and (32) hold for $0 \le t < t_1$. Observe also that $P(t)$ remains positive definite for all $0 \le t < t_1$ (see Claim 1 in Appendix, but using (27) instead of (18)).

Consider now the case $t = t_k$, $k > 0$. From (26), we notice that $J(z;t_k)$ can be written as

$$\begin{aligned} J(z;t_k) = &\min_{\boldsymbol{\alpha}_{j_k}} \Big\{ \min_{\substack{\mathbf{d}:[0,t_k),\boldsymbol{\alpha}_{j_i}\\ i=0,1,\dots k-1}} \Big\{ (x(0)-\hat{x}_0)'P_0(x(0)-\hat{x}_0) + \int_0^{t_k} \|\mathbf{d}(\sigma)\|^2 \, d\sigma \\ &+ \sum_{j\in\mathcal{I}_k} \|\boldsymbol{\alpha}_{j_k} \mathbf{y}_j(t_k) - \bar{C}_j x(t_k) - \bar{d}_j\|^2 + \sum_{i=0}^{k-1} \sum_{j\in\mathcal{I}_i} \|\boldsymbol{\alpha}_{j_i} \mathbf{y}_j(t_i) - \bar{C}_j x(t_i) - \bar{d}_j\|^2 : \\ &x(t_k^-) = x(t_k) = z, \dot{x} = Ax+b+G\mathbf{d} \Big\} \Big\} \\ = &\min_{\boldsymbol{\alpha}_{j_k}} \Big\{ J(z;t_k^-) + \sum_{j\in\mathcal{I}_k} \|\boldsymbol{\alpha}_{j_k} \mathbf{y}_j(t_k) - \bar{C}_j x(t_k) - \bar{d}_j\|^2 \Big\} \end{aligned} \tag{40}$$

For $k=1$ we already saw that $J(z,t_1^-)$ is given by (32). Assuming that it has the same form at time t_1, replacing it in (40), we obtain

$$\begin{aligned}
J(z;t_k) &= \min_{\alpha_j(t_k)} \Big\{ \big(z-\hat{x}(t_k^-)\big)' P(t_k^-)\big(z-\hat{x}(t_k^-)\big) + c(t_k^-) \\
&\quad + \sum_{j\in\mathcal{I}_k} \Big[\|\mathbf{y}_j(t_k)\|^2 \alpha_j^2(t_k) - 2\mathbf{y}_j(t_k)'(\bar{C}_j z + \bar{d}_j)\alpha_j(t_k) + \|\bar{C}_j z + \bar{d}_j\|^2 \Big] \Big\} \\
&= \min_{\alpha_j(t_k)} \Big\{ \big(z-\hat{x}(t_k^-)\big)' P(t_k^-)\big(z-\hat{x}(t_k^-)\big) + c(t_k^-) \\
&\quad + \sum_{j\in\mathcal{I}_k}^{n} \Big[\|\mathbf{y}_j\|^2 \left(\alpha_j(t_k) - \frac{\mathbf{y}_j'(\bar{C}_j z + \bar{d}_j)}{\|\mathbf{y}_j\|^2}\right)^2 + (\bar{C}_j z + \bar{d}_j)' \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right)(\bar{C}_j z + \bar{d}_j) \Big] \Big\} \\
&= z'P(t_k^-)z - 2z'P(t_k^-)\hat{x}(t_k^-) + \hat{x}(t_k^-)'P(t_k^-)x(t_k^-) + c(t_k^-) \\
&\quad + z' \sum_{j\in\mathcal{I}_k} \bar{C}_j' \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right) \bar{C}_j z + 2z' \sum_{j\in\mathcal{I}_k} \bar{C}_j' \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right) \bar{d}_j + \sum_{j\in\mathcal{I}_k} \bar{d}_j \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right) \bar{d}_j.
\end{aligned} \tag{41}$$

Replacing (32) in (41), one obtain

$$\begin{aligned}
& z'\big[P(t_k) - P(t_k^-) - W(t_k)\big]z + 2z'\big[-P(t_k)\hat{x}(t_k) + P(t_k^-)\hat{x}(t_k^-) - w(t_k)\big] \\
& + c(t_k) + \hat{x}(t_k)'P(t_k)x(t_k) - \hat{x}(t_k^-)'P(t_k^-)x(t_k^-) - c(t_k^-) - \sum_{j\in\mathcal{I}_k} \bar{d}_j \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right) \bar{d}_j = 0,
\end{aligned}$$

where the definitions in (31) were used. This equation holds for $k=1$ provided that

$$P(t_k) - P(t_k^-) - W(t_k) = 0, \tag{42}$$

$$-P(t_k)\hat{x}(t_k) + P(t_k^-)\hat{x}(t_k^-) - w(t_k) = 0, \tag{43}$$

$$c(t_k) + \hat{x}(t_k)'P(t_k)x(t_k) - \hat{x}(t_k^-)'P(t_k^-)x(t_k^-) - c(t_k^-) - \sum_{j\in\mathcal{I}_k} \bar{d}_j \left(I - \frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\right) \bar{d}_j = 0. \tag{44}$$

Replacing (42) in (43) and these two equations in (44), we conclude that (29)–(30), and (34) hold.

Notice that $P_1 := P(t_1) = P(t_1^-) + W(t_1)$ is positive definite because $P(t_1^-) > 0$ as it was proved above, and $W(t_i) \geq 0$, $i = 1, \dots, k$. Therefore, replacing the initial condition (36) by

$$J(z;t_1) = (z - \hat{x}_1)'P_1(z - \hat{x}_1), \quad z \in \mathbb{R}^n$$

with $\hat{x}_1 = \hat{x}(t_1)$, and solving the linear partial differential equation (35), we conclude that (27)-(29) hold for $0 \leq t < t_2$. Applying this line of reasoning successively until $i = k$ we conclude that (32) holds and that $\hat{x}(t)$ given by (27)–(30) is indeed the solution to Problem 2. ∎

B. Estimator convergence

In this section we investigate under what conditions the estimate $\hat{x}$ provided by Theorem 3 converges to the true state x of the perspective system. In addition to Assumption 1, the following technical assumption is needed:

Assumption 2: Let $\mathrm{Num}(t,\sigma)$, $0 \leq \sigma < t$ denote the number of time instants at which measurement arrive in the open interval (σ, t). There exist finite positive constants τ_D and N_0, for which the following condition holds:

$$\mathrm{Num}(t,\sigma) \leq N_0 + \frac{t-\sigma}{\tau_D}.$$

The constant τ_D is called the *average dwell-time* and N_0 the *chatter bound*.

This assumption roughly speaking guarantees that the average interval between consecutive arrival of measurements is no less than τ_D. This type of condition typically arises in the context of logic-based switching control (cf., e.g., [28] and references therein). In our context it guarantees that the summation in (26) will not grow unbounded due to "too frequent" measurements. This assumption is purely technical and is only used to simplify the analysis. Moreover, in practice it always holds.

The following result establishes the convergence of the state estimate.

Theorem 4: Assuming that the solution to the process (1), (24) exists globally, the solution to the impulse state estimator (27)–(30) also exits globally. Moreover, when Assumptions 1-2 hold and P^{-1} remains uniformly bounded, there exist positive constants c, $r < 1$, γ_d, $\gamma_1, \dots, \gamma_N$ such that

$$\|\tilde{x}(t_k)\| \leq c\, r^k \|\tilde{x}(0)\| + \gamma_d \sup_{\tau \in (0,t_k)} \|\mathbf{d}(\tau)\| + \sum_{j=1}^{N} \gamma_j \sup_{\tau \in (0,t_k)} \|\bar{\mathbf{n}}_j(\tau)\|, \quad t_k > 0 \tag{45}$$

where $\tilde{x}(t) := \hat{x}(t) - x(t)$ denotes the state estimation error.

Proof: See the Appendix. ■

As before, some condition on the observability of (1), (24) should be needed to achieve convergence of the estimated state $\hat{x}$ to the process state x. In Theorem 4 this condition appears in the form of the requirement that P^{-1} remains bounded. The following result provides a condition under which this happens.

Lemma 2: The matrix P^{-1} remains uniformly bounded along trajectories of the system (1), (24), and the state-estimator (27)–(30), provided that there exist positive constants $\mathbf{N}$, ϵ such that the following persistence of excitation condition

$$\frac{1}{\mathbf{N}} \sum_{j=0}^{\mathbf{N}} \Phi(t_{i+j}, t_i)' W(t_{i+j}) \Phi(t_{i+j}, t_i)) \geq \epsilon I > 0, \tag{46}$$

$i = 0, 1, \dots, k$, holds, where $\Phi(t, \tau)$ denotes the state transition matrix of $\dot{z} = A(u)z$.

Proof: See the Appendix. ■

Combining Theorem 4 and Lemma 2 we obtain the following:

Corollary 2: When Assumptions 1 and 2 hold, and there exist constants $\mathbf{N}$, ϵ such that the persistence of excitation condition (46) holds, the state-estimate $\hat{x}$ converges to the state x in the absence of disturbance input and measurement noise. When the disturbance and noise are bounded but nonzero, $\hat{x}$ may not converge to x but remains bounded away from it.

IV. Rigid body motion estimation using CCD cameras

In this section we show how one can estimate the position and orientation of a mobile robot using a CCD camera mounted on the robot that observes the apparent motion on the image of stationary points. We do this by reducing the problem to the estimation of the state of a system with perspective outputs.

Consider a coordinate frame $\{b\}$ attached to a rigid body that moves with respect to an inertial frame $\{i\}$. We denote[3] by $(p_{ib}, R_{ib}) \in \mathrm{SE}(3)$ the configuration of the frame $\{b\}$ with respect to $\{i\}$. Thus, if q_1^i and q_1^b denote the coordinates of a point Q_1 in the frames $\{i\}$ and $\{b\}$, respectively, we have that

$$q_1^i = p_{ib} + R_{ib} q_1^b. \tag{47}$$

Moreover, if q_j^i and q_j^b denote the coordinates of another point Q_j in the frames $\{i\}$ and $\{b\}$, respectively, we conclude that

$$q_j^b = R_{ib}' q_j^i - R_{ib}' p_{ib} = R_{ib}'(q_j^i - q^i) + q_1^b.$$

We denote by $(v_{ib}^b, \Omega_{ib}^b) \in \mathrm{se}(3)$ the twist that defines the velocity of frame $\{b\}$ with respect to $\{i\}$, expressed in the frame $\{b\}$, i.e.,

$$\Omega_{ib}^b = R_{ib}' \dot{R}_{ib}, \qquad v_{ib}^b = R_{ib}' \dot{p}_{ib}.$$

From this and (47), we obtain

$$\dot{q}_1^b = -\Omega_{ib}^b q_1^b - v_{ib}^b + R_{ib}' \dot{q}_1^i, \qquad \dot{R}_{ib} = R_{ib} \Omega_{ib}^b.$$

[3]We denote by $\mathrm{SE}(3)$ the Cartesian product of $\mathbb{R}^3$ with the group $\mathrm{SO}(3)$ of 3×3 rotation matrices; and by $\mathrm{se}(3)$ the Cartesian product of $\mathbb{R}^3$ with the space $\mathrm{so}(3)$ of 3×3 skew-symmetric matrices (cf.,e.g., [29]).

Suppose that a camera attached to the body frame $\{b\}$ sees N points $Q_1, Q_2, \ldots, Q_N$. Denoting by $y_j \in \mathbb{R}^3$ the homogeneous image coordinates provided by the camera of the point Q_j, the dynamics of the system can be described by the following system with N perspective outputs:

$$\dot{q}_1^b = -\Omega_{ib}^b q_1^b + R_{ib}' \dot{q}_1^i - v_{ib}^b, \tag{48}$$

$$\dot{R}_{ib}' = -\Omega_{ib}^b R_{ib}', \tag{49}$$

$$\alpha_j y_j = F\big(p_{cb} + R_{cb} q_1^b + R_{cb} R_{ib}'(q_j^i - q_1^i)\big), \quad \forall j \in \{1, 2, \ldots, N\}, \tag{50}$$

where $(p_{cb}, R_{cb}) \in \mathrm{SE}(3)$ denotes the configuration of the frame $\{b\}$ with respect to the camera's frame $\{c\}$, and F an upper triangular matrix with the cameras intrinsic parameters, of the form

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & f_{21} \\ 0 & 0 & 1 \end{bmatrix},$$

where each f_{ij} denotes a scalar [30, Chapter 3]. Note that F and (p_{cb}, R_{cb}) can be time-varying in case the camera is allowed to zoom or pan and tilt, which is often needed to get good visual information. The normalization constraints (3) are given by

$$[\,0\ 0\ 1\,]\, y_j = 1, \quad \forall j \in \{1, 2, \ldots, N\}.$$

To proceed we use the following notation: Given an $m \times n$-matrix M, we denote by $\mathrm{stack}(M)$ the mn-vector obtained from stacking the columns of M one on top of each other, with the first column on top. Given two matrices $M_i \in \mathbb{R}^{m_i \times n_i}$, $i \in \{1, 2\}$ we denote by $M_1 \otimes M_2 \in \mathbb{R}^{m_1 m_2 \times n_1 n_2}$ the Kronecker product of M_1 by M_2. Using the fact that given three matrices with appropriate dimensions $\mathrm{stack}(A\,X\,B) = (B' \otimes A)\,\mathrm{stack}(X)$ [31], we can re-write (48)–(50) as follows:

$$\begin{aligned} \dot{q}_1^b &= -\Omega_{ib}^b q_1^b + (I_{3\times 3} \otimes \dot{q}_1^i)\,\mathrm{stack}(R_{ib}) - v_{ib}^b, \\ \mathrm{stack}(\dot{R}_{ib}) &= (-\Omega_{ib}^b \otimes I_{3\times 3})\,\mathrm{stack}(R_{ib}), \\ \alpha_j y_j &= F p_{cb} + F R_{cb} q_1^b + (F R_{cb} \otimes (q_j^i - q_1^i)')\,\mathrm{stack}(R_{ib}). \end{aligned}$$

Thus, defining x to be a 12-dimensional vector whose first 3 entries are the entries of q_1^b and the remaining 9 entries are the columns of R_{ib} stacked on top of each other, that is,

$$x := \begin{bmatrix} q_1^b \\ \mathrm{stack}(R_{ib}) \end{bmatrix} \in \mathbb{R}^{12},$$

and

$$A := \begin{bmatrix} -\Omega_{ib}^b & I_{3\times 3} \otimes \dot{q}_1^i \\ 0 & -\Omega_{ib}^b \otimes I_{3\times 3} \end{bmatrix}, \qquad b := \begin{bmatrix} -v_{ib}^b \\ 0 \end{bmatrix}, \qquad C_j := \begin{bmatrix} F R_{cb} & F R_{cb} \otimes (q_j^i - q_1^i)' \end{bmatrix}, \qquad d_j := F p_{cb},$$

the system (48)–(50) can be expressed in the form (1)-(2) and therefore the estimators given in Theorems 1 and 3 can be constructed. Note that once we compute estimates $\hat{R}_{ib}$ and $\hat{q}_1^b$ for R_{ib} and q_1^b, respectively, we can also estimate p_{ib} using

$$\hat{p}_{ib} = q_1^i - \hat{R}_{ib} \hat{q}_1^b.$$

A. *Singular configurations*

Depending on the configurations of the points $Q_1, Q_2, \ldots, Q_N$, the state of (48)–(50) may not be observable. In fact, when all points are coplanar or collinear we can, respectively, find 9- or 6-dimensional realization for the system. However, even in this case it may still be possible to recover the position and orientation of the rigid body from the camera measurements by using the fact that R_{ib} is a rotation matrix. To this effect let $M \in \mathbb{R}^{3\times m}$ be a matrix whose columns are a basis for the vector space generated by the $N-1$ vectors $\{q_j^i - q_1^i : j = 2, \ldots, N\}$ and let $\tilde{q}_j$, $j \in \{2, \ldots, N\}$ be such that

$$q_j^i - q_1^i = M \tilde{q}_j.$$

In this case the system (48)–(50) can be re-written as

$$\dot{q}_1^b = -\Omega_{ib}^b q_1^b + R_{ib}' \dot{q}_1^i - v_{ib}^b, \tag{51}$$

$$\dot{\bar{R}} = -\Omega_{ib}^b \bar{R} \tag{52}$$

$$\alpha_j y_j = F\big(p_{cb} + R_{cb} q_1^b + R_{cb} \bar{R} \tilde{q}_j\big), \quad \forall j \in \{1, 2, \ldots, N\}, \tag{53}$$

where $\bar{R} := R'_{ib} M \in \mathbb{R}^{3\times m}$. Note that when $m = \operatorname{rank} M < 3$, the system (48)–(50) is not observable because its input-output map is consistent with that of the lower-dimensional model (51)–(53).

To compute an estimate $\hat{R}_{ib}$ of R_{ib} from the estimate $\hat{\bar{R}}$ of $\bar{R}$, the following two cases should be considered separately. For simplicity we assume that M was chosen orthonormal, i.e., that $M'M = I$.

1) $\operatorname{rank} M = 3$, which corresponds to the existence of 4 non-coplanar points. In this case $\hat{R}_{ib}$ can be recovered directly from $\hat{\bar{R}}$ using $\hat{R}'_{ib} = \hat{\bar{R}} M^{-1}$.

2) $\operatorname{rank} M = 2$, which corresponds to all points being coplanar but not collinear. Denoting by $M^{\perp} \in \mathbb{R}^{3\times(3-m)}$ a matrix whose columns form an orthonormal basis for the orthogonal complement of the image of M (i.e., a full rank matrix such that ${M^{\perp}}' M^{\perp} = I$ and $M' M^{\perp} = 0$), the general solution to $M' \hat{R}_{ib} = \hat{\bar{R}}'$, is of the form

$$\hat{R}_{ib} = M \hat{\bar{R}}' + M^{\perp} \mu',$$

for some vector $\mu \in \mathbb{R}^{3\times(3-m)}$. This vector needs to be determined from the fact that $\hat{R}'_{ib}$ is orthonormal. Since

$$\hat{R}'_{ib} \hat{R}_{ib} = (\hat{\bar{R}} M' + \mu {M^{\perp}}')(M \hat{\bar{R}}' + M^{\perp} \mu') = \hat{\bar{R}} \hat{\bar{R}}' + \mu \mu',$$

we conclude that μ needs to be chosen to make $\hat{\bar{R}} \hat{\bar{R}}' + \mu\mu'$ as close to the identity as possible.

A straightforward way to compute the vector μ is to minimize the Frobenius norm $\|\cdot\|_F$ of

$$\begin{aligned} \|\hat{\bar{R}}\hat{\bar{R}}' + \mu\mu' - I\|_F^2 &:= \operatorname{trace}(\hat{\bar{R}}\hat{\bar{R}}' + \mu\mu' - I)'(\hat{\bar{R}}\hat{\bar{R}}' + \mu\mu' - I) \\ &= \operatorname{trace} \mu\mu'\mu\mu' + 2\operatorname{trace}(\hat{\bar{R}}\hat{\bar{R}}' - I)\mu\mu' + \operatorname{trace}(\hat{\bar{R}}\hat{\bar{R}}' - I)^2 \\ &= \|\mu\|^4 + 2\mu'(\hat{\bar{R}}\hat{\bar{R}}' - I)\mu + \operatorname{trace}(\hat{\bar{R}}\hat{\bar{R}}' - I)^2. \end{aligned}$$

Denoting by λ the most negative eigenvalue of $\hat{\bar{R}}\hat{\bar{R}}' - I$ (which must be negative since any vector in the kernel of $\hat{\bar{R}}'$ is an eigenvector corresponding to the eigenvalue -1) and by v the corresponding unit-norm eigenvector, the previous expression has a minimum at $\mu' = \alpha v$, which is equal to

$$\alpha^4 + 2\alpha^2\lambda + \operatorname{trace}(\hat{\bar{R}}\hat{\bar{R}}' - I)^2,$$

where α is a scalar. The minimum is then obtained for $\alpha^2 = -\lambda$ and therefore $\mu = \pm\sqrt{-\lambda}\, v$,

$$\hat{R}_{ib} = M \hat{\bar{R}}' \pm \sqrt{-\lambda}\, M^{\perp} v'.$$

The sign for the square root can be determined from the constrain that the determinant of $\hat{R}_{ib}$ be positive. Note that μ is unique as long as $\hat{\bar{R}}\hat{\bar{R}}' - I$ does not have more than one eigenvector associated with the most negative eigenvalue.

When all points are collinear there may a fundamental loss of observability that cannot be overcome without extra information. Indeed, when there is a direction v such that all line segments between all points are aligned with v, i.e.,

$$q^i_{j_1} - q^i_{j_2} = \beta_j v, \qquad \forall j_1, j_2 \in \{1, 2, \ldots, N\},\ \beta_{j_1 j_2} \in \mathbb{R},$$

and the velocity of the points (if any) is also aligned with v, i.e., $\dot{q}^i_j = \gamma_j v$, $\forall j \in \{1, 2, \ldots, N\}$,$\gamma_j \in \mathbb{R}$, the matrix $R_v \in \mathrm{SO}(3)$ that corresponds to a fixed rotation around v has the property that

$$R_v \dot{q}^i_1 = \dot{q}^i_1, \qquad\qquad R_v(q^i_j - q^i_1) = q^i_j - q^i_1, \quad \forall j \in \{1, 2, \ldots, N\}.$$

In this case, the system (51)–(53) with $\bar{R}'_{ib} := R'_{ib} R_v$ has exactly the same input-output map as (48)–(50). This means that R_{ib} can only be recovered from the perspective outputs up to a rotation around v.

Remark 4: In [32], we incorporated quadratic state constrains in the minimum-energy formulation. When this is done, there is no need to consider lower dimensional realizations for (48)–(50) when the points are coplanar but not collinear.

B. Unknown inertial coordinates

Suppose now the that the inertial coordinates of the points $Q_1, Q_2, \ldots, Q_N$ are not known. In this case, one can still estimate x by using three of the points to define an inertial coordinate system. To this effect, let

$$S := R'_{ib} \begin{bmatrix} q^i_2 - q^i_1 & q^i_3 - q^i_1 & \cdots & q^i_N - q^i_1 \end{bmatrix}.$$

We can re-write (48)–(50) as

$$\dot{q}_1^b = -\Omega_{ib}^b q_1^b + R'_{ib}\dot{q}_1^i - v_{ib}^b, \tag{54}$$

$$\dot{S} = -\Omega_{ib}^b S \tag{55}$$

$$\alpha_j y_j = F\big(p_{cb} + R_{cb} q_1^b + R_{cb} S \tilde{q}_j\big), \quad \forall j \in \{1, 2, \ldots, N\}, \tag{56}$$

where e_j denotes the jth column of the $(N-1) \times (N-1)$ identity matrix.

To recover an estimate $\hat{R}_{ib}$ of R_{ib} from the estimate $\hat{S}$ of S, we can use the QR decomposition to obtain a rotation matrix $\hat{R}_{ib}$ and an upper triangular matrix $\hat{U}$ such that

$$\hat{S} = \hat{R}'_{ib}\hat{U},$$

and then defining $q_1^i := 0$ and q_j^i, $j \in \{2, 3, \ldots, N\}$ equal to the $(j-1)$th column of $\hat{U}$. This corresponds to the following convention to construct the inertial coordinate system: the origin of $\{i\}$ is the point Q_1; its first axis is defined by the direction from Q_1 to Q_2; its second axis is orthogonal to the first one and lies on the plane defines by Q_1, Q_2, and Q_3; and its third axis is defined by the cross product of the first two.

C. Experimental results

The theoretical results presented in the previous sections were experimentally validated by applying them to estimate the position and orientation of a mobile robot using measurements from an on-board CCD camera. This section describes the experimental setup and presents the results obtained for three types of experiments: Robot stopped, following a circular path, and parking.

Fig. 1. Experimental setup: Pionner 2-DXE mobile robot with CCD camera mounted on top and visual landmark.

The experiments were carried out on a Pionner 2-DXE mobile robot from ActivMedia [33]. The vehicle, shown in Fig. 1, has two rear wheels which are powered by two independent high torque, reversible-DC motors, and one passive rear caster to balance the robot. The vehicle is equipped with a Sony EVI D30 pan-tilt-zoom (PTZ) color video camera mounted on the top of the robot with its optical axis oriented towards the forward direction (when pan and tilt angles are zero). To simplify the image processing, in these experiments we used the corners of a black square as visual landmarks (see Fig. 1). The location of these points were obtained by detecting the edges of the square and then computing their intersections. A pan controller was also implemented to keep as much as possible the visual landmark in the center of the image.

1) Robot stopped experiment: To validate the minimum-energy state estimators described in Sections II–III, several open-loop tests were carried out. A simple test consists of running an estimator with the vehicle stopped and commanding the pan angle of the camera.

Fig. 2 shows the experimental results of this test using the estimator proposed in Section II. The robot is at position $(x, y) = (-0.8\,m, 0\,m)$ with $\theta = 0\,rad$. The pan angle is set to zero. At $t = 20\,s$ a ramp-like signal is applied in pan until $t = 30\,s$. As shown in the figure, the estimator converges to the true values and is not affected by the camera's pan motion.

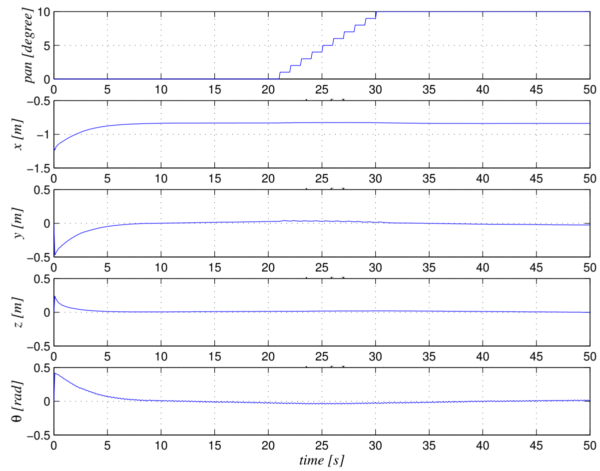


Fig. 2. Robot stopped experiment: Time evolution of pan angle and estimator outputs $\hat{p}_{ib}$ and $\hat{\theta}$.

2) Following a circular path experiment: To illustrate the estimator proposed in Section III, we present here results for a test in which the vehicle follows a circular path with linear velocity $v = 0.06\,m/s$ and angular velocity $\omega = 0.09\,rad/s$. Since the robot is describing a circular trajectory and the pan angle is limited to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the visual landmarks periodically leave the camera's field of view. While this happens, the estimator does not receive any visual measurements.

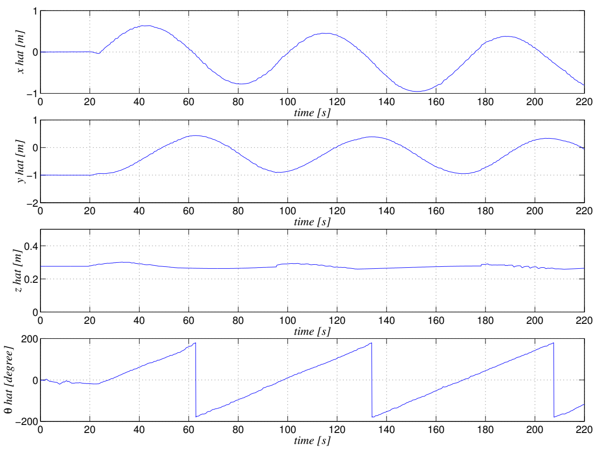
Fig. 3(a)–3(c) show the experimental results. We can see that the output of the estimator converges to the values correspondent to a circular trajectory. Observe also the time evolution of the minimum and maximum singular values of P when the estimator is receiving ($\gamma = 0$) or not receiving ($\gamma = 1$) measurements. Another interesting observation is the behavior of the pan controller that is always trying to compensate the motion of the robot in order to keep the features in the image 3(b).

3) Parking experiment: The nonholonomic kinematics model of the Pionner 2-DXE is given by

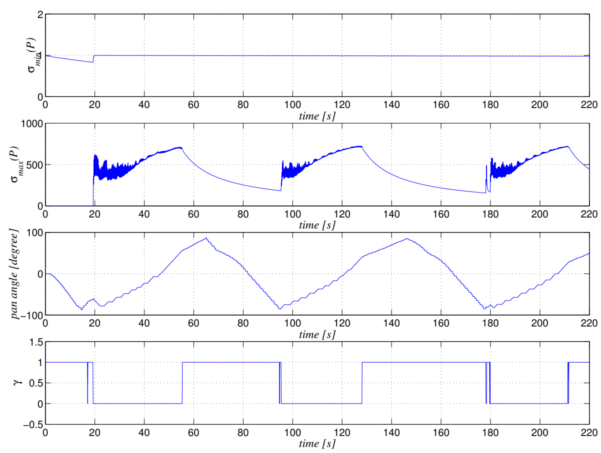
$$\dot{x} = v\cos(\theta), \quad \dot{y} = v\sin(\theta), \quad \dot{\theta} = \omega,$$

where v and ω denote the linear and angular velocity. In this section we present experimental results obtained using the minimum-energy state estimator described in Section II combined with a pan controller and the point stabilization controller presented in [34]. The stabilization control law used is a piecewise time-invariant continuous feedback law and it is based on a non-smooth state transformation inspired by the polar description of the kinematics of the robot. This particular controller was chosen because its control strategy is intuitive, simple to implement, offers a good performance, and the resulting paths are fairly natural, *i.e.*, similar to what a human operator would attempt. However, as reported in [35], this controller is inherently sensitive to sensor noise and small perturbations around the origin.

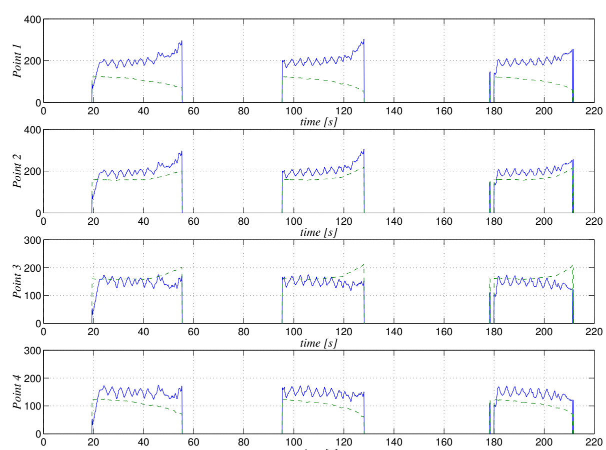
Fig. 4(a)–4(c) show the experimental results for the case where the vehicle starts stationary at $(x, y) = (-0.8\,m, 0.4\,m)$ and with heading $\theta = 0$. The objective was to park the vehicle at position $(x, y) = (-0.5\,m, 0\,m)$ with heading $\theta = 0$. Fig. 4(a) displays the resulting vehicle trajectory computed based on dead-reckoning data measurements. Note that dead-reckoning only gives the pose of the mobile robot with respect to the starting initial condition and is therefore not useful to park at a point specified in a referential frame defined by visual landmarks. Fig. 4(b) shows the time evolution of estimator outputs $\hat{p}_{ib}$ and $\hat{\theta}$, and Fig. 4(c) the time evolution of control signals u_1, u_2, and u_{pan}. In this experiment we imposed that the point stabilization control algorithm only starts to operate at time $t = 5\,s$ and the pan controller at time $t = 1\,s$. From the figures it can be seen that the vehicle converges to a very small neighborhood of the desired pose. From Fig. 4(c) it can be observed that the pan controller is indeed able to compensate the motion of the robot in order to keep the features in the center of the image.



(a)

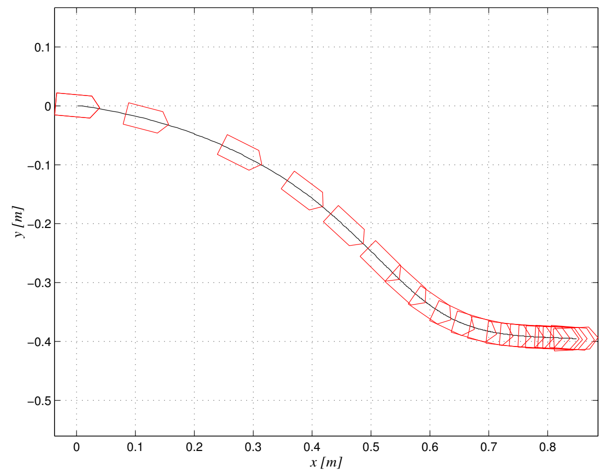


(b)

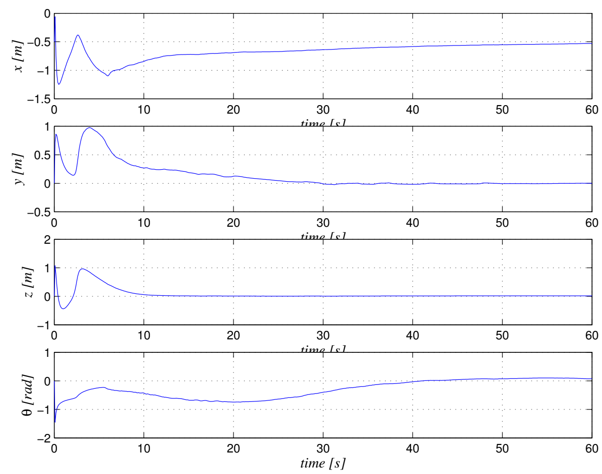


(c)

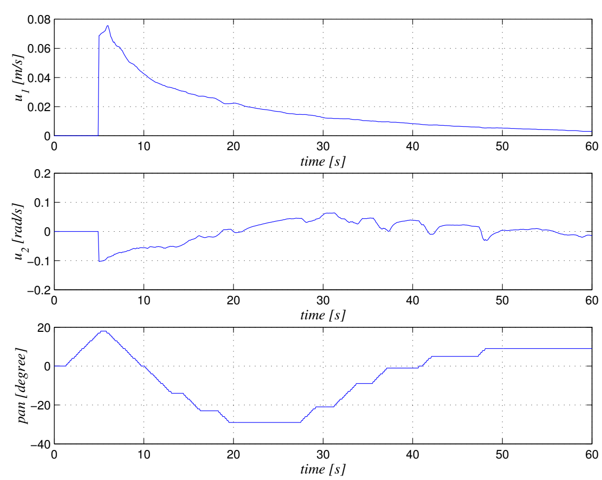
Fig. 3. Following a circular path experiment: Time evolution of 3(a) the estimated position $(\hat{x}, \hat{y}, \hat{z})$, and orientation $\hat{\theta}$; 3(b) the minimum and maximum singular values of P, respectively; the pan angle; and the variable γ which indicates when the estimator is receiving ($\gamma = 0$) or not ($\gamma = 1$) measurements; and 3(c) the position in the image of the corners of the visual landmark (x – , y - -). When the points are out of the camera's field of view, the points' coordinates are not shown.



(a)



(b)



(c)

Fig. 4. Parking experiment: 4(a) Resulting trajectory of the mobile robot in the xy-plane using dead-reckoning data measurements; Time evolution of 4(b) estimator outputs $\hat{p}_{ib}$ and $\hat{\theta}$; and 4(c) control signals $u_1(t)$, $u_2(t)$, and $u_{pan}(t)$.

V. Conclusions

We considered the problem of estimating the state of a system with perspective outputs either available continuously or only at discrete time instants with some delay and perhaps incomplete. We designed an estimator that is globally convergent under appropriate observability assumptions and can therefore be used to design output-feedback controllers. We extended this design methodology to solve the same estimation problem but now considering that the measurements arrive at discrete-time instants, are time-delayed, noisy, and may not be complete. We applied these results to estimate the position and orientation of a mobile robot using measurements from an attached CCD camera. The estimator proposed requires the robot's linear and angular velocities. Adaptive estimation techniques can probably be used to estimate these parameters when they are not available. This is the subject of our future research. Another topic for future research is to incorporate algebraic constrains on the state in the estimation algorithm. Preliminary results regarding this issue can be found in [32], where we incorporated quadratic state constrains in the minimum-energy formulation.

Appendix

Claim 1: The matrix $P(t)$ governed by (18) is positive definite for all $t \geq 0$.

Proof: Observe that (18) can also be written as

$$\dot{P} = -P(A + GG'P) - (A + GG'P)'P + PGG'P + W,$$

and therefore

$$P(t) = \Psi(0,t)'P_0\Psi(0,t) + \int_0^t \Psi(\tau,t)'\big(PGG'P + W\big)\Psi(\tau,t)d\tau, \qquad t \geq 0, \tag{57}$$

where $\Psi(t,\tau)$ denotes the state transition matrix of $\dot{z} = (A + GG'P)z$. This can be verified by taking derivatives of the candidate expression for P:

$$\begin{aligned}\dot{P} &= -(A + GG'P)'\Psi(0,t)'P_0\Psi(0,t) - \Psi(0,t)'P_0\Psi(\tau,t)(A + GG'P) + PGG'P + W \\ &\quad - (A + GG'P)'\int_0^t \Psi(\tau,t)'\big(PGG'P + W\big)\Psi(\tau,t)d\tau - \int_0^t \Psi(\tau,t)'\big(PGG'P + W\big)\Psi(\tau,t)d\tau(A + GG'P) \\ &= -(A + GG'P)'P - P(A + GG'P) + PGG'P + W.\end{aligned}$$

Here we used the fact that, for every fixed τ,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(\tau,t) &= -\Psi(t,\tau)^{-1}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t,\tau)\Big)\Psi(t,\tau)^{-1} \\ &= -\Psi(t,\tau)^{-1}(A + GG'P)\Psi(t,\tau)\Psi(t,\tau)^{-1} = -\Psi(\tau,t)(A + GG'P).\end{aligned}$$

Now, since $\Psi(t,0)P_0\Psi(t,0)' > 0$ and $PGG'P + W \geq 0$, from (57) we conclude that $P(t)$ remains positive definite for all $t \geq 0$. ■

Proof: [Theorem 2] From (1) and (9) we conclude that

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} - QW\hat{x} - G\mathbf{d} - Qw \\ &= (A - QW)\tilde{x} - G\mathbf{d} - Q(Wx + w) \\ &= (A - QW)\tilde{x} - G\mathbf{d} - Q\sum_{j=1}^{N} C_j'Y_j'Y_j(C_jx + d_j) \\ &= (A - QW)\tilde{x} - G\mathbf{d} - Q\sum_{j=1}^{N} C_j'Y_j'Y_j(\alpha_j y_j - \mathbf{n}_j) \\ &= (A - QW)\tilde{x} - G\mathbf{d} + Q\sum_{j=1}^{N} C_j'Y_j'Y_j\mathbf{n}_j,\end{aligned} \tag{58}$$

where each Y_j is a matrix for which $I - \frac{y_j y_j'}{\|y_j\|^2} = Y_j' Y_j$. Such matrices always exist because $I - \frac{y_j y_j'}{\|y_j\|^2} \geq 0$. Defining $V(\tilde{x}) := \tilde{x}' P \tilde{x}$, $P := Q^{-1}$, computing its time-derivative and using (18), yields

$$\begin{aligned}\dot{V} &= \tilde{x}'(\dot{P} + PA + A'P - 2W)\tilde{x} - 2\tilde{x}'PG\mathbf{d} - 2\tilde{x}' \sum_{j=1}^{N} C_j' Y_j' Y_j \mathbf{n}_j \\ &= -\tilde{x}'(PGG'P + W)\tilde{x} - 2\tilde{x}'PG\mathbf{d} - 2\tilde{x}' \sum_{j=1}^{N} C_j' Y_j' Y_j \mathbf{n}_j\end{aligned}$$

By completing the squares, we further conclude that

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\tilde{x}'(PGG'P + W)\tilde{x} - \frac{1}{2}\|G'Px + 2\mathbf{d}\|^2 + 2\|\mathbf{d}\|^2 - \sum_{j=1}^{N} \frac{1}{2}\|Y_j C_j \tilde{x} + 2Y_j \mathbf{n}_j\|^2 + 2\sum_{j=1}^{N} \mathbf{n}_j' Y_j' Y_j \mathbf{n}_j \\ &\leq -\frac{1}{2}\tilde{x}'(PGG'P + W)\tilde{x} + 2\|\mathbf{d}\|^2 + 2\sum_{j=1}^{N} \mathbf{n}_j' Y_j' Y_j \mathbf{n}_j \\ &\leq -\frac{1}{2}\tilde{x}'(PGG'P + W)\tilde{x} + 2\|\mathbf{d}\|^2 + 2\sum_{j=1}^{N} \|\mathbf{n}_j\|^2, \end{aligned} \tag{59}$$

where we used the fact that the largest eigenvalue of $Y_j' Y_j$ is always smaller than 1. Since for every finite time P is positive definite, V must be finite on any finite interval and therefore so must be $\tilde{x}$ and $\hat{x}$. Global existence of solution follows. In case Q in uniformly bounded, P is "uniformly" positive definite and so is $PGG'P$ (by Assumption 1). In this case, it is straightforward to conclude from (59) that the ISS-like bound (21) holds (cf., e.g., [27, Section 5.3]). ■

Proof: [Lemma 1] To prove this lemma we show that

$$P(t) \geq \delta I > 0, \qquad \forall t \geq 0, \tag{60}$$

for some positive constant δ. To this effect we pick an arbitrary vector $x \in \mathbb{R}^n$ and compute $x'P(t)x$ using (57):

$$x'P(t)x = z(0)'P_0 z(0) + \int_0^t z(\tau)' \big(P(\tau)GG'P(\tau) + W(\tau)\big) z(\tau) d\tau,$$

where $z(\tau) := \Psi(\tau, t)x$, $\tau \leq t$ is the solution to

$$\dot{z}(\tau) = (A + GG'P)z(\tau), \qquad z(t) = x, \qquad 0 \leq \tau \leq t. \tag{61}$$

Since $P_0 > 0$, then for $t \geq T$, we conclude that

$$x'P(t)x \geq \alpha(t)^2 + \beta(t)^2, \qquad \forall t \geq T, \tag{62}$$

where $\alpha(t) := \|W(\tau)^{\frac{1}{2}} z(\tau)\|_{(t-T,t)}$, and $\beta(t) := \|G'P(\tau)z(\tau)\|_{(t-T,t)}$. Here, given a positive semidefinite matrix M we denote by $M^{\frac{1}{2}}$ any matrix such that $(M^{\frac{1}{2}})'M^{\frac{1}{2}} = M$ and given a signal x we denote by $\|x\|_{(a,b)}$ the $\mathcal{L}_2$-norm of x truncated to the interval (a, b), i.e., $\|x\|_{(a,b)} := \big(\int_a^b \|x(\tau)\|^2 d\tau\big)^{\frac{1}{2}}$. We now proceed to compute a lower-bound for $x'P(t)x$ by computing a lower-bound for the right-hand-side of (62). Fix some $t \geq T$. Rewriting (61) as $\dot{z} = Az + GG'Pz$ and using the variation of constants formula, it follows that

$$z(\tau) = \Phi(\tau, t)x + \int_t^\tau \Phi(\tau, s)GG'P(s)z(s)ds, \qquad 0 \leq \tau \leq t.$$

Therefore,

$$\|W(\tau)^{\frac{1}{2}}\Phi(\tau, t)x\|_{(t-T,t)} \leq \alpha + \Big\| \int_t^\tau W(\tau)^{\frac{1}{2}}\Phi(\tau, s)GG'P(s)z(s)ds \Big\|_{(t-T,t)}. \tag{63}$$

Moreover, using the Schwartz inequality [36], we conclude that

$$\begin{aligned}\Big\| \int_t^\tau W(\tau)^{\frac{1}{2}}\Phi(\tau, s)GG'P(s)z(s)ds \Big\|_{(t-T,t)}^2 &= \int_{t-T}^t \Big\| \int_t^\tau W(\tau)^{\frac{1}{2}}\Phi(\tau, s)GG'P(s)z(s)ds \Big\|^2 d\tau \\ &\leq \int_{t-T}^t \Big(\int_\tau^t \|W(\tau)^{\frac{1}{2}}\Phi(\tau, s)G\|^2 ds \int_\tau^t \|G'P(s)z(s)\|^2 ds \Big) d\tau\end{aligned}$$

and because $\|W(\tau)^{\frac{1}{2}}\Phi(\tau,s)G\|$ is uniformly bounded for $s\in[\tau,t]$, we further conclude that

$$\Big\|\int_t^\tau W(\tau)^{\frac{1}{2}}\Phi(\tau,s)GG'P(s)z(s)ds\Big\|^2_{(t-T,t)}\leq c\int_{t-T}^t\int_\tau^t\|G'P(s)z(s)\|^2dsd\tau\leq cT\beta(t)^2, \tag{64}$$

for an appropriately defined constant c. From (63) and (64), it then follows that

$$\|W(\tau)^{\frac{1}{2}}\Phi(\tau,t)x\|_{(t-T,t)}\leq\alpha(t)+\sqrt{cT}\beta(t), \qquad \forall t\geq T.$$

From this, (62), and assuming without loss of generality that $cT\geq 1$, it is straightforward to conclude that

$$\begin{aligned}x'P(t)x\geq\alpha^2+\beta^2\geq\frac{(\alpha+\sqrt{cT}\beta)^2}{2cT}&\geq=\frac{1}{2cT}\int_{t-T}^t x'\Phi(\tau,t)'W(\tau)\Phi(\tau,t)xd\tau\\&=\frac{1}{2cT}x'\Phi(t-T,t)'\Big(\int_0^T x'\Phi(t-T+s,t-T)'W(t-T+s)\Phi(t-T+s,t-T)ds\Big)\Phi(t-T,t)x,\end{aligned}$$

for $t\geq T$. From this and (22), we obtain

$$x'P(t)x\geq\frac{\epsilon}{2c}\|\Phi(t-T,t)x\|^2\geq\frac{\epsilon}{2c\|\Phi(t,t-T)\|^2}\|x\|^2, \qquad \forall t\geq T.$$

This proves that (60) holds with δ equal to the smallest of $\frac{\epsilon}{2c\|\Phi(t,t-T)\|^2}$ and the smallest eigenvalue of $P(t)$ on the closed interval $[0,T]$. The latter is strictly positive since $P(t)$ is positive definite for any finite time t. ■

Proof: [Theorem 4] From (1) and (28) we conclude that for all $t_i\leq t<t_{i+1}$, the state estimation error evolves according to

$$\dot{\tilde{x}}=A(u)\tilde{x}-G(u)\mathbf{d}.$$

Defining, $V:=\tilde{x}P\tilde{x}$, and computing its time derivative, it follows that

$$\dot{V}=\tilde{x}'(\dot{P}+PA+A'P)\tilde{x}-2\tilde{x}PG\mathbf{d}=-\tilde{x}'(PGG'P)\tilde{x}-2\tilde{x}'PG\mathbf{d}, \quad t_i\leq t<t_{i+1}$$

By completing the squares and using Assumption 1, we conclude that

$$\dot{V}=-\frac{1}{2}\tilde{x}'(PGG'P)\tilde{x}-\frac{1}{2}\|G'P\tilde{x}+2\mathbf{d}\|^2+2\|\mathbf{d}\|^2\leq-\frac{1}{2}\delta\lambda_{min}(P)V+2\|\mathbf{d}\|^2, \quad t_i\leq t<t_{i+1}$$

where $\lambda_{min}(P)$ denotes the smallest eigenvalue of P. Using the assumption that P is lower bounded by a positive value, and defining $\gamma:=\frac{1}{2}\delta\inf_{\tau\in[t_i,t_{i+1})}\lambda_{min}(P(\tau))$, we further conclude that for all $t_i\leq t<t_{i+1}$,

$$V(t)\leq V(t_i)e^{-\gamma(t-t_i)}+\frac{2}{\gamma}\sup_{\tau\in[t_i,t)}\|\mathbf{d}(\tau)\|^2. \tag{65}$$

Consider now $t=t_{i+1}$. From (29)-(30), the estimation error $\tilde{x}$ at time $t=t_{i+1}$ can be written as

$$\tilde{x}(t_{i+1})=\big[I-P(t_{i+1})^{-1}W(t_{i+1})\big]\tilde{x}(t_{i+1}^-)+P(t_{i+1})^{-1}\eta, \tag{66}$$

where $\eta:=\sum_{j\in\mathcal{I}_{i+1}}\bar{C}_j'Y_j'Y_j\bar{\mathbf{n}}_j$, and each Y_j is a matrix for which $I-\frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}=Y_j'Y_j$. Such matrices always exist because $I-\frac{\mathbf{y}_j\mathbf{y}_j'}{\|\mathbf{y}_j\|^2}\geq 0$. Thus,

$$\begin{aligned}V(t_{i+1})=&\,\tilde{x}(t_{i+1}^-)'\Big[P(t_{i+1})-2W(t_{i+1})+W(t_{i+1})P(t_{i+1})^{-1}W(t_{i+1})\Big]\tilde{x}(t_{i+1}^-)\\&+\eta'P(t_{i+1})^{-1}\eta+2\tilde{x}(t_{i+1}^-)'\left[I-W(t_{i+1})P(t_{i+1})^{-1}\right]\eta.\end{aligned} \tag{67}$$

Observe that using (29) and resorting to the matrix inversion lemma[4], and simplifying the notation by dropping the time dependence, $P(t_{i+1})$ can be written as

$$\begin{aligned}P(t_{i+1})-2W+WP(t_{i+1})^{-1}W&=P-W+W[P+W]^{-1}W\\&=P-W^{\frac{1}{2}}\left[I-W^{\frac{1}{2}}\left[P+W^{\frac{1}{2}}IW^{\frac{1}{2}}\right]^{-1}W^{\frac{1}{2}}\right]W^{\frac{1}{2}}\\&=P-W^{\frac{1}{2}}FW^{\frac{1}{2}}\end{aligned}$$

[4]Let A, C, and $A^{-1}+B'C^{-1}B$ be non-singular matrices, then $(A^{-1}+B'C^{-1}B)^{-1}=A-AB'(BAB'+C)^{-1}BA$. Another useful matrix identity is the following $(A^{-1}+B'C^{-1}B)^{-1}B'C^{-1}=AB'(BAB'+C)^{-1}$.

where $F := \left[I + W^{\frac{1}{2}}P^{-1}W^{\frac{1}{2}}\right]^{-1}$ and $P = P(t_{i+1}^-)$. In this setting, given a positive semidefinite matrix M, we denote by $M^{\frac{1}{2}}$ any matrix such that $(M^{\frac{1}{2}})'M^{\frac{1}{2}} = M$. The others terms in (67) can be written as

$$P(t_{i+1})^{-1} = (P+W)^{-1} = P^{-1} - P^{-1}W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1},$$
$$I - W(t_{i+1})P(t_{i+1})^{-1} = I - W^{\frac{1}{2}}IW^{\frac{1}{2}}\left[P + W^{\frac{1}{2}}IW^{\frac{1}{2}}\right]^{-1} = I - W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1}.$$

Thus,

$$V(t_{i+1}) = \tilde{x}'P\tilde{x} - \tilde{x}'W^{\frac{1}{2}}FW^{\frac{1}{2}}\tilde{x} + \eta'(P^{-1} - P^{-1}W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1})\eta + 2\tilde{x}'(I - W^{\frac{1}{2}}FW^{\frac{1}{2}}P^{-1})\eta.$$

By completing the squares, we further conclude that

$$V(t_{i+1}) \leq (1+\epsilon)V(t_{i+1}^-) + \Big(1 + \frac{1}{\epsilon}\Big)\eta'P^{-1}\eta,$$

where ϵ is an arbitrary small positive constant. Therefore, resorting to (65), $V(t_{i+1})$ satisfies

$$V(t_{i+1}) \leq (1+\epsilon)V(t_i)e^{-\gamma(t_{i+1}-t_i)} + \frac{1}{\epsilon}a_{i+1} + b_{i+1},$$

where

$$a_{i+1} := \lambda_{max}(P^{-1})\|\eta\|^2, \qquad\qquad b_{i+1} := (1+\epsilon)\frac{2}{\gamma}\sup_{\tau\in[t_i,t_{i+1})}\|\mathbf{d}(\tau)\|^2 + a_{i+1}.$$

Furthermore, solving this inequality recursively, we get

$$V(t_k) \leq (1+\epsilon)^k e^{-\gamma(t_k-t_0)}V(t_0) + \sum_{j=0}^{k-1}(1+\epsilon)^j e^{-\gamma(t_k-t_{k-j})}\Big(\frac{1}{\epsilon}a_{k-j} + b_{k-j}\Big).$$

Applying Assumption 2, we first notice that

$$t_k - t_{k-j} \geq [j - N_0]\tau_D,\ j = 0, 1, \ldots, k-1.$$

Consequently

$$V(t_k) \leq \left[(1+\epsilon)e^{-\gamma\tau_D}\right]^k e^{\gamma N_0\tau_D}V(t_0) + \sum_{j=0}^{k-1}\left[(1+\epsilon)e^{-\gamma\tau_D}\right]^j\Big(\frac{1}{\epsilon}a_{k-j} + b_{k-j}\Big)e^{\gamma N_0\tau_D}. \tag{68}$$

From this inequality, we further conclude that by picking $\epsilon > 0$ such that

$$r := (1+\epsilon)e^{-\gamma\max_j\tau_D} < 1,$$

it follows that V is bounded and $V(t_k) \to \frac{1}{1-r}(\frac{1}{\epsilon}\max_j a_j + \max_j b_j)e^{\gamma\max_j\{N_0\tau_D\}}$ as $k \to \infty$. Since for every finite time P is positive definite, V must be finite on any finite interval and therefore so must be $\tilde{x}$ and $\hat{x}$. Global existence of solution follows. It is also straightforward to conclude from (68) that the ISS-like bound (45) holds. ■

Proof: [Lemma 2] First observe that using (27), and (29), $P(t)$ can be written as

$$P(t) = \Psi(0,t)'P_0\Psi(0,t) + \int_0^t \Psi(\tau,t)'P(\tau)GG'P(\tau)\Psi(\tau,t)\,d\tau + \sum_{i=1}^k \Psi(t_i,t)'W(t_i)\Psi(t_i,t) \tag{69}$$

where for any $\tau \in [t_i, t_{i+1})$ and $\sigma \in [t_j, t_{j+1})$

$$\Psi(\tau,\sigma) := \begin{cases} \Psi_i(\tau,\sigma), & i=j \\ \Psi_i(\tau,t_{i+1})\Psi_{i+1}(t_{i+1},t_{i+2})\cdots\Psi_j(t_j,\sigma), & i<j \end{cases}$$

and $\Psi_i(t,\tau)$ denotes the state transition matrix of $\dot{z} = (A + GG'P)z$ for $t_i \leq \tau \leq t < t_{i+1}$. To prove this Lemma we will show that

$$P(t) \geq \delta I > 0, \qquad \forall t \geq 0, \tag{70}$$

for some positive constant δ. To this effect, we start by picking an arbitrary vector $x \in \mathbb{R}^n$ and using (69) it follows that

$$x'P(t)x \geq \int_0^t z(\tau)'P(\tau)GG'P(\tau)z(\tau)\,d\tau + \sum_{i=1}^k z(t_i)'W(t_i)z(t_i)$$

where $z(\tau) := \Psi(\tau, t)x$, and satisfies

$$\frac{d}{d\tau} z(\tau, t) = (A + GG'P)z(\tau), \quad 0 \leq \tau \leq t. \tag{71}$$

Since $P_0 > 0$, then for $t \geq t_{k-\mathbf{N}}$, we conclude that

$$x'P(t)x \geq \alpha(t)^2 + \beta(t)^2, \tag{72}$$

where $\alpha(t) := \left(\sum_{i=k-\mathbf{N}}^{k} \|W(t_i)^{\frac{1}{2}} z(t_i)\|^2\right)^{\frac{1}{2}}$, $\beta(t) := \left(\int_{t-t_{k-\mathbf{N}}}^{t} \|G'P(\tau)z(\tau)\|^2 \, d\tau\right)^{\frac{1}{2}}$. We now proceed to compute a lower-bound for $x'P(t)x$ by computing a lower-bound for the right-hand-side of (72). Fix some $t \geq t_{k-\mathbf{N}}$. Applying the variation of constants formula to (71), we get

$$z(\tau) = \Phi(\tau, t)x + \int_t^{\tau} \Phi(\tau, \sigma)GG'P(\sigma)z(\sigma) \, d\sigma, \ 0 \leq \tau \leq t.$$

Therefore,

$$\left(\sum_{i=k-\mathbf{N}}^{k} \|W(t_i)^{\frac{1}{2}} \Phi(t_i, t)x\|^2\right)^{\frac{1}{2}} \leq \alpha + \left(\sum_{i=k-\mathbf{N}}^{k} \|W(t_i)^{\frac{1}{2}} \int_t^{t_i} \Phi(t_i, \sigma)GG'P(\sigma)z(\sigma) \, d\sigma\|^2\right)^{\frac{1}{2}} \tag{73}$$

Moreover, using the Schwartz inequality, we conclude that

$$\sum_{i=k-\mathbf{N}}^{k} \|W(t_i)^{\frac{1}{2}} \int_t^{t_i} \Phi(t_i, \sigma)GG'P(\sigma)z(\sigma) \, d\sigma\|^2 \leq \sum_{i=k-\mathbf{N}}^{k} \int_{t_i}^{t} \|W(t_i)^{\frac{1}{2}} \Phi(t_i, \sigma)G\|^2 \, d\sigma \int_{t_i}^{t} \|G'P(\sigma)z(\sigma)\|^2 \, d\sigma$$

and because $\|W(t_i)^{\frac{1}{2}} \Phi(t_i, \sigma)G\|$ is uniformly bounded for $\sigma \in [t_i, t]$, we further conclude that the right-hand-side term is bounded by

$$\sum_{i=k-\mathbf{N}}^{k} c \int_{t_i}^{t} \|G'P(\sigma)z(\sigma)\|^2 \, d\sigma \leq c\mathbf{N}\beta(t)^2, \tag{74}$$

for an appropriately defined constant c. From (73) and (74), it then follows that

$$\left(\sum_{i=k-\mathbf{N}}^{k} \|W(t_i)^{\frac{1}{2}} \Phi(t_i, t)x\|^2\right)^{\frac{1}{2}} \leq \alpha(t) + \sqrt{c\mathbf{N}}\beta(t), \ \forall t \geq t_{k-\mathbf{N}}.$$

From this, (72), and assuming without of generality that $c\mathbf{N} \geq 1$, it is straightforward to conclude that

$$x'P(t)x \geq \alpha^2 + \beta^2 \geq \frac{(\alpha + \sqrt{c\mathbf{N}}\beta)^2}{2c\mathbf{N}} \geq \frac{1}{2c\mathbf{N}} \sum_{i=k-\mathbf{N}}^{k} \|W(t_i)^{\frac{1}{2}} \Phi(t_i, t)x\|^2.$$

Performing the change of coordinates $j = i - k + N$, yields

$$\begin{aligned} x'P(t)x &\geq \frac{1}{2c\mathbf{N}} \sum_{j=0}^{\mathbf{N}} x'\Phi(t_{j+k-\mathbf{N}}, t)'W(t_{j+k-\mathbf{N}})\Phi(t_{j+k-\mathbf{N}}, t)x \\ &= \frac{1}{2c\mathbf{N}} x'\Phi(t_{k-\mathbf{N}}, t) \left(\sum_{j=0}^{\mathbf{N}} \Phi(t_{j+k-\mathbf{N}}, t_{k-\mathbf{N}})'W(t_{j+k-\mathbf{N}})\Phi(t_{j+k-\mathbf{N}}, t_{k-\mathbf{N}})\right) \Phi(t_{k-\mathbf{N}}, t)x. \end{aligned}$$

From this and (46), we finally obtain

$$x'P(t)x \geq \frac{\epsilon}{2c} \|\Phi(t_{k-\mathbf{N}}, t)x\|^2 \geq \frac{\epsilon}{2c\|\Phi(t, t_{k-\mathbf{N}})\|^2} \|x\|^2, \qquad \forall t \geq t_{k-\mathbf{N}}.$$

This proves that (70) holds with δ equal to the smallest of $\frac{\epsilon}{2c\|\Phi(t, t_{k-\mathbf{N}})\|^2}$ and the smallest eigenvalue of $P(t)$ on the closed interval $[0, t_{k-\mathbf{N}}]$. The latter is strictly positive since $P(t)$ is positive definite for any finite time t. ∎

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