Stochastic Impulsive Systems Driven by Renewal Processes
Extended version

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***Abstract*— Stochastic impulsive systems are defined by a diffusion process with jumps triggered by a renewal process, i.e., the intervals between jumps are independent and identically distributed. We construct a model for such systems based on jump-diffusion equations and provide Lyapunov-based conditions that guarantee their mean-square stability.**

As an application, we show that stochastic impulsive systems can be used to model networked control systems with stochastic inter-sampling times and packet drops. Conditions for mean-square stability of the resulting systems are provided. For linear dynamics, these conditions can be formulated in terms of Linear Matrix Inequalities.

We use two benchmark examples that previously appeared in the literature to illustrate the use of our results and to investigate their conservativeness.

***Keywords*— Impulsive systems, Stochastic systems, Jump-diffusion processes, Renewal processes, Networked Control Systems**

I. Introduction

Impulsive systems can be viewed as a continuous flow that is interrupted at the so called jump-times. In *stochastic impulsive systems*, the continuous flow is characterized by a stochastic differential equation of the form

$$dx_t = f(x_t)dt + \sigma(x_t)dw_t, \quad t \geq 0,\ x \in \mathbb{R}^{n_x}, \tag{1a}$$

where the driving term $w_t \in \mathbb{R}^m$ is a standard Wiener process. The *jump times* are a sequence of times $t_0 := 0 < t_1 < t_2 < \cdots$, at which the system's state is reset according to a law of the form

$$x_{t_k} = \rho(k-1, x_{t_k^-}, z_k), \quad \forall k \in \mathbb{N}, \tag{1b}$$

where x_{t^-} denotes the limit from below of x_τ as $\tau \uparrow t$, and z_k an exogenous sequence of *jump points* taking values in some set $\mathcal{Z}$.

Stochastic impulsive systems exhibit three sources of randomness: the Wiener process w_t, the jump times $\{t_k : k \in \mathbb{N}\}$, and the jump points $\{z_k : k \in \mathbb{N}\}$. We are interested in stochastic impulsive systems for which the intervals $\{t_{k+1} - t_k : k \in \mathbb{N}\}$ between consecutive jumps are independent and identically distributed (i.i.d.). In this case, the process

$$r_t := \max\{k \in \mathbb{N} : t_k \leq t\}, \quad \forall t \geq 0,$$

that counts the number of jump times t_k in the interval $(0, t]$ is a *renewal process* and we say that the stochastic impulsive system is *driven by a renewal process.* This terminology implicitly subsumes that the jump points $\{z_k : k \in \mathbb{N}\}$ are i.i.d and that all sources of randomness mentioned so far are statistically independent.

In this paper, we show that a stochastic impulsive system driven by a renewal process can be modeled by a jump-diffusion equation with state-dependent intensity of the form

$$dx_t = f(x_t)dt + \sigma(x_t)dw_t + \int_{\mathcal{Z}} (\rho(r_{t^-}, x_{t^-}, z) - x_{t^-}) n(dt, dz), \tag{2a}$$

$$dr_t = \int_{\mathcal{Z}} n(dt, dz), \tag{2b}$$

$$d\tau_t = dt - \int_{\mathcal{Z}} (\tau_{t^-})\, n(dt, dz), \tag{2c}$$

where $n(dt, dz)$ is an integer-valued random measure with jump intensity given by $\lambda_{\text{haz}}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)$. The function $\lambda_{\text{haz}}(\cdot)$ is the hazard rate of the renewal process that characterizes the jump times t_k and $\zeta_{\mathcal{Z}}(\cdot)$ is the probability measure of the jump-points z_k (details can be found in Section IV). This characterization of a stochastic impulsive system allow us to use the Itô formula for semimartingales and Lyapunov-based arguments to derive sufficient conditions for the stability of stochastic impulsive system driven by renewal processes.

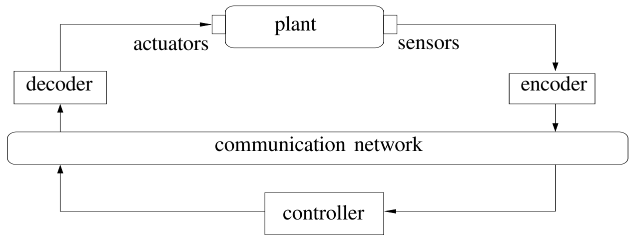

Fig. 1. Networked control system

Spatially distributed systems controlled over a communication network — often referred to as *networked control systems* — provide our main motivation to study impulsive system. Figure 1 shows a simple networked control system for which a remote controller receives process sensor information and send actuation signals through a communication network. In this figure, the *encoder* block map measurements into streams of "symbols" that can be transmitted across the network, and the *decoder* block perform the task of mapping the streams of symbols received from the network into continuous actuation signals. However, the network may drop symbols, which means that the strings received by the controller/decoder may be a strict subset of the strings sent by the encoder/controller.

We show that when the interval between consecutive sample times is i.i.d., networked control systems can be modeled by stochastic impulsive system driven by renewal processes. The construction of the impulsive system mimics the one proposed in a deterministic setting by Nesic and Teel [6]. We then provide sufficient conditions for the stability of networked control systems that explore the special structure of the impulsive systems that arise in this context.

This paper is organized as follows: Section II reviews background material on jump-diffusion processes and the Itô formula for semimartingales. In Section III we show how a renewal processes can be obtained as solution to a jump-diffusion equation. In Section IV, we show how a stochastic impulsive system driven by a renewal process can be obtained as a solution to a jump-diffusion equation and provide a Lyapunov-based sufficient condition for stability of the impulsive system. In Section V, we show how a networked control system can be modeled as a stochastic impulsive system and provide a sufficient conditions for its stability.

Notation: In this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space and ω an element of Ω. To simplify notation, we often omit the dependence on ω for random variables, functions, processes, or measures. When we need to emphasize that a particular symbol denotes a random variable or function, the dependence on ω is included but separated from the remaining arguments by a semicolon. We recall that a stochastic process is called *cadlag* if all its realizations are right-continuous and admit left-limits.

II. Jump-diffusions with state-dependent intensities

A jump-diffusion process x_t is defined by a stochastic differential equation of the form

$$dx_t = f(x_t)dt + \sigma(x_t)dw_t, \qquad t \geq 0,\ x_t \in \mathbb{R}^{n_x}, \tag{3a}$$

and a jump equation of the form

$$x_{t_k} = x_{t_k^-} + \xi(x_{t_k^-}, z_k), \qquad k \in \mathbb{N},\ z_k \in \mathcal{Z}, \tag{3b}$$

where w_t is a standard Wiener process, the $\{t_k(\omega) \geq 0 : k \in \mathbb{N}\}$ form a random sequence of jump times, and the $\{z_k(\omega) : k \in \mathbb{N}\}$ form a random sequence of jump points taking values in some set $\mathcal{Z}$.

In the simplest jump-diffusion processes, the intervals between consecutive jump times $\{t_{k+1}(\omega) - t_k(\omega) : k \in \mathbb{N}\}$ are exponentially distributed i.i.d. random variables with (constant) mean $1/\lambda$. Moreover, these random variables are independent of all the z_k and of the Weiner process w_t. The constant $\lambda > 0$ is called the *intensity* of the jump diffusion process and it is well known that the expected value of the number of jump times that fall in an interval $I := (t_1, t_2]$, $t_2 > t_1 \geq 0$ is given by

$$\mathrm{E}\left[n(\omega; I)\right] = \int_I \lambda\, dt = \lambda(t_2 - t_1), \tag{4}$$

Research supported by the Institute for Collaborative Biotechnologies grant DAAD19-03-D-0004 from the U.S. Army Research Office, by the National Science Foundation grant No. CCR-0311084, and by the U.S. Army Research Office grant DAAD19-03-1-0144.

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where the random variable $n(\omega; I)$ denotes the number of jump times in the set I. These are called *jump-diffusion processes with constant (or Poisson) intensities* and can essentially be viewed as stochastic impulsive systems driven by a Poisson renewal process. There are two minor differences between jump-diffusion processes with constant intensity and stochastic impulsive systems driven by a Poisson renewal process: (i) $\rho(\cdot)$ in the impulsive system equation (1b) is equal to the value of x immediately after the jump, whereas $\xi(\cdot)$ in the jump-diffusion equation (3b) is equal to the discontinuity in x at time t_k; and (ii) in impulsive systems we allow $\rho(\cdot)$ to depend on the jump index k, whereas $\xi(\cdot)$ does not.

To model stochastic impulsive systems driven by more general renewal processes, we need jump-diffusion processes with state-dependent intensities. In these processes, (4) is generalized to

$$\mathrm{E}\left[n(\omega; I, A)\right] = \mathrm{E}\Big[\int_I \lambda(x_{t^-}(\omega), A)dt\Big], \tag{5}$$

where $\{\lambda(x,\cdot) : x \in \mathbb{R}^{n_x}\}$ denotes a family of measures on $\mathcal{Z}$; $I \subset [0,\infty)$ an arbitrary Lebesgue-measurable set in $[0,\infty)$; $A \subset \mathcal{Z}$ an arbitrary $\lambda(x,\cdot)$-measurable set; and $n(\omega; I, A)$ the number of jump times t_k in I for which the corresponding jump points z_k belong to A. Note that n is an integer-valued random measure on $[0,\infty)\times\mathcal{Z}$, in the sense of [4, Definition II.1.13, p. 68]. In particular, setting A to be the whole set $\mathcal{Z}$, we conclude that the expected number of jump times that fall in an interval $I := (t_1, t_2]$, $t_2 > t_1 \geq 0$ is now given by

$$\mathrm{E}\left[n(\omega; I, \mathcal{Z})\right] = \mathrm{E}\Big[\int_{(t_1,t_2]} \lambda(x_{t^-}(\omega), \mathcal{Z})dt\Big].$$

According to this equation, the (expected) number of jump points per unit time will generally vary with time because $\lambda(x_{t^-}, \mathcal{Z})$ varies with x_t. Moreover, jump times and jump points may not be independent since $\lambda(x_{t^-},\cdot)$ may favor distinct distributions for the jump points, depending on whether jump times are more or less likely to occur.

The two equations (3a)–(3b) are often combined into a single *jump-diffusion equation*, written as

$$dx_t = f(x_t)dt + \sigma(x_t)dw_t + \int_{\mathcal{Z}} \xi(x_{t^-}, z)n(dt, dz). \tag{6}$$

The random measure $\lambda(x_{t^-}(\omega),\cdot)$ in (5) — which determines both the rate at which n is incremented and where the corresponding jump points lie — is called the *intensity* of the jump process.

We proceed to justify the use of the single equation (6) to denote (3), which requires the notion of stochastic integral against a random measure. However, if one simply regards (6) as a short-hand notation for (3), and takes (5) as the definition of intensity, one can skip the following paragraphs and jump directly to the statement of Theorem 1, which is the key result that will be needed in subsequent sections.

In the remainder of this section we assume that the reader is familiar with stochastic integration and martingale theory at the level of [4, Chapters I, II]. By a *weak solution to the equation* (6) *with jump intensity* $\lambda(x_{t^-}(\omega), dz)$ *and initial probability measure* μ_0 *on* $\mathbb{R}^{n_x}$, we mean a probability space $(\Omega, \mathcal{F}, \mathrm{P})$; a filtration $\mathcal{F}_t$; an $\mathcal{F}_t$-adapted standard Wiener process w_t; an integer-valued random measure n on $[0,\infty)\times\mathcal{Z}$ with compensator

$$\nu(\omega; dt, dz) := \lambda(x_{t^-}(\omega), dz)dt; \tag{7}$$

and a locally bounded, cadlag $\mathcal{F}_t$-adapted process x_t such that x_0 has measure μ_0 and[1]

$$x_t - x_0 = \int_{[0,t]} f(x_{s^-})ds + \int_{[0,t]} \sigma(x_{s^-})dw_s + \int_{[0,t]\times\mathcal{Z}} \xi(x_{s^-}, z)n(ds, dz). \tag{8}$$

The process x_t is called the *weak solution-process*. We recall that the compensator $\nu(\omega; dt, dz)$ of an integer-valued random measure $n(\omega; dt, dz)$ on $[0,\infty)\times\mathcal{Z}$ is a predictable random measure for which

$$\mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z)n(\omega; ds, dz)\Big] = \mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z)\nu(\omega; ds, dz)\Big] \tag{9}$$

for every nonnegative optional function h [4, Theorem II.1.8, p. 66].

[1] The first integral in (7) can be understood in the sense of [4, Equation I.3.4, p. 28] or [4, Theorem I.4.31, p. 46], the second must be understood in the sense of [4, Theorem I.4.31, p. 46], and the last one in the sense of [4, Equations II.1.5, p. 66 or II.1.15, p. 69]. In integration against the Lebesgue measure dt, one does not need to worry about the integrand being left- or right- continuous. However, for consistency we will generally provide predictable integrands for which integration against semimartingales is well defined [4, Theorem I.4.31, p. 46].

To understand the connection between this definition and (3), we recall that denoting by $\{(t_k, z_k) : k \in \mathbb{N}\}$ the points at which the random measure n is incremented, we have that

$$\int_{[0,t]\times\mathcal{Z}} h(\omega; s, z) n(\omega; ds, dz) = \sum_{k: t_k \in (0,t]} h(\omega; t_k(\omega), z_k(\omega)), \tag{10}$$

for every optional function h [4, Proposition II.1.14, p. 68]. In particular, for $h(\omega; s, z) = \xi(x_{s^-}(\omega), z)$ we conclude that

$$\int_{[0,t]\times\mathcal{Z}} \xi(x_{s^-}, z) n(ds, dz) = \sum_{k: t_k \in (0,t]} \xi\big(x_{t_k^-}, z_k\big).$$

This shows that the last term in (8) is instantaneously incremented by $\xi\big(x_{t_k^-}, z_k\big)$ at each time t_k, which is consistent with the jump equation (3b). Moreover, at all other times, the last term in (8) remains constant and the evolution of x_t is determined by the first two integrals, which is consist with the stochastic differential equation (3a).

To verify that the compensator in (9) leads precisely to equation (5), we take an arbitrary Lebesgue-measurable set $I \subset [0, \infty)$, an arbitrary $\lambda(x, \cdot)$-measurable set $A \subset \mathcal{Z}$, and define $h(\omega; s, z)$ equal to one if $t \in I$, $z \in A$ and equal to zero otherwise. We then conclude from (9) and (7) that

$$\begin{aligned} \mathrm{E}\big[n(\omega; I, A)\big] &= \mathrm{E}\Big[\int_{I\times A} n(\omega; ds, dz)\Big] = \mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z) n(\omega; ds, dz)\Big] \\ &= \mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z)\lambda(x_{t^-}(\omega), dz) ds\Big] = \mathrm{E}\Big[\int_I \lambda(x_{t^-}(\omega), A) ds\Big]. \end{aligned}$$

The following theorem is a consequence of the fact that any weak solution-process to (6) is a semimartingale, for which one can apply the Itô formula for semimartingales. A detailed proof of this result (with extensive references to [4]) can be found in the Appendix.

Theorem 1: Let x_t denote a weak solution-process to (6), $V : \mathbb{R}^{n_x} \to \mathbb{R}$ a twice continuously differentiable function, and define

$$\mathcal{L}V(x) := \nabla_x V(x) \cdot f(x) + \frac{1}{2}\operatorname{trace}[\sigma(x)' H_x V(x) \sigma(x)] + \int_{\mathcal{Z}} \Big(V\big(x + \xi(x, z)\big) - V(x)\Big)\lambda(x, dz), \tag{11}$$

$\forall x \in \mathbb{R}^{n_x}$, where $\nabla_x V(x)$ and $H_x V(x)$ denote the gradient vector and the Hessian matrix of $V(x)$, respectively. Then $V(x_t) - V(x_0) - \int_{[0,t]} \mathcal{L}V(x_{s^-}) ds$ is a martingale and, when the function V is nonnegative and $\mathrm{E}[V(x_0)] < \infty$,

$$\mathrm{E}\big[V(x_t)\big] = \mathrm{E}\big[V(x_0)\big] + \int_{[0,t]} \mathrm{E}\Big[\mathcal{L}V(x_s)\Big] ds, \quad \forall t \ge 0. \tag{12}$$

□

Remark 1: Conditions for existence and uniqueness of solution to (6) are often formulated for Poisson intensities (i.e., not state dependent) [4, Section III.2c]. However, these results can be used to derive existence and uniqueness conditions for state-dependent intensities using a thinning procedure [3]. An extended discussion on existence of uniqueness results for jump-diffusion equations is beyond the scope of this paper. However, it is worth while pointing out that Theorem 1 holds whether or not weak solutions are unique. Moreover, we will show that the jump-diffusion equations introduced in Section III to generate renewal processes always have weak solutions so existence is not a significant issue. □

III. Renewal processes

Consider two sequences of random variable $\{t_k(\omega) \in [0, \infty) : k \in \mathbb{N}\}$ and $\{z_k(\omega) \in \mathcal{Z} : k \in \mathbb{N}\}$, which we call the sequence of *jump times* and the sequence of *jump points*, respectively, with the following three properties:

(i) $t_0 = 0$ with probability one and the random variables $\{t_{k+1} - t_k : k \in \mathbb{N}\}$ are independent and identically distributed nonnegative random variables with cumulative distribution F_τ and mean $\mu_\tau > 0$. The distribution of the increments $\{t_{k+1} - t_k : k \in \mathbb{N}\}$ can also be characterized in terms of its *hazard rate*, which is defined by

$$\lambda_{\mathrm{haz}}(s) := \frac{F_\tau'(s)}{1 - F_\tau(s)}, \quad \forall s \in [0, T), \tag{13}$$

where $T \in (0, +\infty]$ denotes the maximum inter-jump time, and therefore $F_\tau(s) < 1$, $\forall s \in [0, T)$. For simplicity, we assume that the hazard rate is continuous on $[0, T)$.

(ii) The random variables $\{z_k \in \mathcal{Z} : k \in \mathbb{N}\}$ are independent and identically distributed with measure $\zeta_{\mathcal{Z}}$.

(iii) All random variables $t_{k+1} - t_k$, z_ℓ, $\forall k, \ell \in \mathbb{N}$ are independent.

These sequences of random variables can be used to define a *renewal process*

$$r_t := \max\{k \in \mathbb{N} : t_k \leq t\}, \quad \forall t \geq 0,$$

which counts the number of jump times t_k in the interval $(0, t]$, and a *timer process*

$$\tau_t := t - t_k, \quad \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N},$$

which provides the time elapsed since the last t_k. The following theorem shows that renewal processes can be viewed as solutions to appropriately defined jump-diffusion equations with state-dependent intensities.

Theorem 2: The renewal and timer processes r_t and τ_t are weak solution-processes to the jump-diffusion equation

$$dr_t = \int_{\mathcal{Z}} n(dt, dz), \qquad\qquad d\tau_t = dt - \int_{\mathcal{Z}} (\tau_{t^-})\, n(dt, dz). \tag{14}$$

with jump intensity $\lambda_{\text{haz}}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)$, and initialization $r_0 = \tau_0 = 0$ with probability one. Moreover, the corresponding integer-valued random measure $n(\omega; dt, dz)$ satisfies

$$\int_{[0,t]\times\mathcal{Z}} h(\omega; t, z) n(\omega; dt, dz) = \sum_{k: t_k(\omega)\in[0,t]} h\big(\omega; t_k(\omega), z_k(\omega)\big), \tag{15}$$

for every optional function h. □

The following proposition is needed to prove Theorem 2. It lists well-known properties of renewal processes.

Proposition 1: The renewal process r_t satisfies the following properties

$$\lim_{t\to\infty} \frac{\mathrm{E}[r_t]}{t} = \frac{1}{\mu_\tau} \tag{16}$$

and, for every $t \geq 0$,

$$\begin{aligned} \mathrm{P}\left(r_{t+dt} - r_t = 0 \,\middle|\, \mathcal{F}_t\right) &= 1 - \lambda_{\text{haz}}(\tau_t)dt + o(dt) \\ \mathrm{P}\left(r_{t+dt} - r_t = 1 \,\middle|\, \mathcal{F}_t\right) &= \lambda_{\text{haz}}(\tau_t)dt + o(dt) \\ \mathrm{P}\left(r_{t+dt} - r_t > 1 \,\middle|\, \mathcal{F}_t\right) &= o(dt) \\ \mathrm{E}\left[r_{t+dt} - r_t \,\middle|\, \mathcal{F}_t\right] &\leq K dt, \end{aligned}$$

where $\mathcal{F}_t$ denotes any filtration such that $\mathcal{F}_t \supset \sigma\{r_s : s \leq t\} = \sigma\{\tau_s : s \leq t\}$. □

Proof of Theorem 2. To construct the desired weak solution to (14), we start with the probability space $(\Omega, \mathcal{F}, P)$ on which the $t_k(\omega)$, $z_k(\omega)$, $k \in \mathbb{N}$ are defined and choose any filtration $\mathcal{F}_t$ to which r_t and τ_t are adapted. The next step is to find an integer-valued random measure $n(\omega; dt, dz)$ with compensator $\lambda_{\text{haz}}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)dt$ for which

$$r_t = \int_{[0,t]\times\mathcal{Z}} n(dt, dz), \qquad\qquad \tau_t = t - \int_{[0,t]\times\mathcal{Z}} (\tau_{t^-})\, n(dt, dz). \tag{17}$$

Let $\mathcal{F}_t$ be any filtration such that $\mathcal{F}_t \supset \sigma\{r_s : s \leq t\} = \sigma\{\tau_s : s \leq t\}$. The renewal process r_t is a cadlag counting process, $\mathcal{F}_t \supset \sigma\{r_s : s \leq t\} = \sigma\{\tau_s : s \leq t\}$, and $\mathrm{E}[r_t] < \infty$, $\forall t \geq 0$, because of (16). Therefore we can use [7, Theorem 1 and equation (3.19)] to conclude that

$$r_t = m_t + a_t, \qquad\qquad a_t := \int_{[0,t]} \lambda_{\text{haz}}(\tau_{t^-})dt,$$

where m_t is a (zero-mean) martingale on $\mathcal{F}_t$ and a_t is an increasing $\mathcal{F}_t$-predictable process. This shows that a_t is the compensator for r_t [4, Theorem I.3.17i, p. 32] and that r_t is a special semimartingale [4, Definition I.4.21, p. 43]. Since $D := \big\{\big(\omega, t_k(\omega)\big) : \omega \in \Omega, k \in \mathbb{N}\big\}$ is a thin set and the process

$$p_t := z_k, \quad \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}$$

is optional, we can define the following integer-valued random measure on $[0, \infty)$

$$n(\omega; dt, dz) := \sum_{k\in\mathbb{N}} \epsilon_{(t_k(\omega), p_{t_k(\omega)})}(dt, dz) \qquad \Big(= \sum_{k: t_k(\omega)\in dt, p_{t_k(\omega)}\in dz} 1\Big)$$

where ϵ_a denotes the Dirac measure at point a, for which (15) holds for every optional function h (cf. [4, Proposition II.1.14, p. 68]). Note that $n(\omega; dt, dz)$ is σ-finite in the sense of [4, Definition II.1.6, p. 66] because r_t is a locally integrable increasing process (cf. argument used in [4, Example II.1.7, p. 66]).

To verify that the compensator of $n(\omega; dt, dz)$ is indeed given by the (predictable) measure $\lambda_{\rm haz}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)dt$, it suffices to check that

$$\mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z) n(\omega; dt, dz)\Big] \\ = \mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z)\lambda_{\rm haz}(\tau_{t^-}(\omega))\zeta_{\mathcal{Z}}(dz)dt\Big], \quad (18)$$

$\forall t > 0$ and for every function h of the form $h(\omega; s, z) = 1_B(\omega)1_{(s_1,s_2]}(s)1_C(z)$, for arbitrary $0 \le s_1 < s_2$, $B \in \mathcal{F}_{s_1}$, and $\zeta_{\mathcal{Z}}$-measurable set $C \subset \mathcal{Z}$, where $1_X(x)$ denotes the indicator function of X. For such h, the left-hand side of (18) is given by

$$\begin{aligned}\mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} h(\omega; s, z) n(\omega; ds, dz)\Big] \\ &= \mathrm{E}\Big[\sum_{k:t_k(\omega)\in[0,\infty)} 1_B(\omega)1_{(s_1,s_2]}(t_k(\omega))1_C(z_k(\omega))\Big] \\ &= \zeta_{\mathcal{Z}}(C)\,\mathrm{E}\Big[\sum_{k:t_k(\omega)\in[0,\infty)} 1_B(\omega)1_{(s_1,s_2]}(t_k(\omega))\Big] = \zeta_{\mathcal{Z}}(C)\,\mathrm{E}\Big[\sum_{k:t_k(\omega)\in(s_1,s_2]} 1_B(\omega)\Big] \\ &= \zeta_{\mathcal{Z}}(C)\,\mathrm{E}\Big[1_B(\omega)\big(r_{s_2}(\omega) - r_{s_1}(\omega)\big)\Big] \\ &= \zeta_{\mathcal{Z}}(C)\,\mathrm{E}\Big[\int_{[0,\infty)\times\mathcal{Z}} 1_B(\omega)1_{(s_1,s_2]}(s)dr_t(\omega)\Big], \quad (19)\end{aligned}$$

where, to factor out $\zeta_{\mathcal{Z}}(C)$, we used the fact that the random variables $1_C(z_k(\omega))$, $t_k > s_1$ are identically distributed and independent of $1_B(\omega)$ and of the $t_k(\omega)$. On the other hand, the right-hand side of (18) is given by

$$\begin{aligned}\mathrm{E}\Big[\int_{[0,t]\times\mathcal{Z}} h(\omega; s, z)\lambda_{\rm haz}(\tau_{t^-}(\omega))\zeta_{\mathcal{Z}}(dz)dt\Big] \\ &= \mathrm{E}\Big[\int_{[0,t]\times\mathcal{Z}} 1_B(\omega)1_{(s_1,s_2]}(s)1_C(z)\lambda_{\rm haz}(\tau_{t^-}(\omega))\zeta_{\mathcal{Z}}(dz)dt\Big] \\ &= \zeta_{\mathcal{Z}}(C)\,\mathrm{E}\Big[\int_{[0,t]\times\mathcal{Z}} 1_B(\omega)1_{(s_1,s_2]}(s)\lambda_{\rm haz}(\tau_{t^-}(\omega))dt\Big] \\ &= \zeta_{\mathcal{Z}}(C)\,\mathrm{E}\Big[\int_{[0,t]\times\mathcal{Z}} 1_B(\omega)1_{(s_1,s_2]}(s)da_t(\omega)\Big]. \quad (20)\end{aligned}$$

Equality of (19) and (20) then follows from the fact that a_s is the compensator for r_s [4, Theorem I.3.17iii, p. 32].

To finish the proof, it remains to show that the cadlag $\mathcal{F}_t$-adapted processes r_t, τ_t satisfy (17). This is a trivial matter since the left-hand-side and the right-hand-side equations in (17) follow immediately from (15) with $h(\omega; s, z) = 1$ and $h(\omega; s, z) = \tau_{t^-}(\omega)$, respectively. ■

IV. IMPULSIVE SYSTEMS DRIVEN BY RENEWAL PROCESSES

We are now ready to construct a jump-diffusion equation with state-dependent intensity that models a given stochastic impulsive system driven by a renewal process. The following result is a straightforward consequence of Theorem 2.

Corollary 1: Let x_t denote any process that satisfies (1) for sequences of jump times and jump points defined as in Section III, and let r_t and τ_t denote the corresponding renewal and timer processes, respectively. Then x_t, r_t, τ_t are weak solution-processes to (2) with jump intensity $\lambda_{\rm haz}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)$ and initialization $r_0 = \tau_0 = 0$ with probability one. □

Proof of Corollary 1. To construct the desired weak solution to (2), we start with the probability space $(\Omega, \mathcal{F}, P)$ on which the $t_k(\omega)$, $z_k(\omega)$, $k \in \mathbb{N}$, x_t and w_t are defined and choose any filtration $\mathcal{F}_t$ to which x_t, w_t, r_t, and τ_t are adapted. We then define $n(\omega; dt, dz)$ to be the integer-valued random measure whose existence is guaranteed by Theorem 2. This random measure has the desired compensator $\lambda_{\rm haz}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)dt$ and

$$r_t = \int_{[0,t]\times\mathcal{Z}} n(dt, dz), \qquad \tau_t = t - \int_{[0,t]\times\mathcal{Z}} (\tau_{t^-})\, n(dt, dz). \quad (21)$$

To verify that (2a) also holds, note that one can conclude from (15) that

$$\int_{[0,t]\times\mathcal{Z}} (\rho(r_{t^-}, x_{t^-}, z) - x_{t^-}) n(dt, dz) = \sum_{k: t_k \in [0,t]} (\rho(r_{t_k^-}, x_{t_k^-}, z_k) - x_{t_k^-})$$
$$= \sum_{k: t_k \in [0,t]} (\rho(k-1, x_{t_k^-}, z_k) - x_{t_k^-}) \tag{22}$$

and therefore the solution-process x_t to (2a) have jumps at the times t_k equal to $\rho(k-1, x_{t_k^-}, z_k) - x_{t_k^-}$, which corresponds exactly to (1b). Between the t_k, (22) remains constant and therefore x_t simply flows according to (1a). ■

The following theorem is the main result of this section. It combines Theorem 1 and Corollary 1 to obtain a sufficient condition for the stability of a stochastic impulsive system driven by a renewal process.

Theorem 3: Assume that the following two conditions hold:

C1 There exists a nonnegative function $W : \mathbb{N} \times \mathbb{R}^{n_x} \to [0, \infty)$ and constants $L \in \mathbb{R}$, $c, \ell \geq 0$ for which

$$\nabla_x W(r, x) \cdot f(x) + \frac{1}{2} \operatorname{trace}[\sigma(x)' H_x W(r, x) \sigma(x)] \leq L\, W(r, x) + c, \tag{23a}$$

$$\int_{\mathcal{Z}} W(r+1, \rho(r, x, z)) \zeta_{\mathcal{Z}}(dz) \leq \ell\, W(r, x), \tag{23b}$$

$$W(r, x) \geq \alpha \|x\|^2, \qquad \forall r \in \mathbb{N},\ x \in \mathbb{R}^{n_x}. \tag{23c}$$

C2 There exists a continuously differentiable function $\gamma : [0, T) \to [0, \infty)$ and constants $\epsilon > 0$, $0 < a \leq b < \infty$ such that $\gamma(0) = 1$ and

$$\gamma'(s) \leq (\lambda_{\text{haz}}(s) - L - \epsilon)\gamma(s) - \ell \lambda_{\text{haz}}(s), \qquad \gamma(s) \in [a, b], \qquad \forall s \in [0, T). \tag{24}$$

Then every weak solution-process x_t to (2) for which $\mathrm{E}[W(0, x_0)] < \infty$ is mean-square stable and satisfies

$$\mathrm{E}\left[\|x_t\|^2\right] \leq \frac{e^{-\epsilon t}}{\alpha\, a} \mathrm{E}\left[W(0, x_0)\right] + \frac{c\, b}{\epsilon\, \alpha\, a}, \quad \forall t \geq 0. \tag{25}$$

□

The proof of this result can be found in Section IV-C. In the remaining of this section we discuss the two conditions that appear in Theorem 3.

A. *Condition C2*

Intuitively, stability of the solutions to the stochastic impulsive system (2) will either rely on the continuous flow being "stabilizing," which corresponds to $L < 0$ in (23a); or on the jumps being "stabilizing," which corresponds to $\ell < 1$ on (23b). In the former case, long intervals between jumps are desirable and in the latter case short intervals are preferable. The condition C2 in Theorem 3 implicitly expresses this requirements, but it is difficult to verify directly. The next Lemma provides an alternative version of this condition (possibly more conservative) that is generally straightforward to verify.

Lemma 1: Assume that there exists constants $d_1 < \infty$, $d_2 < 1$ such that

$$\int_{s_1}^{s_2} \lambda_{\text{haz}}(\rho) d\rho \geq (L + \epsilon)(s_2 - s_1) - d_1, \tag{26a}$$

$$\ell \frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho} F'_\tau(\rho) d\rho \leq d_2, \tag{26b}$$

$\forall\, 0 \leq s_1 < s_2 < T$. Then condition C2 in Theorem 3 holds. □

Note that (26a) holds trivially for any inter-jump distribution with finite support ($T < \infty$) by setting $d_1 := (L + \epsilon)T$. With Lemma 1, it becomes straightforward to determine whether or not condition C2 in Theorem 3 holds for given distributions of the inter-jump time. The following corollary, considers a few common distributions:

Corollary 2: C2 in Theorem 3 holds for any of the following distributions of the inter-jump times:

(i) F_τ is any distribution with support on $[0, T)$ and $\ell e^{LT} < 1$.

(ii) F_τ is uniformly distributed on $[0, T)$ and $\ell \frac{e^{LT} - 1}{LT} < 1$.

(iii) F_τ is exponentially distributed with mean $\bar{T}$ and $\frac{\ell}{1 - L\bar{T}} < 1$. □

The condition (i) in Corollary 2 is necessarily very conservative because it applies to *every* distribution with finite support. E.g., if we compare conditions (i) and (ii) in Corollary 2, we conclude that when $LT \gg 1$, the knowledge that the distribution is uniform allows ℓ to become almost LT larger.

Remark 2: The inequalities (26) in Lemma 1 are used to show that there exists a constant $a > 0$ such that the solution to the nonlinear scalar differential equation

$$\gamma(0) = 1, \qquad \gamma'(s) = \begin{cases} (\lambda_{\rm haz}(s) - L - \epsilon)\gamma(s) - \ell\lambda_{\rm haz}(s) & \gamma(s) < 1 \\ \min\left\{0, (\lambda_{\rm haz}(s) - L - \epsilon)\gamma(s) - \ell\lambda_{\rm haz}(s)\right\} & \gamma(s) \geq 1 \end{cases}$$

remains larger than or equal to a for every $s \in [0, T)$. For specific hazard rates, one may verify that this is so by numerically solving the above differential equation. However, a numerical verification does not permit the derivation of "clean" conditions between ℓ and L such as the ones provided by Corollary 2. □

B. Condition C1

For linear systems and quadratic functions W, the condition C1 in Theorem 3 can be verified numerically in an efficient manner. To this effect, we restrict our attention to system dynamics of the form[2]

$$f(x) = Ax, \qquad \sigma(x) = B, \qquad \rho(r, x, z) = R_{r,z}x, \qquad \forall r \in \mathbb{N},\ x \in \mathbb{R}^{n_x},$$

and a function W of the form $W(r, x) = x'P_r x$, $P_r = P_r'$, $\forall r \in \mathbb{N}$, $x \in \mathbb{R}^{n_x}$. In this case, the inequalities in (23) take the form

$$x'(P_rA + A'P_r - LP_r)x \leq c - \operatorname{trace}\left[B'P_rB\right], \tag{27a}$$

$$x'\Big(\int_{\mathcal{Z}} R'_{r,z}P_{r+1}R_{r,z}\zeta_{\mathcal{Z}}(dz) - \ell P_r\Big)x \leq 0, \tag{27b}$$

$$x'(P_r - \alpha I)x \geq 0, \qquad \forall r \in \mathbb{N},\ x \in \mathbb{R}^{n_x}. \tag{27c}$$

The existence of the constants c, L, ℓ and the matrices P_r, $r \in \mathbb{N}$ for which (27) holds is equivalent to the feasibility of the following infinite family of matrix inequalities in the unknowns L, ℓ, α, $P_r = P_r'$, $r \in \mathbb{N}$:

$$P_rA + A'P_r - LP_r \leq 0, \tag{28a}$$

$$\int_{\mathcal{Z}} R'_{r,z}P_{r+1}R_{r,z}\zeta_{\mathcal{Z}}(dz) - \ell P_r \leq 0, \tag{28b}$$

$$P_r - \alpha I \geq 0, \qquad \forall r \in \mathbb{N}. \tag{28c}$$

This is because (27b)–(27c) follow directly from (28b)–(28c) and then (27a) holds with $c := \sup_{r\in\mathbb{N}} \operatorname{trace}\left[B'P_rB\right]$. Conversely, if the inequalities in (28) do not hold, then it is straightforward to show that the inequalities in (27) cannot hold for any finite constant c. When the reset matrices $R_{r,z}$ are periodic with respect to the integer r, i.e., $R_{r,z} = R_{r+N,z}$, $\forall r \in \mathbb{N}$, $z \in \mathcal{Z}$, we can restrict the search to periodic matrices P_r and (28) becomes a finite system of linear matrix inequalities (LMIs) in the unknowns $P_1, P_2, \ldots, P_N$.

As seen in Section IV-A, condition C2 implicitly imposes a constraint between ℓ and L, for which small values of ℓ favor stability. We can therefore use a line search as L ranges over $[0, \infty)$ to verify if C2 holds for the minimum value of ℓ for which (28) is feasible. Finding such ℓ is a quasi-convex generalized eigenvalue minimization problem (GEVP), which can be solved very efficiently [1].

C. Proofs of Theorem 3, Lemma 1, and Corollary 2

Proof of Theorem 3. Consider the jump-diffusion process in Corollary 1. Applying the corresponding operator $\mathcal{L}$ defined in Theorem 1 to the nonnegative function $V(x, r, \tau) := \gamma(\tau)W(r, x)$, $\forall x \in \mathbb{R}^{n_x}$, $r \in \mathbb{N}$, $\tau \in [0, T)$ yields

$$\begin{aligned}\mathcal{L}V(x, r, \tau) &= \nabla_\tau V(x, r, \tau) + \nabla_x V(x, r, \tau) \cdot f(x) + \frac{1}{2}\operatorname{trace}[\sigma(x)'H_xV(x, \tau)\sigma(x)] \\ &\quad + \int_{\mathcal{Z}} \Big(V(\rho(r, x, z), r + 1, 0) - V(x, r, \tau)\Big)\lambda_{\rm haz}(\tau)\zeta_{\mathcal{Z}}(dz) \\ &= \gamma'(\tau)W(r, x) + \gamma(\tau)\Big(\nabla_x W(r, x) \cdot f(x) + \frac{1}{2}\operatorname{trace}[\sigma(x)'H_xW(r, x)\sigma(x)]\Big) \\ &\quad + \Big(\lambda_{\rm haz}(\tau)\int_{\mathcal{Z}} W(r + 1, \rho(r, x, z))\zeta_{\mathcal{Z}}(dz)\Big) - \gamma(\tau)W(r, x)\lambda_{\rm haz}(\tau).\end{aligned}$$

Using (23a)–(23b) we conclude that

$$\begin{aligned}\mathcal{L}V(x, r, \tau) \leq\ & L\gamma(\tau)W(r, x) + c\gamma(\tau) + \gamma'(\tau)W(r, x) + \ell W(r, x)\lambda_{\rm haz}(\tau) \\ & - \gamma(\tau)W(r, x)\lambda_{\rm haz}(\tau)\end{aligned}$$

[2]It would be straightforward to generalize the results in this section to functions $f(x)$ and $\sigma(x)$ that are affine on x.

$$= \Big(L\gamma(\tau) + \gamma'(\tau) + \ell\lambda_{\text{haz}}(\tau) - \gamma(\tau)\lambda_{\text{haz}}(\tau)\Big)W(r,x) + c\gamma(\tau),$$

and our choice of γ satisfying (24), simplifies the above to

$$\mathcal{L}V(x,r,\tau) \le -\epsilon V(x,r,\tau) + c\,b, \quad \forall x \in \mathbb{R}^{n_x},\ r \in \mathbb{N},\ \tau \in [0,T). \tag{29}$$

From Theorem 1, we then conclude that

$$\mathrm{E}\left[V(x_t,r_t,\tau_t)\right] \le \mathrm{E}\left[W(0,x_0)\right] + c\,b\,t, \quad \forall t \ge 0,$$

which establishes the finiteness of $\mathrm{E}\left[V(x_t,r_t,\tau_t)\right]$ for every finite time $t \ge 0$. Taking derivatives to both sides of (12) with respect to t and using (29), we further conclude that

$$\frac{\mathrm{d}\,\mathrm{E}\left[V(x_t,r_t,\tau_t)\right]}{\mathrm{d}t} = \mathrm{E}\left[\mathcal{L}V(x_t,r_t,\tau_t)\right] \le -\epsilon\,\mathrm{E}\left[V(x_t,r_t,\tau_t)\right] + c\,b,\ \ \forall t \ge 0.$$

From this and the Comparison Principle [5, Lemma 3.4], one obtains

$$\begin{aligned}\mathrm{E}\left[V(x_t,r_t,\tau_t)\right] &\le e^{-\epsilon t}\,\mathrm{E}\left[W(0,x_0)\right] + c\int_{[0,t]} e^{-\epsilon(t-s)}\gamma(\tau_s)ds \\ &\le e^{-\epsilon t}\,\mathrm{E}\left[W(0,x_0)\right] + \frac{c\,b}{\epsilon}, \quad \forall t \ge 0.\end{aligned}$$

and the inequality (25) then follows from the observation that

$$\mathrm{E}[\|x_t\|^2] \le \frac{1}{\alpha}\,\mathrm{E}[W(r_t,x_t)] = \frac{1}{\alpha}\,\mathrm{E}\left[\frac{V(x_t,r_t,\tau_t)}{\gamma(\tau_t)}\right] \le \frac{\mathrm{E}\left[V(x_t,r_t,\tau_t)\right]}{\alpha\,a}.$$

■

Proof of Lemma 1. The desired function γ can be constructed by the following differential equation:

$$\gamma(0) = 1, \qquad \gamma'(s) = \begin{cases}(\lambda_{\text{haz}}(s) - L - \epsilon)\gamma(s) - \ell\lambda_{\text{haz}}(s) & \gamma(s) < 1 \\ \min\left\{0, (\lambda_{\text{haz}}(s) - L - \epsilon)\gamma(s) - \ell\lambda_{\text{haz}}(s)\right\} & \gamma(s) \ge 1\end{cases} \quad \forall s \in [0,T). \tag{30}$$

Although the right-hand-side of (30) is discontinuous, the solution to this differential equation exists globally on $[0,T)$ and consists of segments on which γ is constant and equal to one, interlaced with segments on which it follows the top branch, which is linear on γ. By construction, $\gamma'(s)$ satisfies the inequality in (24) and $\gamma(s) \le 1$, $\forall s \in [0,T)$. To complete the proof, it remains to show that there exists a constant $a > 0$ such that $\gamma(s) \ge a$, $\forall s \in [0,T)$. To this end consider an arbitrary interval (s_1,s_2) on which $\gamma(s) < 1$, $\forall s \in (s_1,s_2)$ and $\gamma(s_1) = 1$. On such interval, γ evolves according to the following linear differential equation

$$\gamma(s_1) = 1, \qquad \gamma'(s) = (\lambda_{\text{haz}}(s) - \epsilon - L)\gamma(s) - \ell\lambda_{\text{haz}}(s), \quad \forall s \in (s_1,s_2),$$

whose solution is given by

$$\begin{aligned}\gamma(s) &= e^{-(L+\epsilon)(s-s_1)+\int_{s_1}^{s}\lambda_{\text{haz}}(\eta)d\eta} - \ell\int_{s_1}^{s} e^{-(L+\epsilon)(s-\rho)+\int_{\rho}^{s}\lambda_{\text{haz}}(\eta)d\eta}\lambda_{\text{haz}}(\rho)d\rho, \\ &= e^{-(L+\epsilon)(s-s_1)+\int_{s_1}^{s}\lambda_{\text{haz}}(\eta)d\eta}\Big(1 - \ell e^{-(L+\epsilon)s_1}\int_{s_1}^{s} e^{(L+\epsilon)\rho+\int_{\rho}^{s_1}\lambda_{\text{haz}}(\eta)d\eta}\lambda_{\text{haz}}(\rho)d\rho\Big), \qquad \forall s \in [s_1,s_2).\end{aligned} \tag{31}$$

Using (13), we conclude that

$$e^{\int_{\rho}^{s}\lambda_{\text{haz}}(\eta)d\eta} = e^{\int_{\rho}^{s}\frac{F_\tau'(\eta)}{1-F_\tau(\eta)}d\eta} = e^{\int_{F_\tau(\rho)}^{F_\tau(s)}\frac{dF}{1-F}} = e^{\log\frac{1-F_\tau(\rho)}{1-F_\tau(s)}} = \frac{1-F_\tau(\rho)}{1-F_\tau(s)},\ \ \forall\, 0 \le \rho \le s < T, \tag{32}$$

which can be substituted in (31) to obtain

$$\gamma(s) = e^{-(L+\epsilon)(s-s_1)+\int_{s_1}^{s}\lambda_{\text{haz}}(\eta)d\eta}\Big(1 - \frac{\ell e^{-(L+\epsilon)s_1}}{1-F_\tau(s_1)}\int_{s_1}^{s} e^{(L+\epsilon)\rho}F_\tau'(\rho)d\rho\Big), \quad \forall s \in [s_1,s_2). \tag{33}$$

From this and (26), we conclude that

$$\gamma(s) \ge a := e^{-d_1}(1-d_2) > 0, \quad \forall s \in [s_1,s_2).$$

■

Proof of Corollary 2. To verify condition (i) we use directly the condition C2 in Theorem 3. To this end, pick $\epsilon > 0$ sufficiently small so that

$$\ell e^{(L+\epsilon)T} \le 1. \tag{34}$$

The desired function γ can be implicitly defined by the following (linear) differential equation

$$\gamma(0) = 1, \qquad \gamma'(s) = -(\epsilon + L)\gamma(s), \qquad \forall s \in [0, T),$$

whose solution is given by $\gamma(s) = e^{-(\epsilon+L)s}$, $\forall s \in [0, T)$. Using (34), we conclude that $\ell \leq \gamma(s) \leq 1$, $\forall s \in [0, T)$ and therefore

$$\gamma'(s) \leq \lambda_{\text{haz}}(s)(\gamma(s) - \ell) - (\epsilon + L)\gamma(s) = (\lambda_{\text{haz}}(s) - \epsilon - L)\gamma(s) - \ell\lambda_{\text{haz}}(s), \quad \forall s \in [0, T).$$

Therefore (24) holds with $a := \ell$, $b := 1$.

To verify condition (ii), it suffices to show that (26b) holds, because this distribution has finite support. To this end, pick $\epsilon > 0$ sufficiently small so that

$$\ell \frac{e^{(L+\epsilon)T} - 1}{(L + \epsilon)T} < 1. \tag{35}$$

For the desired distribution, $F_\tau(s) = s/T$ and $F'_\tau(s) = 1/T$, $\forall s \in [0, T)$. In this case,

$$\begin{aligned} \ell \frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho} F'_\tau(\rho) d\rho &= \ell \frac{e^{(L+\epsilon)(s_2 - s_1)} - 1}{(L + \epsilon)(T - s_1)} \\ &\leq \ell \frac{e^{(L+\epsilon)(T - s_1)} - 1}{(L + \epsilon)(T - s_1)} \leq \ell \frac{e^{(L+\epsilon)T} - 1}{(L + \epsilon)T} < 1, \quad \forall 0 \leq s_1 < s_2 < T, \end{aligned}$$

where we used the fact that $\frac{e^x - 1}{x}$ is a monotone increasing function of x and (35).

To verify condition (iii), we pick $\epsilon > 0$ sufficiently small so that

$$\frac{\ell}{1 - (L + \epsilon)\bar{T}}. \tag{36}$$

For the desired distribution, $\lambda_{\text{haz}}(s) = 1/\bar{T}$, $F_\tau(s) = 1 - e^{-s/\bar{T}}$ and $F'_\tau(s) = e^{-s/\bar{T}}/\bar{T}$, $\forall s \geq 0$. In this case, (26a) holds because $1/\bar{T} - L - \epsilon > 0$. As for (26b):

$$\begin{aligned} \ell \frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho} F'_\tau(\rho) d\rho &= \frac{\ell}{\bar{T}} \int_{s_1}^{s_2} e^{(L+\epsilon-\frac{1}{\bar{T}})(\rho - s_1)} d\rho \\ &= \frac{\ell\big(1 - e^{(L+\epsilon-\frac{1}{\bar{T}})(s_2 - s_1)}\big)}{1 - (L + \epsilon)\bar{T}} \leq \frac{\ell}{1 - (L + \epsilon)\bar{T}} < 1, \quad \forall s_2 > s_1 \geq 0, \end{aligned}$$

where we used (36). ■

V. NETWORKED CONTROL SYSTEMS

Consider a nonlinear plant and remote controller with exogenous disturbances of the following form:

$$dx_P = f_P(x_P, \hat{u})dt + g_P(x_P, \hat{u})dw, \qquad y = g_P(x_P), \tag{37a}$$

$$dx_C = f_C(x_C, \hat{y})dt + g_C(x_C, \hat{y})dw, \qquad u = g_C(x_C), \tag{37b}$$

where x_P and x_C are the states of the plant and the controller; $\hat{u}$ and y the plant's input and output; $\hat{y}$ and u the controller's input and output; and w a standard Wiener process. The plant and the controller are connected through a *two-channel feedback NCS* as in Fig. 1. Ignoring network delay, between the sampling times $\{t_k : k \in \mathbb{N}\}$ both $\hat{u}$ and $\hat{y}$ are held constant:

$$\hat{u}_t = \hat{u}_{t_k}, \qquad \hat{y}_t = \hat{y}_{t_k}, \qquad \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \tag{38}$$

The signals u_t and y_t are not necessarily both sampled and sent to the network at every sampling time. Moreover, the samples sent may be dropped by the network with a given probability[3]. This is captured by the following model

$$\hat{u}_{t_k} = \begin{cases} u_{t_k} & u \text{ sampled at time } t_k \text{ and not dropped} \\ \hat{u}_{t_k^-} & u \text{ not sampled at time } t_k \text{ or dropped} \end{cases} \qquad \forall k \in \mathbb{N}, \tag{39a}$$

$$\hat{y}_{t_k} = \begin{cases} y_{t_k} & y \text{ sampled at time } t_k \text{ and not dropped} \\ \hat{y}_{t_k^-} & y \text{ not sampled at time } t_k \text{ or dropped} \end{cases} \qquad \forall k \in \mathbb{N}. \tag{39b}$$

[3]This model ignores network quantization. However, it would be straightforward to include it.

This sampling model with drops can be written compactly as

$$\hat{u}_{t_k} = u_{t_k} + z_k(\hat{u}_{t_k^-} - u_{t_k}) + (1 - z_k)\rho_u\big(k - 1, e_{t_k^-}\big), \tag{40a}$$

$$\hat{y}_{t_k} = y_{t_k} + z_k(\hat{y}_{t_k^-} - y_{t_k}) + (1 - z_k)\rho_y\big(k - 1, e_{t_k^-}\big), \tag{40b}$$

where $z_k \in \mathcal{Z} := \{0, 1\}$ is equal to one if the sample sent at time t_k is dropped and equal to zero otherwise, $e = \left[\begin{smallmatrix} e_u \\ e_y \end{smallmatrix}\right] := \left[\begin{smallmatrix} \hat{u}-u \\ \hat{y}-y \end{smallmatrix}\right]$, and

$$\rho_u(k-1, e) := \begin{cases} 0 & u \text{ sampled at time } t_k \\ e_u & u \text{ not sampled at time } t_k \end{cases}$$

$$\rho_y(k-1, e) := \begin{cases} 0 & y \text{ sampled at time } t_k \\ e_y & y \text{ not sampled at time } t_k \end{cases}$$

Nesic and Teel [6], Walsh et al. [11] actually consider a sampling model more general than (39), as they allow for a subset of the entries of the vectors u and y to be transmitted through the network at each sampling time. In practice, this means that at each sampling time only some entries of $\rho_u(\cdot)$ and $\rho_y(\cdot)$ are set equal to zero. Following Nesic and Teel [6], we capture this by generalizing (40) to

$$e_{t_k} = z_k e_{t_k^-} + (1 - z_k)\rho(k - 1, e_{t_k^-}), \qquad \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \tag{41}$$

where $\rho(k - 1, \cdot)$ specifies which entries of u and y are sampled at the kth sampling time. The function $\rho(\cdot)$ can be regarded as implementing a *network access protocol* that decides which input/output channels should be sampled at each time t_k, $k \in \mathbb{N}$. When this decision is based on the current mismatches between u and $\hat{u}$ and/or between y and $\hat{y}$, we have a *dynamic protocol*, such as the try-once-discard protocol in [6, 11]. Otherwise, we have a *static protocol*, such as the round-robin protocol in [6, 11, 13].

We are interested in networked control systems for which the sampling times $\{t_k \in [0, \infty) : k \in \mathbb{N}\}$ and the drops $\{z_k \in \mathcal{Z} : k \in \mathbb{N}\}$ are the jump times and the jump points, respectively, of a renewal process r_t with hazard rate λ_{haz} and measure $\zeta_{\mathcal{Z}}$ for the jump points. The measure $\zeta_{\mathcal{Z}}$ corresponds to a Bernoulli random variable with probability of drop (i.e., $z_k = 1$) equal to p. Defining $x := \begin{bmatrix} x'_P & x'_C \end{bmatrix}'$, the NCS described by (37), (38), and (41) can be modeled by a stochastic impulsive system of the form

$$dx_t = g(x_t, e_t)dt + \eta(x_t, e_t)dw_t, \qquad \forall t \geq 0,\ x \in \mathbb{R}^{n_x},\ w \in \mathbb{R}^{n_w}, \tag{42a}$$

$$de_t = f(x_t, e_t)dt + \sigma(x_t, e_t)dw_t \qquad \forall t \in [t_k, t_{k+1}),\ e \in \mathbb{R}^{n_e}, \tag{42b}$$

$$e_{t_k} = z_k e_{t_k^-} + (1 - z_k)\rho(k - 1, e_{t_k^-}), \qquad \forall k \in \mathbb{N}. \tag{42c}$$

In view of the results in Section IV, the stability of (42) can be deduced by analyzing the weak solution-processes to the following jump-diffusion equation

$$dx_t = g(x_t, e_t)dt + \eta(x_t, e_t)dw_t, \tag{43a}$$

$$de_t = f(x_t, e_t)dt + \sigma(x_t, e_t)dw_t + \int_{\mathcal{Z}} (z e_{t^-} + (1 - z)\rho(r_{t^-}, e_{t^-}) - e_{t^-}) n(dt, dz), \tag{43b}$$

$$dr_t = \int_{\mathcal{Z}} n(dt, dz), \tag{43c}$$

$$d\tau_t = dt - \int_{\mathcal{Z}} (\tau_{t^-})\, n(dt, dz), \tag{43d}$$

with jump intensity $\lambda_{\text{haz}}(\tau_{t^-})\zeta_{\mathcal{Z}}(dz)$ and initialization $r_0 = \tau_0 = 0$ with probability one.

Theorem 4: Assume that the following two conditions hold:

C3 There exist nonnegative functions $U, Y : \mathbb{R}^{n_x} \to [0, \infty)$, $W : \mathbb{N} \times \mathbb{R}^{n_e} \to [0, \infty)$ and constants $L_x, L \in \mathbb{R}$, $\delta, \ell, c_1, c_2, \alpha_1, \alpha_2 \geq 0$ for which

$$\nabla_x U(x) \cdot g(x, e) + \frac{1}{2}\operatorname{trace}[\eta(x, e)' H_x U(x) \eta(x, e)] \leq -\delta U(x) + L_x W(r, e) + c_1, \tag{44a}$$

$$\begin{aligned} &\nabla_x U(x) \cdot g(x, e) + \frac{1}{2}\operatorname{trace}[\eta(x, e)' H_x U(x) \eta(x, e)] \\ &\quad + \nabla_e W(r, e) \cdot f(x, e) + \frac{1}{2}\operatorname{trace}[\sigma(x, e)' H_e W(r, e) \sigma(x, e)] \\ &\qquad \leq -\delta U(x) + (L + L_x) W(r, e) + c_1 + c_2, \end{aligned} \tag{44b}$$

$$(1-p)W(r+1,\rho(r,e)) + pW(r+1,e) \le \ell\, W(r,e), \tag{44c}$$

$$U(x) \ge \alpha_1\|x\|^2, \quad W(r,e) \ge \alpha_2\|e\|^2, \quad \forall r \in \mathbb{N},\ e \in \mathbb{R}^{n_e},\ x \in \mathbb{R}^{n_x}. \tag{44d}$$

C4 There exists a continuously differentiable functions $\gamma_1, \gamma_2 : [0,T) \to [0,\infty)$ and constants $\epsilon > 0$, $0 < a \le b < \infty$ such that

$$\gamma_1'(s) \le (\lambda_{\text{haz}}(s) + \delta - \epsilon)\gamma_1(s) - \gamma_1(0)\lambda_{\text{haz}}(s), \tag{45a}$$

$$\gamma_2'(s) \le (\lambda_{\text{haz}}(s) - L - \epsilon)\gamma_2(s) - L_x\gamma_1(s) - \ell\,\gamma_2(0)\lambda_{\text{haz}}(s), \tag{45b}$$

$$a \le \gamma_2(s) \le \gamma_1(s) \le b, \qquad \forall s \in [0,T). \tag{45c}$$

Then every weak solution-process x_t, e_t to (43) for which $\mathrm{E}[U(x_0)] < \infty$ and $\mathrm{E}[W(0,e_0)] < \infty$ is mean-square stable. □

A. Condition C4

Intuitively, the stability of the solutions to the stochastic impulsive system (43) will rely on frequent jumps that keep the error process e_t small. The condition C4 in Theorem 4 implicitly expresses this requirements but it is difficult to verify directly. The next Lemma provides an alternative version of this condition (possibly more conservative) that is generally straightforward to verify.

Lemma 2: Assume that there exists constants $d_1 < \infty$, $d_2 < 1$ such that

$$\int_{s_1}^{s} \lambda_{\text{haz}}(\rho)d\rho \ge (L+\epsilon)(s-s_1) - d_1, \tag{46a}$$

$$\frac{e^{-(L+\epsilon)s_1}}{1-F_\tau(s_1)}\int_{s_1}^{s_2} e^{(L+\epsilon)\rho}\Big(\ell F_\tau'(\rho) + L_x\big(1-F_\tau(\rho)\big)\Big)d\rho \le d_2, \tag{46b}$$

$\forall\, 0 \le s_1 < s_2 < T$. Then condition C4 in Theorem 4 holds. □

Note that when $L_x = 0$ the assumptions of Lemma 2 are exactly the same as the ones of Lemma 1. This corresponds to the (somewhat unlikely) situation of a process x_t that is stable regardless of the evolution of e_t [cf. (44a)].

Lemma 2 allows one to determine whether or not condition C4 in Theorem 4 holds for given distributions of the inter-jump time. The following corollary, considers a few common distributions:

Corollary 3: C4 in Theorem 4 holds for any of the following distributions of the inter-jump times:

(i) F_τ is any distribution with support on $[0,T)$ and

$$\frac{\ell + \frac{L_x}{L}}{1+\frac{L_x}{L}} e^{LT} < 1. \tag{47}$$

(ii) F_τ is uniformly distributed on $[0,T)$ and

$$\frac{\ell + \frac{L_x}{L}}{1+\frac{L_x}{L}} \frac{e^{LT}-1}{LT} < 1 \tag{48}$$

(iii) F_τ is exponentially distributed with mean $\bar{T}$ and

$$\frac{\ell + L_x\bar{T}}{1-L\bar{T}} < 1. \tag{49}$$

The condition (47) is analogous to the one found by Nesic and Teel [6] for deterministic worst-case sampling times and no drops, but the constants involved do not have the same numerical values and it does seem easy to prove that one condition is strictly less conservative than the other. However, for all the examples in Section VI, condition (47) does lead to less conservative results than the one found by Nesic and Teel [6]. The most significant advantage of this result with respect to the ones derived for deterministic worst-case sampling times (e.g., by Nesic and Teel [6], Walsh et al. [10, 11]) is realized when the inter-sampling time distribution is available. E.g., when $LT \gg 1$ and the distribution is uniform, (48) allows ℓ to become almost LT larger than (47). This result also applies to Bernoulli drops, which is not possible in a deterministic setting.

Remark 3: The inequalities (46) in Lemma 2 are used to show that there exists a constant $a > 0$ such that the solution to the nonlinear scalar differential equation

$$\gamma_2'(s) = \begin{cases} (\lambda_{\text{haz}}(s) - L - \epsilon)\gamma_2(s) - L_x - \ell\,\lambda_{\text{haz}}(s) & \gamma_2(s) < 1 \\ \min\big\{0, (\lambda_{\text{haz}}(s) - L - \epsilon)\gamma_2(s) - L_x - \ell\,\lambda_{\text{haz}}(s)\big\} & \gamma_2(s) \ge 1 \end{cases}$$

with initial condition $\gamma_2(0) = 1$ remains larger than or equal to a for every $s \in [0,T)$. For specific hazard rates, one may verify that this is so by numerically solving the above differential equation. However, a numerical verification does not permit the derivation of "clean" conditions between ℓ, L, and L_x such as the ones provided by Corollary 3. □

B. Condition C3

For linear systems and quadratic functions U, W, Y, the condition C3 in Theorem 4 can be verified numerically in an efficient manner. To this effect, we restrict our attention to system dynamics of the form[4]

$$\begin{aligned} &g(x,e) = A_{xx}x + A_{xe}e, && f(x,e) = A_{ex}x + A_{ee}e, \\ &\eta(x,e) = B_x, && \sigma(x,e) = B_e, && \rho(r,e) = R(r,e)e, \qquad \forall r \in \mathbb{N},\ x \in \mathbb{R}^{n_x},\ e \in \mathbb{R}^{n_e}, \end{aligned}$$

and functions U and W of the form $U(x) = x'Qx$, $Q = Q'$, $\forall x \in \mathbb{R}^{n_x}$ and $W(r,e) = e'P_re$, $P_r = P_r'$, $\forall r \in \mathbb{N}$, $e \in \mathbb{R}^{n_e}$, respectively. In this case, the inequalities in (44) take the form

$$\begin{bmatrix} x' & e' \end{bmatrix} \begin{bmatrix} QA_{xx} + A_{xx}'Q + \delta Q & QA_{xe} \\ A_{xe}'Q & -L_xP_r \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \leq c_1 - \mathrm{trace}[B_x'QB_x] \tag{51a}$$

$$\begin{bmatrix} x' & e' \end{bmatrix} \begin{bmatrix} QA_{xx} + A_{xx}'Q + \delta Q & QA_{xe} + A_{ex}'P_r \\ A_{xe}'Q + P_rA_{ex} & P_rA_{ee} + A_{ee}'P_r - (L+L_x)P_r \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \leq c_1 + c_2 - \mathrm{trace}[B_x'QB_x + B_e'P_rB_e] \tag{51b}$$

$$x'(Q - \alpha_1 I)x \geq 0, \quad e'(P_r - \alpha_2 I)e \geq 0, \tag{51c}$$

$$e'\big((1-p)R(r,e)'P_{r+1}R(r,e) + pP_{r+1} - \ell\, P_r\big)e \leq 0, \quad \forall r \in \mathbb{N},\ e \in \mathbb{R}^{n_e},\ x \in \mathbb{R}^{n_x}. \tag{51d}$$

The existence of the constants c_1, c_2, L, L_x, ℓ, α_1, α_2 and the matrices $Q = Q'$, $P_r = P_r'$, $r \in \mathbb{N}$ for which (51a)–(51c) hold is equivalent to the feasibility of the following family of matrix inequalities in the unknowns L, L_x, ℓ, α_1, α_2, $Q = Q'$, $P_r = P_r'$, $r \in \mathbb{N}$:

$$\begin{bmatrix} QA_{xx} + A_{xx}'Q + \delta Q & QA_{xe} \\ A_{xe}'Q & -L_xP_r \end{bmatrix} \leq 0 \tag{52a}$$

$$\begin{bmatrix} QA_{xx} + A_{xx}'Q + \delta Q & QA_{xe} + A_{ex}'P_r \\ A_{xe}'Q + P_rA_{ex} & P_rA_{ee} + A_{ee}'P_r - (L+L_x)P_r \end{bmatrix} \leq 0, \tag{52b}$$

$$Q - \alpha_1 I \geq 0, \quad P_r - \alpha_2 I \geq 0, \qquad \forall r \in \mathbb{N}. \tag{52c}$$

This is because whenever (52a)–(52c) hold, then (51a)–(51c) also hold with $c_1 := \mathrm{trace}\,[B_x'QB_x]$, $c_2 := \sup_{r\in\mathbb{N}} \mathrm{trace}\,[B_e'P_rB_e]$. Conversely, if (52a)–(52c) do not hold, then it is straightforward to show that (51a)–(51c) cannot hold for finite constants c_1, c_2.

For protocols such as the *round-robin protocol* in [6, 11, 13], the matrices $R(r,e)$ are independent of e and periodic with respect to the integer r, i.e.,

$$R(r + kN, e) = R_r, \quad \forall r \in \{1, 2, \dots, N\},\ k \in \mathbb{N}, \tag{53}$$

for some finite period $N \in \mathbb{N}$. In this case, we can restrict the search to periodic matrices P_r and (51d) also becomes a matrix inequality of the form[5]

$$(1-p)R_r'P_{r+1}R_r + pP_{r+1} \leq \ell\, P_r, \quad \forall r \in \mathbb{N}. \tag{54}$$

Alternatively, for protocol such as the *try-once-discard protocol* in [6, 11], the matrices $R(r,e)$ do not depend on the integer r and instead are chosen from a finite collection of symmetric projection matrices[6] R_i, $i \in \{1, 2, \dots, N\}$, based on the current value of e. We recall that a matrix R is called a *projection matrix* if $R^2 = R$. In particular,

$$R(r,e) = \arg \min_{R_i : i \in \{1,\dots,n\}} e'R_ie, \quad \forall r \in \mathbb{N},\ e \in \mathbb{R}^{n_e}. \tag{55}$$

In this case, a sufficient condition for (51d) to hold with $P_r = P$, $\forall r \in \mathbb{N}$ is that

$$e'R_ie \leq e'R_je,\ \ \forall j \quad \Rightarrow \quad e'\big((1-p)R_i'PR_i + pP - \ell\, P\big)e \leq 0, \quad \forall i \in \{1, 2, \dots, N\},\ e \in \mathbb{R}^{n_e},$$

which is further implied by the following family of matrix inequalities on the unknowns $\mu_{ij} \geq 0$

$$(1-p)R_i'PR_i + pP + \sum_{j \neq i} \mu_{ij}(R_j - R_i) \leq \ell\, P. \tag{56}$$

As seen in Section V-A, condition C4 implicitly imposes a constraint between ℓ, L, and L_x, for which small values of ℓ favor stability. We can therefore let L and L_x range over $[0, \infty)$ and verify if C4 holds for the smallest constant ℓ for which (52), (54) [or (52), (56)] is feasible. These are quasi-convex generalized eigenvalue minimization problem (GEVP), which can be solved very efficiently [1].

[4]It would be straightforward to generalize the results in this section to functions f, q, σ, η that are affine on x and e.

[5]Although (52), (54) exhibit a quantification over $r \in \mathbb{N}$, because of the periodicity in r, one only has to verify N inequalities.

[6]A matrix R is called a *projection matrix* if $R^2 = R$.

The inequality (44c) should be viewed as a requirement on the network access protocol specified by the function $\rho(\cdot)$. In practice, $L > 0$ and to keep e_t bounded one needs $\ell < 1$. In this case, (44c) requires the protocol to define an exponentially stable auxiliary stochastic discrete-time system

$$y_{k+1} = \begin{cases} \rho(k, y_k) & \text{with probability } 1-p \\ y_k & \text{with probability } p \end{cases}$$

Motivated by Nesic and Teel [6], we use the terminology *uniformly exponentially stable protocol* to denote any protocol that satisfies (44c).

Remark 4: It is worth noting that if the x_t dynamics in (43a) with $e_t = 0$ (ideal network) are asymptotically stable — which corresponds to A_{xx} Hurwitz — then the matrix inequalities in (52) are always feasible for sufficiently large constants $L_x, L > 0$. In this case, as long as the protocol can guarantee that (51d) holds for some $\ell < 1$, any of the inequalities in Corollary 3 can be made to hold by selecting L sufficiently large and T (or $\bar{T}$) sufficiently small. This confirms the intuitive notion that if the closed-loop system is stable for an ideal network and sampling is sufficiently frequent, then the NCS will remain stable. □

C. Proofs of Theorem 4, Lemma 2, and Corollary 3

Proof of Theorem 4. Applying the operator $\mathcal{L}$ defined in Theorem 1 to the nonnegative function $V(x,e,r,\tau) := \gamma_1(\tau)U(x) + \gamma_2(\tau)W(r,e)$, $\forall x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $r \in \mathbb{N}$, $\tau \in [0,T)$ yields

$$\begin{aligned}
\mathcal{L}V(x,e,r,\tau) &= \nabla_\tau V(x,e,r,\tau) + \nabla_x V(x,e,r,\tau)\cdot g(x,e) + \nabla_e V(x,e,r,\tau)\cdot f(x,e) \\
&\quad + \frac{1}{2}\operatorname{trace}[\eta(x,e)' H_x V(x,e,r,\tau)\eta(x,e)] \\
&\quad + \frac{1}{2}\operatorname{trace}[\sigma(x,e)' H_e V(x,e,r,\tau)\sigma(x,e)] \\
&\quad + \lambda_{\text{haz}}(\tau)\int_{\mathcal{Z}} \Big(V(x,\rho(r,e,z),r+1,0) - V(x,e,r,\tau)\Big)\zeta_{\mathcal{Z}}(dz) \\
&= \gamma_1'(\tau)U(x) + \gamma_2'(\tau)W(r,e) \\
&\quad + \gamma_1(\tau)\Big(\nabla_x U(x)\cdot g(x,e) + \frac{1}{2}\operatorname{trace}[\eta(x,e)' H_x U(x)\eta(x,e)]\Big) \\
&\quad + \gamma_2(\tau)\Big(\nabla_e W(r,e)\cdot f(x,e) + \frac{1}{2}\operatorname{trace}[\sigma(x,e)' H_e W(r,e)\sigma(x,e)]\Big) \\
&\quad + \lambda_{\text{haz}}(\tau)\int_{\mathcal{Z}} \Big(\gamma_1(0)U(x) + \gamma_2(0)W(r+1, ze + (1-z)\rho(r,e)) \\
&\qquad - \gamma_1(\tau)U(x) - \gamma_2(\tau)W(r,e)\Big)\zeta_{\mathcal{Z}}(dz) \\
&= \gamma_1'(\tau)U(x) + \gamma_2'(\tau)W(r,e) \\
&\quad + \gamma_1(\tau)\Big(\nabla_x U(x)\cdot g(x,e) + \frac{1}{2}\operatorname{trace}[\eta(x,e)' H_x U(x)\eta(x,e)]\Big) \\
&\quad + \gamma_2(\tau)\Big(\nabla_e W(r,e)\cdot f(x,e) + \frac{1}{2}\operatorname{trace}[\sigma(x,e)' H_e W(r,e)\sigma(x,e)]\Big) \\
&\quad + \lambda_{\text{haz}}(\tau)\Big(\gamma_1(0)U(x) + \gamma_2(0)\int_{\mathcal{Z}} W(r+1, ze + (1-z)\rho(r,e))\zeta_{\mathcal{Z}}(dz) \\
&\qquad - \gamma_1(\tau)U(x) - \gamma_2(\tau)W(r,e)\Big) \\
&= \gamma_1'(\tau)U(x) + \gamma_2'(\tau)W(r,e) \\
&\quad + \gamma_1(\tau)\Big(\nabla_x U(x)\cdot g(x,e) + \frac{1}{2}\operatorname{trace}[\eta(x,e)' H_x U(x)\eta(x,e)]\Big) \\
&\quad + \gamma_2(\tau)\Big(\nabla_e W(r,e)\cdot f(x,e) + \frac{1}{2}\operatorname{trace}[\sigma(x,e)' H_e W(r,e)\sigma(x,e)]\Big) \\
&\quad + \lambda_{\text{haz}}(\tau)\Big(\gamma_1(0)U(x) + \gamma_2(0)(1-p)W(r+1,\rho(r,e)) + \gamma_2(0)pW(r+1,e) \\
&\qquad - \gamma_1(\tau)U(x) - \gamma_2(\tau)W(r,e)\Big) \\
&= \gamma_1'(\tau)U(x) + \gamma_2'(\tau)W(r,e) \\
&\quad + \big(\gamma_1(\tau) - \gamma_2(\tau)\big)\Big(\nabla_x U(x)\cdot g(x,e) + \frac{1}{2}\operatorname{trace}[\eta(x,e)' H_x U(x)\eta(x,e)]\Big) \\
&\quad + \gamma_2(\tau)\Big(\nabla_e W(r,e)\cdot f(x,e) + \frac{1}{2}\operatorname{trace}[\sigma(x,e)' H_e W(r,e)\sigma(x,e)]
\end{aligned}$$

$$
\begin{aligned}
&+ \nabla_x U(x) \cdot g(x,e) + \frac{1}{2}\operatorname{trace}[\eta(x,e)' H_x U(x) \eta(x,e)]\Big) \\
&+ \lambda_{\rm haz}(\tau)\Big(\gamma_1(0)U(x) + \gamma_2(0)(1-p)W(r+1,\rho(r,e)) + \gamma_2(0)pW(r+1,e) \\
&- \gamma_1(\tau)U(x) - \gamma_2(\tau)W(r,e)\Big).
\end{aligned}
$$

Using (44a)–(44c) we conclude that

$$
\begin{aligned}
\mathcal{L}V(x,e,r,\tau) &\le \gamma_1'(\tau)U(x) + \gamma_2'(\tau)W(r,e) \\
&\quad + \big(\gamma_1(\tau) - \gamma_2(\tau)\big)\big(-\delta U(x) + L_x W(r,e) + c_1\big) \\
&\quad + \gamma_2(\tau)\big(-\delta U(x) + (L+L_x)W(r,e) + c_1 + c_2\big) \\
&\quad + \lambda_{\rm haz}(\tau)\big(\ell\gamma_1(0)W(r,e) - \gamma_1(\tau)U(x) - \gamma_2(\tau)W(r,e)\big) \\
&= \Big(\gamma_1'(\tau) - \delta\gamma_1(\tau) + \lambda_{\rm haz}(\tau)\big(\gamma_1(0) - \gamma_1(\tau)\big)\Big)U(x) \\
&\quad \Big(\gamma_2'(\tau) + L_x\gamma_1(\tau) + L\gamma_2(\tau) + \lambda_{\rm haz}(\tau)\big(\ell\gamma_2(0) - \gamma_2(\tau)\big)\Big)W(r,e) \\
&\quad + c_1\gamma_1(\tau) + c_2\gamma_2(\tau)
\end{aligned}
$$

Our choice of γ_1, γ_2 satisfying (45) simplifies the above to

$$
\mathcal{L}V(x,e,r,\tau) \le -\epsilon V(x,e,r,\tau) + (c_1+c_2)b,
$$

$\forall x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $r \in \mathbb{N}$, $\tau \in [0,T)$. The result then follows from a reasoning completely analogous to the one used in the proof of Theorem 3. ∎

Proof of Lemma 2. Without loss of generality we assume that $\epsilon \le \delta$ and take $\gamma_1(s) = 1$, $\forall s \in [0,T)$, for which (45a) holds trivially. We then construct γ_2 using the following differential equation:

$$
\gamma_2(0) = 1, \qquad \gamma_2'(s) = \begin{cases} (\lambda_{\rm haz}(s) - L - \epsilon)\gamma_2(s) - L_x - \ell\,\lambda_{\rm haz}(s) & \gamma_2(s) < 1 \\ \min\big\{0, (\lambda_{\rm haz}(s) - L - \epsilon)\gamma_2(s) - L_x - \ell\,\lambda_{\rm haz}(s)\big\} & \gamma_2(s) \ge 1 \end{cases} \quad \forall s \in [0,T). \tag{57}
$$

By construction, $\gamma_2(s)$ satisfies (45b) and $\gamma_2(s) \le \gamma_1(s) = 1$, $\forall s \in [0,T)$. To complete the proof, it remains to show that there exists a constant $a > 0$ such that $\gamma_2(s) \ge a$, $\forall s \in [0,T)$. To this end consider an arbitrary interval (s_1, s_2) on which $\gamma_2(s) < 1$, $\forall s \in (s_1,s_2)$ and $\gamma_2(s_1) = 1$. On such interval, γ_2 evolves according to the following linear differential equation

$$
\gamma_2(s_1) = 1, \qquad \gamma_2'(s) = (\lambda_{\rm haz}(s) - L - \epsilon)\gamma_2(s) - L_x - \ell\,\lambda_{\rm haz}(s), \quad \forall s \in (s_1,s_2),
$$

whose solution is given by

$$
\begin{aligned}
\gamma_2(s) &= e^{-(L+\epsilon)(s-s_1) + \int_{s_1}^{s}\lambda_{\rm haz}(\eta)d\eta} - \int_{s_1}^{s} e^{-(L+\epsilon)(s-\rho) + \int_{\rho}^{s}\lambda_{\rm haz}(\eta)d\eta}\big(\ell\,\lambda_{\rm haz}(\rho) + L_x\big)d\rho, \\
&= e^{-(L+\epsilon)(s-s_1) + \int_{s_1}^{s}\lambda_{\rm haz}(\eta)d\eta}\Big(1 - e^{-(L+\epsilon)s_1}\int_{s_1}^{s} e^{(L+\epsilon)\rho + \int_{\rho}^{s_1}\lambda_{\rm haz}(\eta)d\eta}\big(\ell\,\lambda_{\rm haz}(\rho) + L_x\big)d\rho\Big), \\
&= e^{-(L+\epsilon)(s-s_1) + \int_{s_1}^{s}\lambda_{\rm haz}(\eta)d\eta}\Big(1 - \frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)}\int_{s_1}^{s} e^{(L+\epsilon)\rho}\Big(\ell\,F_\tau'(\rho) + L_x\big(1 - F_\tau(\rho)\big)\Big)d\rho\Big),
\end{aligned}
$$

$\forall s \in [s_1, s_2)$ [cf. derivation of (33)]. From this and (46), we conclude that

$$
\gamma_2(s) \ge a := e^{-d_1}(1 - d_2) > 0, \quad \forall s \in [s_1, s_2).
$$

∎

Proof of Corollary 3. To verify condition (i) we use directly the condition C4 in Theorem 4. To this effect, we note that (47) is equivalent to

$$
e^{-LT} > \frac{\ell + \frac{L_x}{L}}{1 + \frac{L_x}{L}} \quad \Leftrightarrow \quad \Big(1 + \frac{L_x}{L}\Big)e^{-LT} - \frac{L_x}{L} > \ell
$$

and therefore we can pick $\epsilon > 0$ sufficiently small so that

$$
\epsilon \le \delta, \qquad \Big(1 + \frac{L_x}{L+\epsilon}\Big)e^{-(L+\epsilon)T} - \frac{L_x}{L+\epsilon} \ge \ell. \tag{58}
$$

We pick $\gamma_1(s) = 1$, $\forall s \in [0, T)$ and γ_2 is implicitly defined by the following (linear) differential equation

$$\gamma_2(0) = 1, \qquad \gamma_2'(s) = -(L+\epsilon)\gamma_2(s) - L_x, \qquad \forall s \in [0, T),$$

whose solution is given by

$$\gamma_2(s) = \Big(1 + \frac{L_x}{L+\epsilon}\Big) e^{-(L+\epsilon)s} - \frac{L_x}{L+\epsilon} \qquad \forall s \in [0, T).$$

Since γ_2 is monotone decreasing, we conclude using (58) that

$$\ell \leq \gamma_2(T) \leq \gamma_2(s) \leq \gamma_2(0) = 1 = \gamma_1(s), \quad \forall s \in [0, T),$$

and therefore

$$\begin{aligned}
&\gamma_1'(s) = 0 \leq \lambda_{\text{haz}}(s)\big(\gamma_1(s) - \gamma_1(0)\big) + (\delta - \epsilon)\gamma_1(s), \\
&\gamma_2'(s) = -(L+\epsilon)\gamma_2(s) - L_x\gamma_1(s) \leq \lambda_{\text{haz}}(s)\big(\gamma_2(s) - \ell\,\gamma_2(0)\big) - (L+\epsilon)\gamma_2(s) - L_x\gamma_1(s).
\end{aligned}$$

Therefore (45) holds with $a := \ell$, $b := 1$.

To verify condition (ii), it suffices to show that (46b) holds, because this distribution has finite support and therefore (46a) holds with $d_1 = (L+\epsilon)T$. To this effect, we note that (48) is equivalent to

$$\Big(\ell + \frac{L_x}{L}\Big)\frac{e^{LT} - 1}{LT} - \frac{L_x}{L} < 1.$$

and therefore we can pick $\epsilon > 0$ sufficiently small so that

$$\Big(\ell + \frac{L_x}{L+\epsilon}\Big)\frac{e^{(L+\epsilon)T} - 1}{(L+\epsilon)T} - \frac{L_x}{L+\epsilon} < 1. \tag{59}$$

For the desired distribution, $F_\tau(s) = s/T$ and $F_\tau'(s) = 1/T$, $\forall s \in [0, T)$. In this case,

$$\begin{aligned}
&\frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho}\Big(\ell F_\tau'(\rho) + L_x\big(1 - F_\tau(\rho)\big)\Big) d\rho \\
&\qquad = \frac{e^{-(L+\epsilon)s_1}}{T - s_1} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho}\Big(\ell + L_x(T - \rho)\Big) d\rho \\
&\qquad = \frac{e^{(L+\epsilon)(s_2 - s_1)}(L_x + (\ell + L_x(T - s_2))(L+\epsilon)) - (L_x + (\ell + L_x(T - s_1))(L+\epsilon))}{(L+\epsilon)^2(T - s_1)} \\
&\qquad = \frac{e^{(L+\epsilon)(s_2 - s_1)}(\ell + \frac{L_x}{L+\epsilon} + L_x(T - s_2)) - (\ell + \frac{L_x}{L+\epsilon} + L_x(T - s_1))}{(L+\epsilon)(T - s_1)}, \quad \forall 0 \leq s_1 < s_2 < T.
\end{aligned}$$

Since $e^{(L+\epsilon)(s_2 - s_1)}(\ell + \frac{L_x}{L+\epsilon} + L_x(T - s_2)$ is a monotone increasing function of s_2, we can construct an upper bound by setting $s_2 = T$, which yields

$$\begin{aligned}
&\frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho}\Big(\ell F_\tau'(\rho) + L_x\big(1 - F_\tau(\rho)\big)\Big) d\rho \\
&\qquad \leq \frac{(\ell + \frac{L_x}{L+\epsilon})e^{(L+\epsilon)(T - s_1)} - (\ell + \frac{L_x}{L+\epsilon} + L_x(T - s_1))}{(L+\epsilon)(T - s_1)} \\
&\qquad = \Big(\ell + \frac{L_x}{L+\epsilon}\Big)\frac{e^{(L+\epsilon)(T - s_1)} - 1}{(L+\epsilon)(T - s_1)} - \frac{L_x}{L+\epsilon} \leq \Big(\ell + \frac{L_x}{L+\epsilon}\Big)\frac{e^{(L+\epsilon)T} - 1}{(L+\epsilon)T} - \frac{L_x}{L+\epsilon} < 1, \quad \forall 0 \leq s_1 < s_2 < T.
\end{aligned}$$

where we used the fact that $\frac{e^x - 1}{x}$ is a monotone increasing function of x and (59).

To verify condition (iii), we pick $\epsilon > 0$ sufficiently small so that

$$\frac{\ell + L_x\bar{T}}{1 - (L+\epsilon)\bar{T}} < 1. \tag{60}$$

For the desired distribution, $\lambda_{\text{haz}}(s) = 1/\bar{T}$, $F_\tau(s) = 1 - e^{-s/\bar{T}}$ and $F_\tau'(s) = e^{-s/\bar{T}}/\bar{T}$, $\forall s \geq 0$. In this case, (46a) holds because $1/\bar{T} - L - \epsilon > 0$. As for (46b):

$$\frac{e^{-(L+\epsilon)s_1}}{1 - F_\tau(s_1)} \int_{s_1}^{s_2} e^{(L+\epsilon)\rho}\Big(\ell F_\tau'(\rho) + L_x\big(1 - F_\tau(\rho)\big)\Big) d\rho = \Big(\frac{\ell}{\bar{T}} + L_x\Big) \int_{s_1}^{s_2} e^{(L+\epsilon - \frac{1}{\bar{T}})(\rho - s_1)} d\rho$$

$$= \frac{\ell + L_x \bar{T}}{1-(L+\epsilon)\bar{T}}\left(1 - e^{(L+\epsilon-\frac{1}{\bar{T}})(s_2-s_1)}\right) \leq \frac{\ell + L_x \bar{T}}{1-(L+\epsilon)\bar{T}} < 1, \quad \forall s_2 > s_1 \geq 0,$$

where we used (60). ■

VI. Examples and Discussion

We investigate the conservativeness of the stability conditions derived above in the context of two benchmark problems that previously appeared in the literature.

A. Batch Reactor

This example appeared in [2, 6, 11] and considers the control of the following linearized model for a two-input/two-output unstable batch reactor:

$$\dot{x}_P = A_P x_P + B_P \hat{u}, \qquad\qquad y = C_P x_P,$$

where

$$A_P := \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \quad B_P := \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_P := \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This system is controlled by the following PI controller:

$$\dot{x}_C = A_C x_C + B_C \hat{y}, \qquad\qquad u = C_C x_C + D_C \hat{y},$$

where

$$A_C := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_C := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C_C := \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}, \qquad D_C := \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}.$$

Following Nesic and Teel [6], Walsh et al. [11], we assume that only the outputs are transmitted over the network, which means that $\hat{u} = u$, $e = \hat{y} - u$. This leads to the following matrices in (50):

$$A_{xx} := \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix}, \qquad A_{xe} := \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix},$$
$$A_{ex} := \begin{bmatrix} -C_P & 0 \end{bmatrix} A_{xx}, \qquad A_{ee} := \begin{bmatrix} -C_P & 0 \end{bmatrix} A_{xe}.$$

In the equations above, we omitted the noise terms since these do not affect the stability conditions (although they do affect the upper bounds on the state and error variances). For the round-robin protocol we assumed that the first entry of the output vector was transmitted at the odd samples, and the second entry at the even samples. This corresponds to

$$R_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad\qquad R_2 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

in (53). The same projection matrices were used in (55) for the try-once-Discard protocol, which then sends the entry of the output vector with largest absolute value. Table I summarizes the results obtained.

TABLE I

COMPARISON OF STABILITY CONDITIONS FOR THE BATCH REACTOR EXAMPLE

	Round-robin		Try-once-discard	
	no drops	$p = 50\%$	no drops	$p = 50\%$
Maximum (deterministic) time interval between samples computed from the results in Walsh et al. [11] (values taken from Walsh et al. [11])	$\approx 10^{-5}$	NA	$\approx 10^{-5}$	NA
Maximum (deterministic) time interval between samples computed from the results in Nesic and Teel [6] (values taken from Nesic and Teel [6])	0.0082	NA	0.01	NA
Maximum (deterministic) time interval between samples computed from the results in Carnevale et al. [2] (values taken from Carnevale et al. [2])	0.009	NA	0.0108	NA
Maximum (deterministic) time interval between samples computed from the results in Tabbara et al. [9] (values taken from Tabbara et al. [9])	0.0123	NA	NA	NA
Maximum support for an arbitrary inter-sampling time distribution from the condition (i) in Corollary 3	0.0279	0.010323	0.0200	0.008804
Maximum support for a uniform inter-sampling time distribution from the condition (ii) in Corollary 3	0.0517	0.019921	0.0372	0.017032
Maximum average for an exponential inter-sampling time distribution from the condition (iii) in Corollary 3	0.0217	0.009239	0.0158	0.007946

TABLE II

COMPARISON OF STABILITY CONDITIONS FOR THE CH-47 TANDEM-ROTOR HELICOPTER EXAMPLE

	Round-robin		Try-once-discard	
	no drops	$p = 50\%$	no drops	$p = 50\%$
Maximum (deterministic) time interval between samples computed from the results in Ye et al. [12] (values taken from Tabbara et al. [9])	3.13×10^{-19}	NA	?	NA
Maximum (deterministic) time interval between samples computed from the results in Nesic and Teel [6] (values taken from Tabbara et al. [9])	1.20×10^{-5}	NA	?	NA
Maximum (deterministic) time interval between samples computed from the results in Tabbara et al. [9] (values taken from Tabbara et al. [9])	2.81×10^{-4}	NA	NA	NA
Maximum support for an arbitrary inter-sampling time distribution from the condition (i) in Corollary 3	8.02×10^{-4}	2.95×10^{-4}	5.38×10^{-4}	2.39×10^{-4}
Maximum support for a uniform inter-sampling time distribution from the condition (ii) in Corollary 3	1.48×10^{-3}	5.69×10^{-4}	1.01×10^{-3}	4.63×10^{-4}
Maximum average for an exponential inter-sampling time distribution from the condition (iii) in Corollary 3	6.21×10^{-4}	2.64×10^{-4}	4.34×10^{-4}	2.17×10^{-4}

B. CH-47 tandem-rotor helicopter

This example appeared in [8] and considers the control of a CH-47 tandem-rotor helicopter in the horizontal plane, around a nominal airspeed of 40knots, which can be modeled by

$$\dot{x}_P = A_P x_P + B_P \hat{u}, \qquad y = C_P x_P,$$

where the output y_1 denotes the vertical velocity (in knots/hour), the output y_2 the pitch altitude (in radians), the input u_1 the collective rotor thrust, the input u_2 the differential rotor thrust, and

$$A_P := \begin{bmatrix} -.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B_P := \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}, \qquad C_P := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{bmatrix}.$$

This system is controlled by the following static controller from [8]

$$u = D_C \hat{y},$$

where

$$D_C := \begin{bmatrix} -12.7177 & -45.0824 \\ 63.5163 & 25.9144 \end{bmatrix}.$$

Following Tabbara et al. [8], we assume that only the outputs are transmitted over the network, which means that $\hat{u} = u$, $e = \hat{y} - u$. This leads to the following matrices in (50):

$$A_{xx} := A_P + B_P D_C C_P, \qquad A_{xe} := B_P D_C,$$
$$A_{ex} := -C_P A_{xx}, \qquad A_{ee} := -C_P A_{xe}.$$

In the equations above, we omitted the noise terms since these do not affect the stability conditions (although they do affect the upper bounds on the state and error variances). For the round-robin protocol we assumed that the first entry of the output vector was transmitted at the odd samples, and the second entry at the even samples. This corresponds to

$$R_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R_2 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

in (53). The same projection matrices were used in (55) for the try-once-Discard protocol, which then sends the entry of the output vector with largest absolute value. Table II summarizes the results obtained.

Several conclusions can be drawn from these two examples:

(i) Theorem 4 appears to be less conservative than its previously published deterministic counterparts. In particular, in the absence of drops, the distribution-independent condition (i) in Corollary 3 allows for supports for the inter-sampling time distribution larger than the maximum time between samples computed from previously published results.

(ii) Information about the inter-sampling time distribution permits less conservative results. E.g., when the distribution is known to be uniform, its support can almost double with respect to the distribution-independent result.

Moreover, even inter-sampling time distributions with infinite-support (such as the exponential distribution) can result in stable NCSs.

(iii) Achieving stability in the presence of drops, generally requires "shorter" inter-sampling times. E.g., with 50% drop probability, the inter-sampling time must generally decrease to smaller than one half.

APPENDIX

Proof of Theorem 1. From (8), we conclude that x_t is the sum of four factors:

(i) A finite-valued and $\mathcal{F}_0$-measurable process x_0.

(ii) A continuous local martingale

$$x_t^c := \int_{[0,t]} \sigma(x_{s^-}) dw_s. \tag{61}$$

First, note that this integral is well defined in the sense of [4, Theorem I.4.31, p. 46] because w_t is a semimartingale [4, Proposition I.4.10, p. 40] and $\sigma(x_-)$ is locally bounded (because x_t is locally bounded) and predictable [4, Proposition I.2.6, p. 17]. Since w_t is actually a continuous martingale [4, Proposition I.4.10, p. 40], we conclude that x_t^c is also a continuous local martingale [4, Corollary I.4.55d, p. 55].

(iii) A purely discontinuous local martingale

$$\int_{[0,t]\times\mathcal{Z}} \xi(x_{s^-}, z) n(ds, dz) - \int_{[0,t]\times\mathcal{Z}} \xi(x_{s^-}, z)\lambda(x_{s^-}, dz) ds \tag{62}$$

Since $\xi(x_s, z)$ is cadlag, $\xi(x_{s^-}, z)$ is predictable [4, Proposition I.2.6, p. 17] and we conclude from [4, Proposition II.1.28, p. 72] that (62) is equal to

$$\int_{[0,t]\times\mathcal{Z}} \xi(x_{s^-}, z)\big(n(ds, dz) - \lambda(x_{s^-}, dz)\big) ds,$$

with the integral defined as in [4, Definition II.1.27, p. 72], which automatically guarantees that this term is a purely discontinuous local martingale.

(iv) A finite-variation predictable process

$$\int_{[0,t]} f(x_{s^-}) ds + \int_{[0,t]\times\mathcal{Z}} \xi(x_{s^-}, z)\lambda(x_{s^-}, dz) ds$$
$$= \int_{[0,t]} \Big(f(x_{s^-}) + \int_{\mathcal{Z}} \xi(x_{s^-}, z)\lambda(x_{s^-}, dz) \Big) ds$$

Since $f(x_s) + \int_{\mathcal{Z}} \xi(x_s, z)\lambda(x_s, dz)$ is cadlag, $f(x_{s^-}) + \int_{\mathcal{Z}} \xi(x_{s^-}, z)\lambda(x_{s^-}, dz)$ is optional [4, Corollary I.1.25, p. 7] and predictable [4, Proposition I.2.6, p. 17]. We therefore conclude that the integral on the right can be defined in the sense of [4, Equation I.3.4, p. 28], which means that it has finite-variation and is predictable.

This shows that x_t is a special semimartingale [4, Definition I.4.21, p. 43] and that x_t^c is the continuous martingale part of x_t [4, Definition I.4.27, p. 45].

From Itô's formula [4, Theorem I.4.57, p. 57], we conclude that $V(x_t)$ is also a semimartingale and

$$V(x_t) = V(x_0) + \int_{[0,t]} \nabla V(x_{s^-}) \cdot dx_s + \frac{1}{2} \int_{[0,t]} \sum_{i,j\le d} \frac{\partial^2 V(x_{s^-})}{\partial x^i \partial x^j} dq_s^{i,j}$$
$$+ \sum_{s\le t} \Big(V(x_{s^-} + \Delta x_s) - V(x_{s^-}) - \nabla V(x_{s^-}) \cdot \Delta x_s \Big), \tag{63}$$

where $\Delta x_s := x_s - x_{s^-}$ and $q^{i,j} := \langle x^{i,c}, x^{j,c} \rangle$. First note that because of (61), [4, Theorem I.4.40d, p. 48], and [4, Definition I.4.9b, p. 39], we have that

$$q_t^{i,j} := \langle x^{i,c}, x^{j,c} \rangle_t = \sum_{k,\ell\le m} \int_{[0,t]} \sigma^{i,k}(x_{s^-}) \sigma^{j,\ell}(x_{s^-}) d\langle w^k, w^\ell \rangle_t$$
$$= \sum_{k\le m} \int_{[0,t]} \sigma^{i,k}(x_{s^-}) \sigma^{j,k}(x_{s^-}) dt,$$

and therefore

$$\int_{[0,t]} \sum_{i,j\le d} \frac{\partial^2 V(x_{s^-})}{\partial x^i \partial x^j} dq_s^{i,j} = \int_{[0,t]} \sum_{k\le m} \sum_{i,j\le d} \frac{\partial^2 V(x_{s^-})}{\partial x^i \partial x^j} \sigma^{i,k}(x_{s^-})\sigma^{j,k}(x_{s^-})ds. \tag{64}$$

Second, since the jump Δx_s of x_s at time s is equal to $\xi(x_{s^-}, z)$, we can re-write the summation in(63) as

$$\int_{[0,t]\times\mathcal{Z}} \Big(V\big(x_{s^-} + \xi(x_{s^-}, z)\big) - V(x_{s^-}) - \nabla V(x_{s^-})\cdot \xi(x_{s^-}, z)\Big) n(ds, dz),$$

which can be re-written as

$$\begin{aligned}
&\int_{[0,t]\times\mathcal{Z}} \Big(V\big(x_{s^-} + \xi(x_{s^-}, z)\big) - V(x_{s^-})\Big) n(ds, dz) \\
&\qquad\qquad - \int_{[0,t]\times\mathcal{Z}} \nabla V(x_{s^-})\cdot \xi(x_{s^-}, z) n(ds, dz) \\
&\qquad = \int_{[0,t]\times\mathcal{Z}} \Big(V\big(x_{s^-} + \xi(x_{s^-}, z)\big) - V(x_{s^-})\Big) \lambda(x_{s^-}, dz) ds \\
&\qquad\qquad - \int_{[0,t]\times\mathcal{Z}} \nabla V(x_{s^-})\cdot \xi(x_{s^-}, z) n(ds, dz) + m_t,
\end{aligned} \tag{65}$$

for some martingale m_t. Therefore, replacing (64) and (65) in (63) and using (11), we conclude that

$$\begin{aligned}
V(x_t) - V(x_0) &= \int_{[0,t]} \nabla V(x_{s^-})\cdot dx_s \\
&\quad + \frac{1}{2}\int_{[0,t]} \sum_{i,j\le d} \sum_{k\le m} \frac{\partial^2 V(x_{s^-})}{\partial x^i \partial x^j} \sigma^{i,k}(x_{s^-})\sigma^{j,k}(x_{s^-})ds \\
&\quad + \int_{[0,t]\times\mathcal{Z}} \Big(V\big(x_{s^-} + \xi(x_{s^-}, z)\big) - V(x_{s^-})\Big) \lambda(x_{s^-}, dz) ds \\
&\quad - \int_{[0,t]\times\mathcal{Z}} \nabla V(x_{s^-})\cdot \xi(x_{s^-}, z) n(ds, dz) + m_t \\
&= \int_{[0,t]} \nabla V(x_{s^-})\cdot dx_s + \int_{[0,t]} \Big(\mathcal{L}V(x_{s^-}) - \nabla V(x_{s^-})\cdot f(x_{s^-})\Big) ds \\
&\quad - \int_{[0,t]\times\mathcal{Z}} \nabla V(x_{s^-})\cdot \xi(x_{s^-}, z) n(ds, dz) + m_t,
\end{aligned}$$

where we used the fact that

$$\sum_{k\le m} \sum_{i,j\le d} \frac{\partial^2 V(y)}{\partial y^i \partial y^j} \sigma^{i,k}(y)\sigma^{j,k}(y) = \mathrm{trace}[\sigma(y)' HV(y)\sigma(y)], \quad \forall y \in \mathbb{R}^{n_x}.$$

We thus conclude that

$$\begin{aligned}
&V(x_t) - V(x_0) - \int_{[0,t]} \mathcal{L}V(x_{s^-}) ds = \\
&\quad = \int_{[0,t]} \nabla V(x_{s^-})\cdot dx_s - \int_{[0,t]} \nabla V(x_{s^-})\cdot f(x_{s^-}) ds \\
&\qquad - \int_{[0,t]} \nabla V(x_{s^-})\cdot \int_{\mathcal{Z}} \xi(x_{s^-}, z) n(ds, dz) + m_t \\
&\quad = \int_{[0,t]} \nabla V(x_{s^-})\cdot \Big(dx_s - f(x_{s^-})ds - \int_{\mathcal{Z}} \xi(x_{s^-}, z) n(ds, dz)\Big) + m_t \\
&\quad = \int_{[0,t]} \nabla V(x_{s^-})\cdot dx_t^c + m_t,
\end{aligned}$$

which is a martingale because x_t^c is a continuous local martingale and $\nabla V(x_{s^-})$ is locally bounded (because x_t is locally bounded) and predictable [4, Corollary I.4.55d, p. 55]. Finally, from the martingale property, we conclude that

$$\mathrm{E}\Big[V(x_t) - V(x_0) - \int_{[0,t]} \mathcal{L}V(x_{s^-}) ds\Big]$$

$$= \mathrm{E}\left[\mathrm{E}\left[V(x_t) - V(x_0) - \int_{[0,t]} \mathcal{L}V(x_{s^-})ds \;\middle|\; \mathcal{F}_0\right]\right] = 0,$$

from which we obtain

$$\mathrm{E}\left[V(x_t) - \int_{[0,t]} \mathcal{L}V(x_{s^-})ds\right] = \mathrm{E}\left[V(x_0)\right] < \infty.$$

When V is uniformly upper-bounded, this must necessarily imply that

$$\mathrm{E}[V(x_t)] = \mathrm{E}\left[V(x_0)\right] + \mathrm{E}\left[\int_{[0,t]} \mathcal{L}V(x_{s^-})ds\right],$$

whether or not the left-most and the right-most expectations are finite. Since $\mathcal{L}V(x_s)$ is cadlag, $\mathcal{L}V(x_{s^-})$ is optional [4, Corollary I.1.25, p. 7] and therefore the integral in the right-hand side can be defined in the sense of [4, Equation I.3.4, p. 28]. The final result then follows from Fubini's Theorem. ■

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