Multi-path Routing for Networked Control Systems

Technical Report

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1 Estimation over a faulty network

Consider the problem of estimating the state of a linear systems

$$x(k+1) = Ax(k) + Bw(k), \quad \forall k \geq 0,$$

at a remote location, based on state measurements transmitted over a faulty network. The input $w(k)$ is assumed to be wide-sense stationary with zero mean and finite covariance matrix. The dynamics of an optimal remote estimator are given by

$$\hat{x}(k+1) = \begin{cases} Ax(k) & \text{successful transmission of } x(k) \\ A\hat{x}(k) & \text{unsuccessful transmission of } x(k), \quad \forall k \geq 0, \end{cases}$$

which lead to the following dynamics for the error $e(k) := x(k) - \hat{x}(k)$

$$\begin{aligned} e(k+1) &= \begin{cases} Bw(k) & \text{successful transmission of } x(k) \\ Ae(k) + Bw(k) & \text{unsuccessful transmission of } x(k) \end{cases} \\ &= \big(1 - \sigma(k)\big)Ae(k) + Bw(k), \quad \forall k \geq 0, \end{aligned} \tag{1}$$

where $\sigma(k) = 1$ if the transmission of $x(k)$ succeeded and $\sigma(k) = 0$ otherwise.

1.1 Single path

Suppose that data is sent over a single link that can fail with probability p and that once the link fails, it remains off for an exponentially distributed amount of time with mean T. The variable

$$\theta(k) := \begin{cases} 0 & \text{link not working at time } k \\ 1 & \text{link working at time } k \end{cases}$$

is a discrete-time Markov chain with finite state $\mathcal{Q} := \{0, 1\}$, transition probability matrix

$$P := \begin{bmatrix} 1 - \frac{1}{T} & \frac{1}{T} \\ p & 1 - p \end{bmatrix},$$

and steady-state distribution π given by

$$\pi P = P \quad \Leftrightarrow \quad \pi = \begin{bmatrix} \frac{pT}{1+pT} & \frac{1}{1+pT} \end{bmatrix}'.$$

The steady-state probability of failure is therefore equal to $\frac{pT}{1+pT}$.

In this scenario, (1) holds with $\sigma(k)=\theta(k)$, $\forall k\geq 0$. From Theorem 1, we conclude that the error is MSS if and only if there exist matrices $P_0, P_1 > 0$ such that[1]

$$\Big(1-\frac{1}{T}\Big)A'P_0A < P_0, \qquad \frac{1}{T}A'P_0A < P_1.$$

Such matrices exist if and only if $\sqrt{\frac{T-1}{T}}A$ is schur, i.e., if

$$|\lambda[A]| < \sqrt{\frac{T}{T-1}} \quad\Leftrightarrow\quad T\big(|\lambda[A]|^2-1\big) < |\lambda[A]|^2 \quad\Leftrightarrow\quad |\lambda[A]|\leq 1 \text{ or } T < \frac{|\lambda[A]|^2}{|\lambda[A]|^2-1}.$$

For $|\lambda[A]| = 1+\epsilon$, with very small ϵ, this leads to

$$T < \frac{(1+\epsilon)^2}{(1+\epsilon)^2-1} \approx \frac{1}{2\epsilon}$$

1.2 Multiple paths

Suppose that the data is sent over either one of two links. Denoting by $\rho(k)\in\{1,2\}$ the link used at time k, we assume that

$$\rho(k) = \begin{cases} 1 & \text{w.p. } 1/2 \\ 2 & \text{w.p. } 1.2, \end{cases}$$

with the $\rho(k)$ i.i.d. As before, each link can fail with probability p and that once a link fails, it remains off for an exponentially distributed amount of time with mean T. The variable

$$\theta(k) := \begin{cases} 0 & \text{none of the links working at time } k \\ 1 & \text{only the link other than } \rho(k) \text{ working at time } k \\ 2 & \text{only the link } \rho(k) \text{ working at time } k \\ 3 & \text{both links working at time } k \end{cases}$$

is a discrete-time Markov chain with finite state $\mathcal{Q} := \{0,1,2,3\}$, transition probability matrix

$$P := \begin{bmatrix} \big(1-\frac{1}{T}\big)^2 & \frac{1}{T}\big(1-\frac{1}{T}\big) & \frac{1}{T}\big(1-\frac{1}{T}\big) & \frac{1}{T^2} \\ \big(1-\frac{1}{T}\big)p & \frac{1}{2}\big(1-\frac{1}{T}\big)(1-p)+\frac{1}{2}\frac{p}{T} & \frac{1}{2}\big(1-\frac{1}{T}\big)(1-p)+\frac{1}{2}\frac{p}{T} & \frac{1-p}{T} \\ \big(1-\frac{1}{T}\big)p & \frac{1}{2}\big(1-\frac{1}{T}\big)(1-p)+\frac{1}{2}\frac{p}{T} & \frac{1}{2}\big(1-\frac{1}{T}\big)(1-p)+\frac{1}{2}\frac{p}{T} & \frac{1-p}{T} \\ p^2 & p(1-p) & p(1-p) & (1-p)^2 \end{bmatrix}$$

and steady-state distribution π given by

$$\pi P = P \quad\Leftrightarrow\quad \pi = \begin{bmatrix} \frac{p^2T^2}{(1+pT)^2} & \frac{pT}{(1+pT)^2} & \frac{pT}{(1+pT)^2} & \frac{1}{(1+pT)^2} \end{bmatrix}'.$$

[1] The other equivalent conditions would be

$$\Big(1-\frac{1}{T}\Big)A'P_0A < P_0, \qquad pA'P_0A < P_1,$$

$$\Big(1-\frac{1}{T}\Big)A'P_0A + \frac{1}{T}A'P_1A < P_0,$$

$$\Big(1-\frac{1}{T}\Big)A'P_0A + pA'P_1A < P_0,$$

The steady-state probability of failure is therefore equal to $\frac{p^2T^2}{(1+pT)^2} + \frac{pT}{(1+pT)^2} = \frac{pT}{1+pT}$, which is exactly the same as in the single-path case.

In this scenario, (1) holds with

$$\sigma(k) = \begin{cases} 0 & \theta(k) \in \{0,1\} \\ 1 & \theta(k) \in \{2,3\} \end{cases} \qquad \forall k \geq 0.$$

From Theorem 1, we conclude that the error is MSS if and only if there exist matrices $P_0, P_1, P_2, P_3 > 0$ such that

$$\begin{aligned} &p_{00}A'P_0A + p_{10}A'P_1A < P_0, && p_{01}A'P_0A + p_{11}A'P_1A < P_1, \\ &p_{02}A'P_0A + p_{12}A'P_1A < P_2, && p_{03}A'P_0A + p_{13}A'P_1A < P_3. \end{aligned}$$

Since no "active" constraints are posed on P_2 and P_3, it suffices to find P_0 and P_1 such that

$$\begin{aligned} &\big(1-\frac{1}{T}\big)^2 A'P_0A + \big(1-\frac{1}{T}\big)pA'P_1A < P_0, \\ &\frac{1}{T}\big(1-\frac{1}{T}\big)A'P_0A + \frac{1}{2}\Big(\big(1-\frac{1}{T}\big)(1-p) + \frac{p}{T}\Big)A'P_1A < P_1. \end{aligned}$$

When $A = a$ is a scalar, $T > 1$ and, without loss of generality, taking $P_0 = 1$, $P_1 = p_1 > 0$ we obtain

$$\begin{aligned} &\big(1-\frac{1}{T}\big)^2 a^2 + \big(1-\frac{1}{T}\big)pa^2p_1 < 1 \\ &\frac{1}{T}\big(1-\frac{1}{T}\big)a^2 + \frac{1}{2}\Big(\big(1-\frac{1}{T}\big)(1-p) + \frac{p}{T}\Big)a^2p_1 < p_1, \end{aligned}$$

which is equivalent to

$$p_1 < \frac{1-\big(1-\frac{1}{T}\big)^2a^2}{\big(1-\frac{1}{T}\big)pa^2}, \qquad \Big(1-\frac{1}{2}\Big(\big(1-\frac{1}{T}\big)(1-p)+\frac{p}{T}\Big)a^2\Big)p_1 > \frac{1}{T}\big(1-\frac{1}{T}\big)a^2,$$

Therefore we need

$$\begin{aligned} &\big(1-\frac{1}{T}\big)^2a^2 < 1, \\ &\frac{1}{2}\Big(\big(1-\frac{1}{T}\big)(1-p)+\frac{p}{T}\Big)a^2 < 1, \\ &\frac{1-\big(1-\frac{1}{T}\big)^2a^2}{\big(1-\frac{1}{T}\big)pa^2} > \frac{\frac{1}{T}\big(1-\frac{1}{T}\big)a^2}{1-\frac{1}{2}\Big(\big(1-\frac{1}{T}\big)(1-p)+\frac{p}{T}\Big)a^2} \end{aligned}$$

or equivalently

$$a < \frac{T}{T-1}, \qquad a^2 < \frac{2T}{(T-1)(1-p)+p}, \qquad \frac{1-\big(\frac{T-1}{T}\big)^2a^2}{\frac{T-1}{T}pa^2} > \frac{\frac{1}{T}\frac{T-1}{T}a^2}{1-\frac{(T-1)(1-p)+p}{2T}a^2}.$$

For very small p, we basically need

$$a < \frac{T}{T-1}, \quad a^2 < \frac{2T}{T-1} \quad \Leftrightarrow \quad T < \frac{a}{a-1}, \quad -a^2 < (2-a^2)T.$$

For $a = 1+\epsilon$, with very small ϵ, this leads to

$$T < \frac{1+\epsilon}{\epsilon} \approx \frac{1}{\epsilon}.$$

A Appendix

A.1 Markov Jump Linear Systems

Consider the following linear system

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}w(k), \qquad \forall k \geq 0, \tag{2}$$

with state $x(k) \in \mathbb{R}^n$ and a wide sense stationary input disturbance $w(k) \in \mathbb{R}^m$ with finite first and second moments, $\mathrm{E}[w(k)] = \mu_w \in \mathbb{R}^m$ and $\mathrm{E}[w(k)w(k)'] = R_w \in \mathbb{R}^{m\times m}$, respectively. The sequence $\theta(k)$ is the state of a discrete-time Markov chain with a finite state-space $\mathcal{Q}$ with N elements and transition probability matrix $P := [p_{ij}]_{N\times N}$, where p_{ij}, $i, j \in \mathcal{Q}$ denotes the probability of transition from state i to state j.

The system (2) is said to be *mean-square stable (MSS)* if for every initial condition $x(0) = x_0 \in \mathbb{R}^n$, every initial distribution for $\theta(0)$, and every wide sense stationary input disturbance $w(k) \in \mathbb{R}^m$ with finite first and second moments, we have that

$$\mathrm{E}[x(k)] \to q, \qquad\qquad \mathrm{E}[x(k)x(k)'] \to Q,$$

for some vector $q \in \mathbb{R}^n$ and matrix $Q \in \mathbb{R}^{n\times n}$.

Theorem 1 (MSS stability [1]). *The system* (2) *is MSS if and only if there exist matrices* $P_i > 0$*,* $i \in \mathcal{Q}$ *such that*

$$\sum_{i\in\mathcal{Q}} p_{ij}A_i'P_iA_i < P_j, \quad \forall j \in \mathcal{Q}.$$

The theorem still holds if the above LMI is replaced by any one of the following three alternatives:

$$\sum_{j\in\mathcal{Q}} p_{ij}A_j'P_jA_j < P_i, \quad \forall i \in \mathcal{Q},$$

$$A_i'\Big(\sum_{j\in\mathcal{Q}} p_{ij}P_j\Big)A_i < P_i, \quad \forall i \in \mathcal{Q},$$

$$A_j'\Big(\sum_{i\in\mathcal{Q}} p_{ij}P_i\Big)A_j < P_j, \quad \forall j \in \mathcal{Q}. \qquad \square$$

References

[1] O. L. V. Costa and M. D. Fragoso. Stability results for discrete-time linear systems with Markovian jumping parameters. *J. Mathematical Anal. and Applications*, 179:154–178, 1993.