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Switched Systems: Mixing Logic with Differential Equations

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slides will be made available at `http://www.ece.ucsb.edu/~hespanha`

Outline

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Logic-based switched systems framework

application areas:

- Congestion control in data networks
- Vision-based control
- Adaptive control

analytical tools:

- Impact maps
- Lyapunov tools
- Interconnection of systems

Logic-based switched systems

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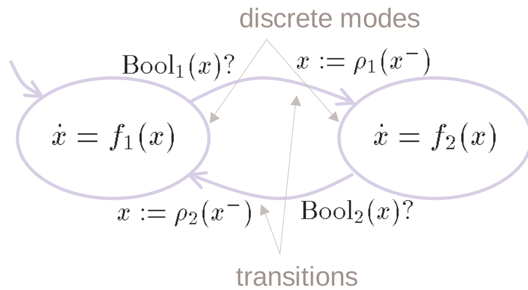
hybrid automaton representation



Logic-based switched systems

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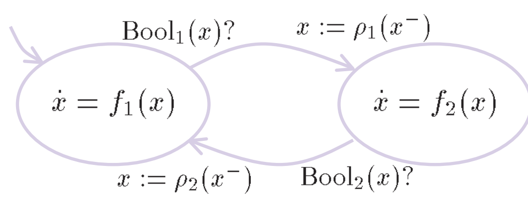
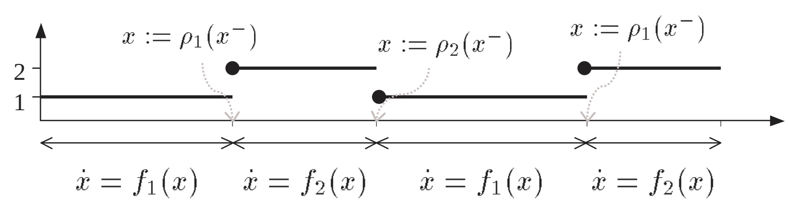
hybrid automaton representation





Logic-based switched systems

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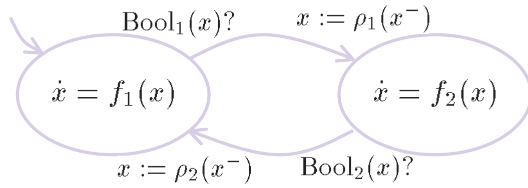

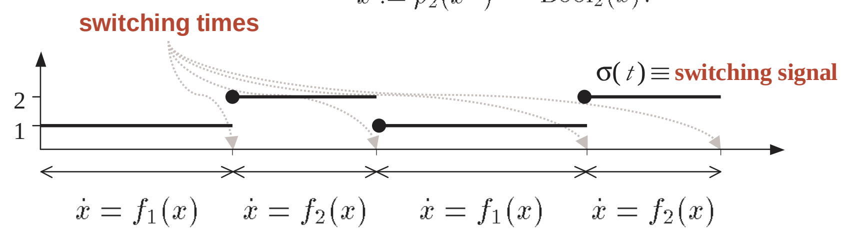

dynamical system representation

$$\begin{cases} \dot{x} = f_\sigma(x) & \text{differential equation} \\ (\sigma, x) = \phi(\sigma^-, x^-) & \text{discrete transition} \end{cases}$$

$$\phi(s, z) = \begin{cases} (2, \rho_1(z)) & s = 1,\ \text{Bool}_1(z) \\ (1, \rho_2(z)) & s = 2,\ \text{Bool}_2(z) \\ s & \text{otherwise} \end{cases}$$

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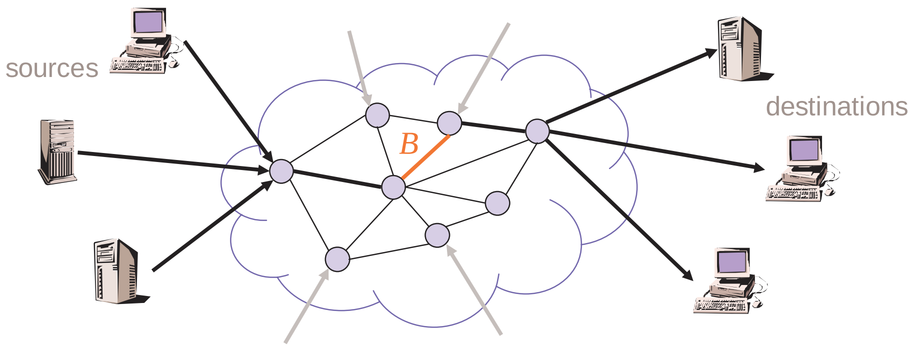
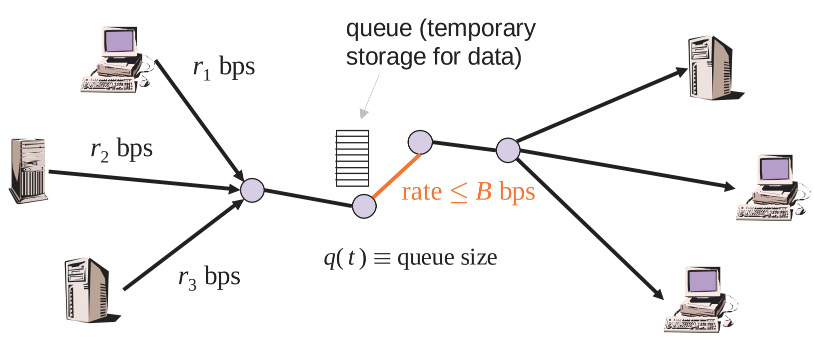
Congestion control in data networks



Congestion control problem:
How to adjust the sending rates of the data sources to make sure that the bandwidth B of the bottleneck link is not exceeded?

B is unknown to the data sources and possibly time-varying

Congestion control in data networks



When $\sum_i r_i$ exceeds B the queue fills and data is lost (drops)

$$\dot{q} = \begin{cases} \sum_i r_i - B & 0 \leq q \leq q_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$q = q_{\max}, \quad \sum_i r_i > B \quad \Rightarrow$ **drop** (discrete event)

Event-based control:
The sources adjust their rates based on the detection of drops

Window-based rate adjustment

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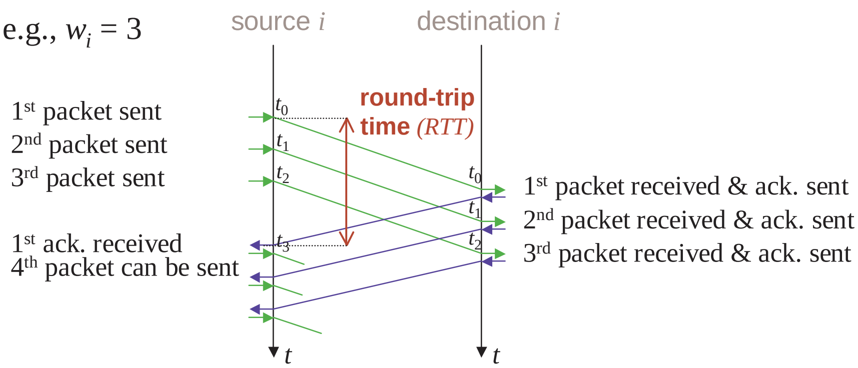
w_i (window size) $\equiv$ number of packets that can remain unacknowledged for by the destination

e.g., $w_i = 3$



w_i effectively determines the sending rate r_i:

$$r_i(t) = \frac{w_i(t)}{RTT(t)}$$

round-trip time

Window-based rate adjustment

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w_i (window size) $\equiv$ number of packets that can remain unacknowledged for by the destination

$$r_i(t) = \frac{w_i(t)}{RTT(t)} \quad \equiv \text{sending rate}$$

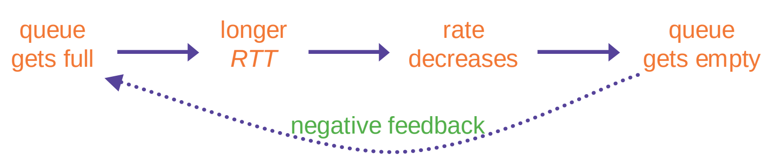
$$RTT(t) = T_{\mathrm{P}} + \frac{1}{B} q(t)$$

total round-trip time

propagation delay

per-packet transmission time

time in queue until transmission



This mechanism is still not sufficient to prevent a catastrophic collapse of the network if the sources set the w_i too large

TCP Reno congestion control

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1. While there are no drops, increase w_i by 1 on each RTT
2. When a drop occurs, divide w_i by 2

(congestion controller constantly probe the network for more bandwidth)

Network/queue dynamics

$$\dot{q} = \begin{cases} \sum_i r_i - B & 0 \le q \le q_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$RTT = T_p + \frac{q}{B}$$

$$q = q_{\max}$$

$$\Downarrow$$

drop occurs

Reno controllers

$$\dot{w}_i = \frac{1}{RTT}$$

$$r_i = \frac{w_i}{RTT}$$

drop detected

(one RTT after occurred)

$$\Downarrow$$

$$w_i \to \frac{w_i}{2}$$

disclaimer: this is a simplified version of Reno that ignores several interesting phenomena…

Switched system model for TCP

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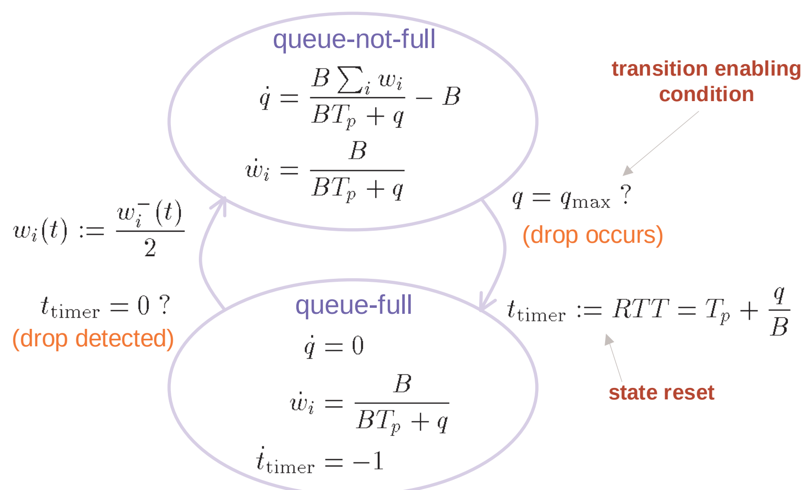

Switched system model for TCP

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queue-not-full ($\sigma = 1$)

$$\dot{q} = \frac{B \sum_i w_i}{BT_p + q} - B$$

$$\dot{w}_i = \frac{B}{BT_p + q}$$

$q = q_{\max}$? (drop occurs)

$t_{\text{timer}} := RTT = T_p + \frac{q}{B}$

queue-full ($\sigma = 2$)

$$\dot{q} = 0$$

$$\dot{w}_i = \frac{B}{BT_p + q}$$

$$\dot{t}_{\text{timer}} = -1$$

$t_{\text{timer}} = 0$? (drop detected)

$w_i(t) := \frac{w_i^-(t)}{2}$

alternatively…

$x := [q \; w_1 \; w_2 \; \cdots \; w_n \; t_{\text{timer}}]'$

$\sigma \in \{1, 2\}$

$$\begin{cases} \dot{x} = f_\sigma(x) & \text{continuous dynamics} \\ (\sigma, x) = \phi(\sigma^-, x^-) & \text{discrete dynamics} \end{cases}$$

Impact maps

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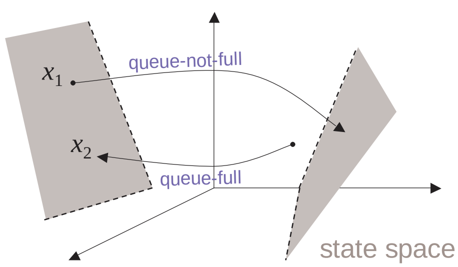
$x_1 \xrightarrow{T} x_2$

$t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6$

t_0–t_1	t_1–t_2	t_2–t_3	t_3–t_4	t_4–t_5	t_5–t_6
queue-not-full	queue full	queue-not-full	queue full	queue-not-full	queue full
$\dot{x} = f_1(x)$	$\dot{x} = f_2(x)$	$\dot{x} = f_1(x)$	$\dot{x} = f_2(x)$	$\dot{x} = f_1(x)$	$\dot{x} = f_2(x)$

$x_k := x(t_{2k})$ $\equiv$ k^{th} time the system enters the queue-not-full mode



$x_{k+1} = T(x_k)$

impact map

Impact maps

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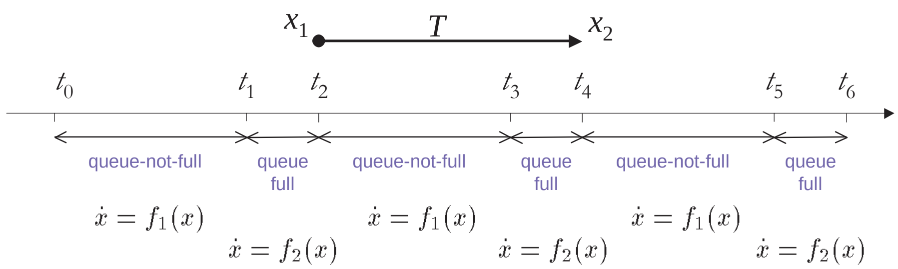

$x_k := x(t_{2k})$ $\equiv k^{\text{th}}$ time the system enters the queue-not-full mode

Theorem [1]**:** The function T is a contraction. In particular,

$$||T(a) - T(b)||_* \leq \tfrac{1}{2}||a - b||_*, \quad \forall a, b$$

Therefore

- $x_k \to x_\infty$ as $k \to \infty$ $\qquad x_\infty \equiv$ constant
- $x(t) \to x_\infty(t)$ as $t \to \infty$ $\qquad x_\infty(t) \equiv$ periodic limit cycle

NS-2 simulation results

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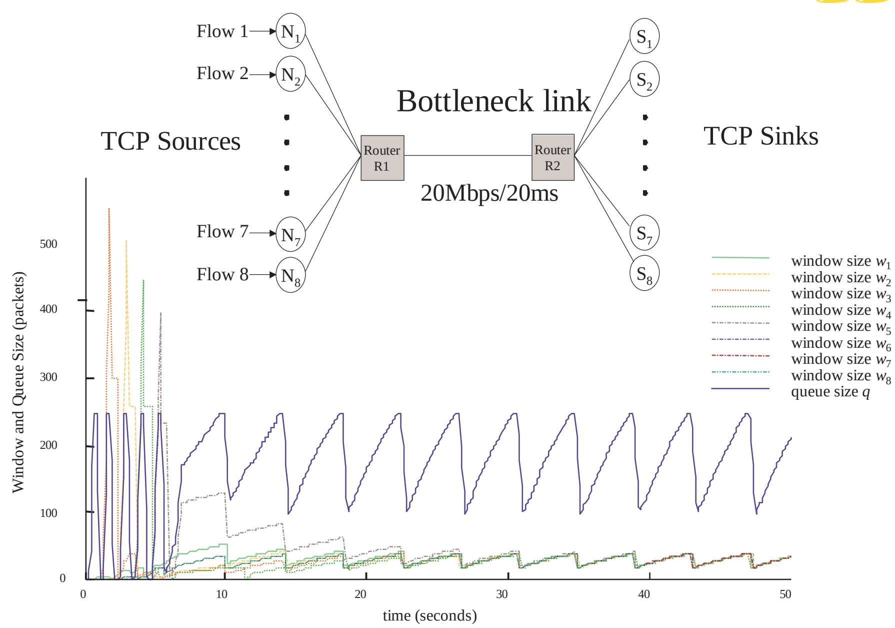

Random early detection (RED)

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Performance could be improved if the congestion controllers were notified of congestion ***before*** *a drop occured*

queue-not-full

$$\dot{q} = \frac{B \sum_i w_i}{BT_p + q} - B$$

$$\dot{w}_i = \frac{B}{BT_p + q}$$

$\mathbf{N}_i > \mathbf{N}_i^-$?

(notification of congestion)

$$w_i(t) := \frac{w_i^-(t)}{2}$$

$\mathbf{N}_i \equiv$ notification counter (incremented whenever a notification of congestion arrives)

In RED, **N** is a *random variable* with

function to be adjusted

$$\lim_{dt \downarrow 0} \frac{\mathrm{P}\left(\mathbf{N}_i(t) - \mathbf{N}_i(t - dt) = n\right)}{dt} = \begin{cases} \frac{w_i}{RTT}\gamma(q) & n = 1 \\ 0 & n > 1 \end{cases}$$

Stochastic switched system

Impact maps

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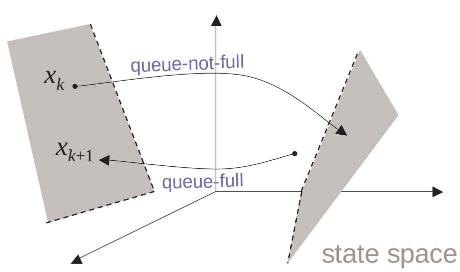

$$x_{k+1} = T(x_k)$$

impact map

Impact maps are difficult to compute because their computation requires:

Solving the differential equations on each mode (in general only possible for linear dynamics)

Intersecting the continuous trajectories with a surface (often transcendental equations)

It is often possible to prove that T is a contraction without an explicit formula for T…

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Vision-based control of a flexible manipulator

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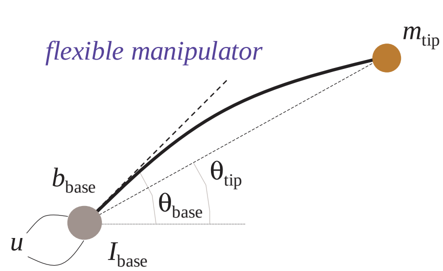

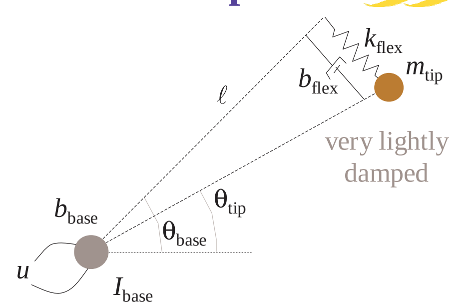

4th dimensional small-bending approximation

$$m_{\text{tip}}\ell^2\ddot\theta_{\text{tip}} = \ell k_{\text{flex}}(\theta_{\text{base}} - \theta_{\text{tip}}) + \ell b_{\text{flex}}(\dot\theta_{\text{base}} - \dot\theta_{\text{tip}})$$

$$I_{\text{base}}\ddot\theta_{\text{base}} = -b_{\text{base}}\dot\theta_{\text{base}} + \ell k_{\text{flex}}(\theta_{\text{tip}} - \theta_{\text{base}}) + \ell b_{\text{flex}}(\dot\theta_{\text{tip}} - \dot\theta_{\text{base}}) + k_{\text{motor}}u$$

Control objective: drive θ_{tip} to zero, using feedback from

$\theta_{base} \rightarrow$ encoder at the base
$\theta_{tip} \rightarrow$ machine vision (essential to increase the damping of the flexible modes in the presence of noise)

Vision-based control of a flexible manipulator

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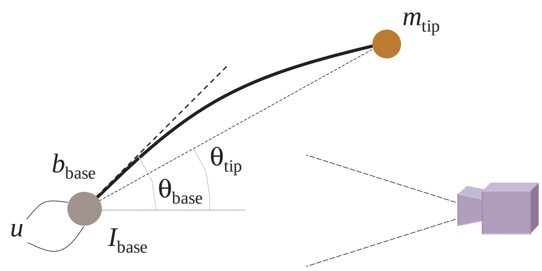

To achieve high accuracy in the measurement of q_{tip} the camera must have a ***small field of view***

feedback output: $$y := \begin{cases} \begin{bmatrix} \theta_{base} & \theta_{tip} \end{bmatrix}' & |\theta_{tip}| \leq \theta_{max} \\ \theta_{base} & |\theta_{tip}| > \theta_{max} \end{cases}$$

Control objective: drive θ_{tip} to zero, using feedback from

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Switched process

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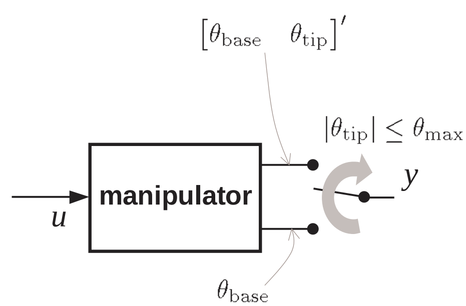
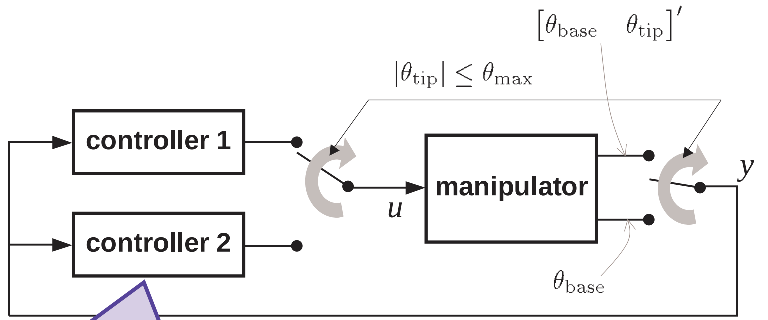


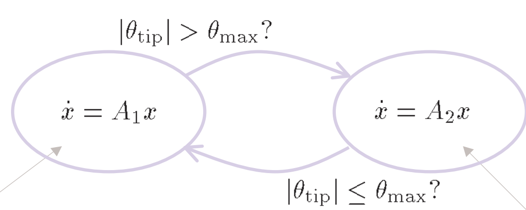
Switched process



controller 1 optimized for feedback from θ_{base} and θ_{tip}
and
controller 2 optimized for feedback only from θ_{base}

E.g., LQG controllers that minimize $\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \theta_{\text{tip}}^2 + \dot{\theta}_{\text{tip}}^2 + \rho u^2 dt\right]$

Switched system



feedback connection
with controller 1
(θ_{base} and θ_{tip} available)

feedback connection
with controller 2
(only θ_{base} available)

How does one check if the overall system is ***stable*** and that

$$x := [\theta_{\text{tip}} \quad \dot{\theta}_{\text{tip}} \quad \theta_{\text{base}} \quad \dot{\theta}_{\text{base}} \quad x_C']'$$

controller's state

eventually ***converges to zero*** ?

Common Lyapunov functions

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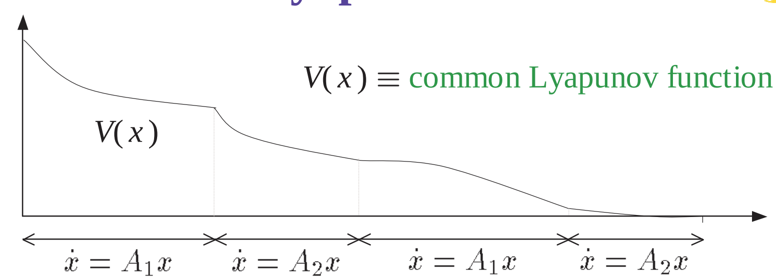

Suppose that there exists a cont. diff. function $V(x)$ such that

$V(x)$ provides a measure of the size of x:

- positive definite
- radially unbounded

$V(x)$ decreases along trajectories of both systems:

$$\frac{\partial V}{\partial x} A_i x < 0 \qquad i \in \{1,2\}, x \neq 0$$

switched system is stable and $x \to 0$ as $t \to \infty$
independently of how switching takes place

Common Lyapunov functions

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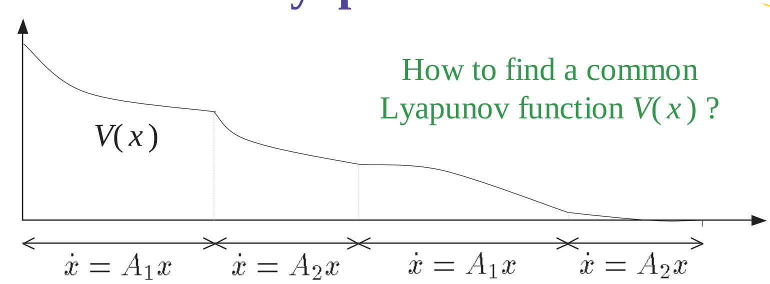

Algebraic conditions for the existence of a common Lyapunov function

- The matrices commute, i.e., $A_1 A_2 = A_2 A_1$ [S1,S2]
- The Lie Algebra generated by $\{A_1, A_2\}$ is solvable
- For all $\lambda \in [0,1]$ the matrices $\lambda A_1 + (1-\lambda) A_2$ and $\lambda A_1 + (1-\lambda) A_2^{-1}$ are asymptotically stable (only for 2×2 matrices)

But, all these conditions fail for the problem at hand …

Multiple Lyapunov functions

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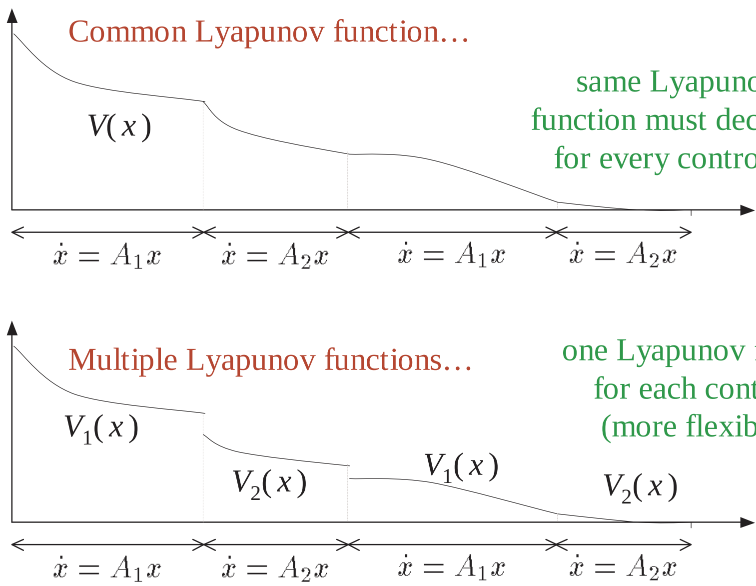

same Lyapunov function must decrease for every controller

one Lyapunov function for each controller (more flexibility)

[2,3, etc.]

Multiple Lyapunov functions

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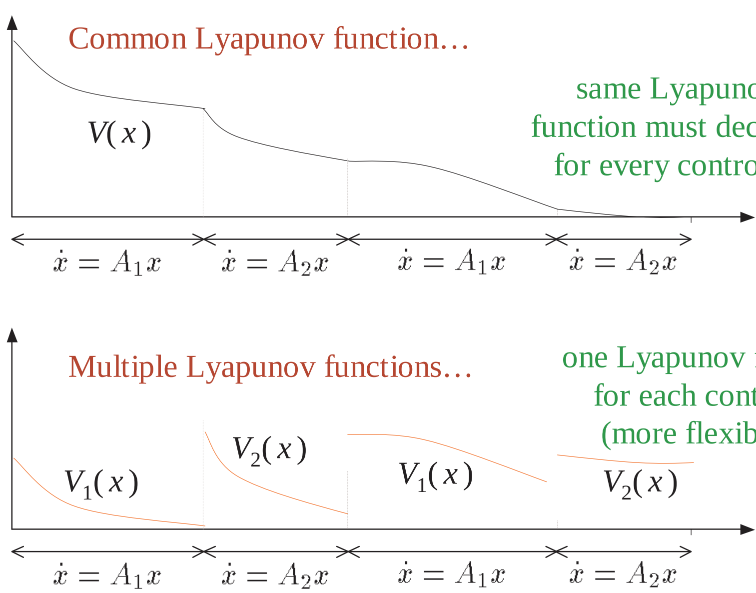

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Multiple Lyapunov functions

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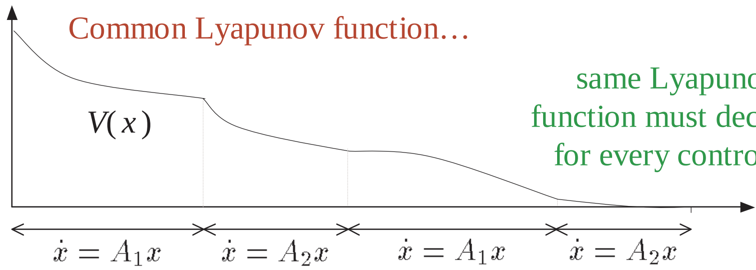

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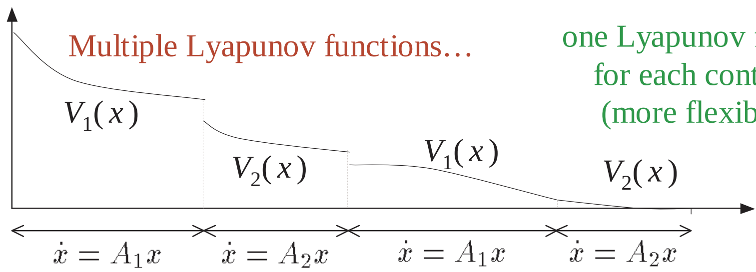

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Multiple Lyapunov functions

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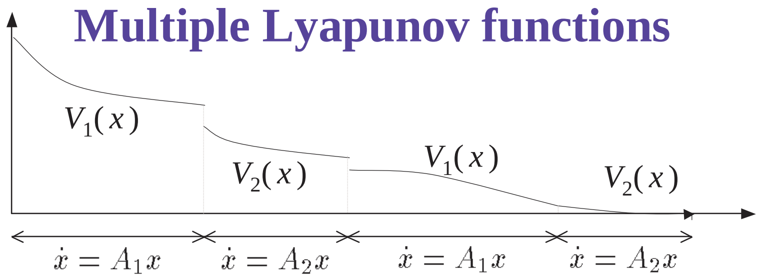

Suppose that exist positive definite, radially unbounded cont. diff. functions $V_1(x)$, $V_2(x)$ such that

$V_i(x)$ decreases along trajectories of A_i:

$$\frac{\partial V_i}{\partial x} A_i x < 0 \quad i \in \{1,2\}$$

$V_i(x)$ does not increase during transitions:

$$V_i(z) \geq V_j(z)$$

at points z where a switching from i to j can occur

switched system is stable and $x \to 0$ as $t \to \infty$

LaSalle-like versions of this results that only require $\frac{\partial V_i}{\partial x} A_i x \leq 0$, $i \in \{1,2\}$ are also available [4]

Multiple Lyapunov functions

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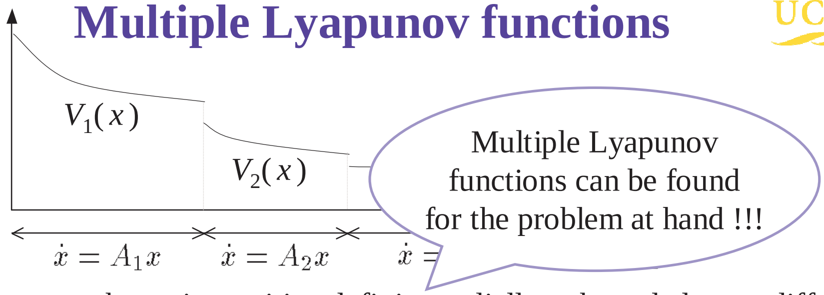

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$V_i(x)$ does not increase during transitions:

$$V_i(z) \geq V_j(z)$$

at points z where a switching from i to j can occur

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[S2]

Closed-loop response

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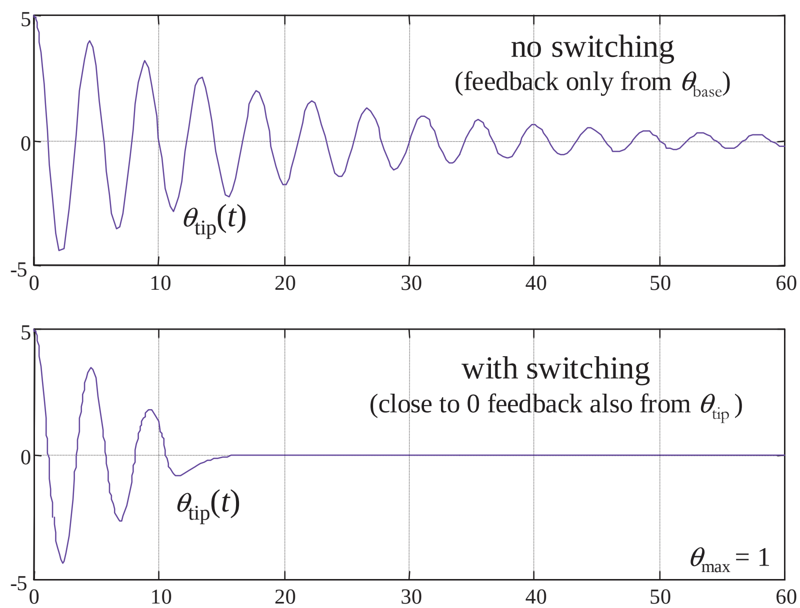

[S2]

Robustness

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When will a small perturbation in the dynamics result in a small perturbation in the switched system's trajectory?

For purely continuous systems (difference or differential equations) Lyapunov stability automatically provides some degree of robustness

This is not necessarily true for switched systems:
When is this a problem?
Is there a notion of stability that automatically provides robustness?

Important for …
1. numerical ***simulation*** of switched systems
2. digital ***implementation*** of switched controllers
3. ***analysis*** and ***design*** based on numerical methods

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Prototype adaptive control problem

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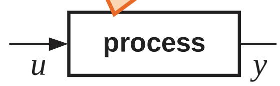
process can either be:

$\mathbf{P}_1: \quad \dot{x} = A_1 x + b_1 u, \quad y = c_1 x$

or

$\mathbf{P}_2: \quad \dot{x} = A_2 x + b_2 u, \quad y = c_2 x$



Control objective: Stabilize process (keep state of the process bounded)

Prototype adaptive control problem

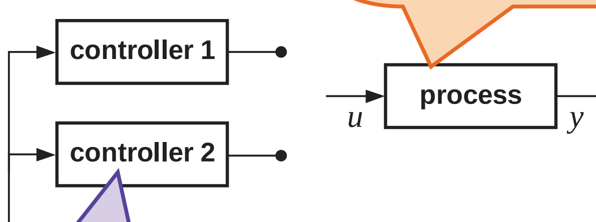
UCSB

process can either be:

$\mathbf{P}_1: \quad \dot{x} = A_1 x + b_1 u, \quad y = c_1 x$

or

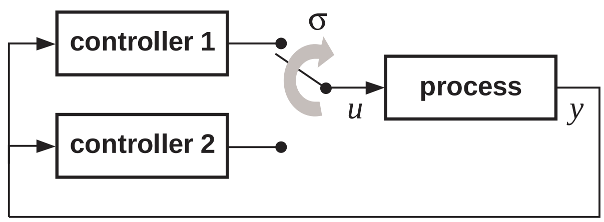
$\mathbf{P}_2: \quad \dot{x} = A_2 x + b_2 u, \quad y = c_2 x$



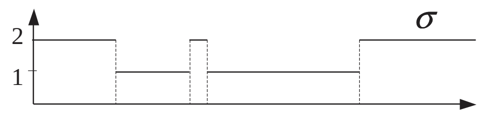
controller 1 stabilizes $\mathbf{P}_1$
and
controller 2 stabilizes $\mathbf{P}_2$

Prototype adaptive control problem

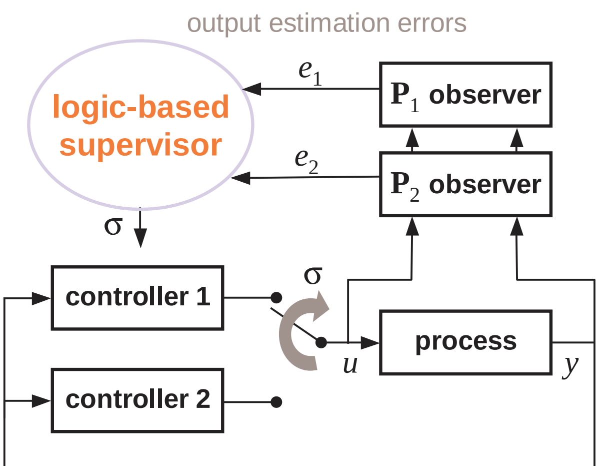
How to choose online which controller to use?



σ ≡ ***switching signal*** taking values on the set {1,2}



Estimator-based architecture



e_1 small ➡ "likelihood" of process being $\mathbf{P}_1$ is high ➡ should use controller 1

Certainty equivalence inspired

Estimator-based architecture

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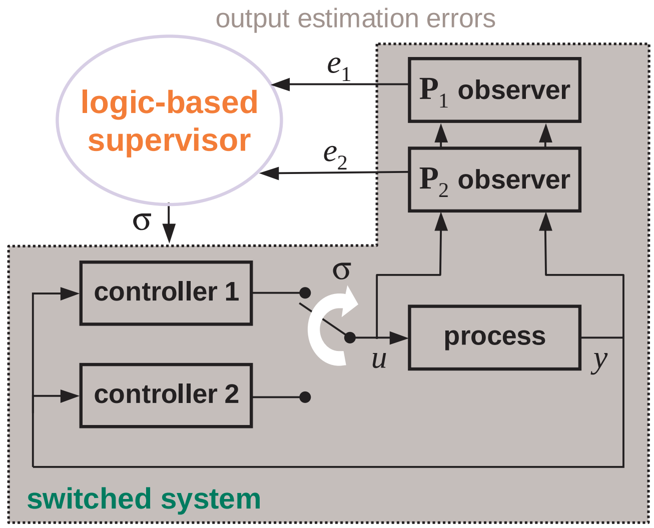

$$\dot{x} = A_\sigma x \qquad e_p = C_p x \quad p \in \{1,2\}$$

Scale-independent hysteresis switching

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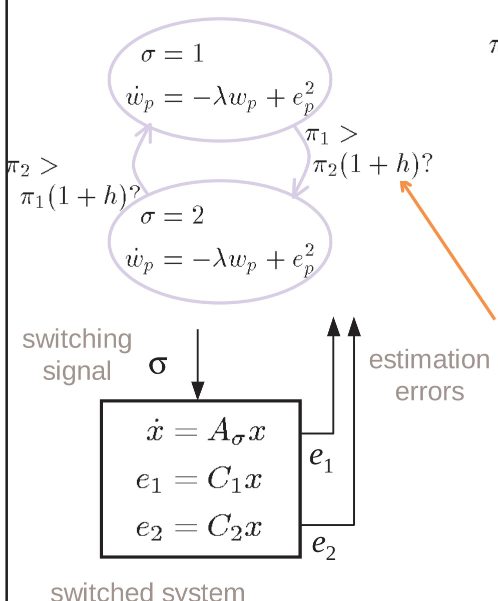

$$\pi_p := w_p + \varepsilon = \int_0^t e^{-\lambda(t-\tau)} e_p^2 d\tau + \varepsilon$$

$$\varepsilon > 0,\ \lambda > 0,\ p \in \{1,2\}$$

performance signals
(measure the "size" of the estimation errors)

hysteresis constant (positive)

Scale-independent hysteresis switching

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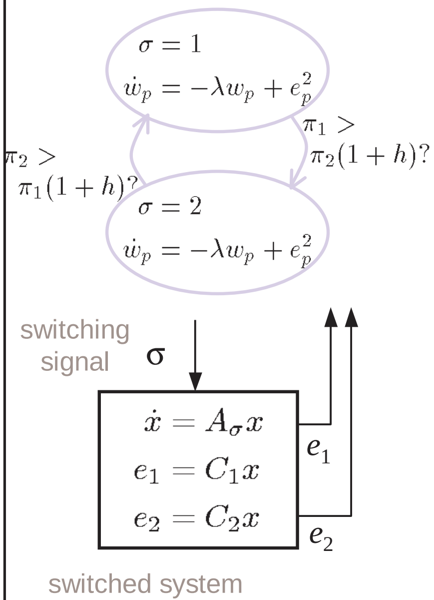

How does one verify if x remains bounded along solutions to the hybrid system?

Analyzing the system as a whole is too difficult. We need to:

1. ***abstract*** the complex behavior of each subsystem (supervisor & switched) to a small set of properties
2. ***infer*** properties of the overall system from the properties of the interconnected subsystems

One-diagram analysis outline

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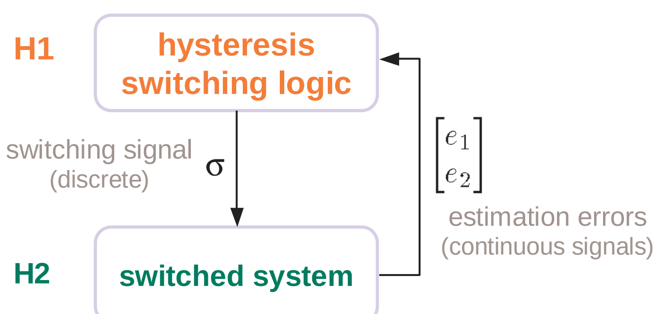

One can show [S3] that…

H1 has the property that

$$\int_0^t e^{\lambda\tau} e_\sigma^2 d\tau \leq \gamma \Big(\min_p \int_0^t e^{\lambda\tau} e_p^2 d\tau \Big)$$

finite "L_2-induced gain" from smallest error to the switched error

H2 has the property that

$$\min_p \int_0^t e^{\lambda\tau} e_p^2 d\tau \leq c_0 + \delta \int_0^t e^{\lambda\tau} e_\sigma^2 d\tau$$

vice-versa ("detectability" through e_σ)

(with δ = 0 when there is no unmodeled dynamics)

One-diagram analysis outline

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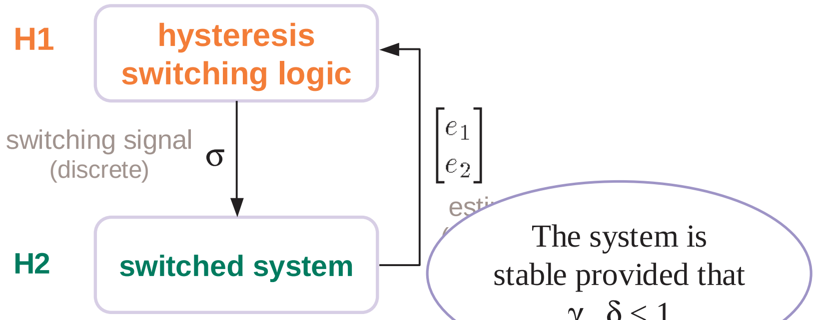

One can show [S3] that…

H1 has the property that

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(with $\delta = 0$ when there is no unmodeled dynamics)

vice-versa ("detectability" through e_s)

Interconnections of switched system

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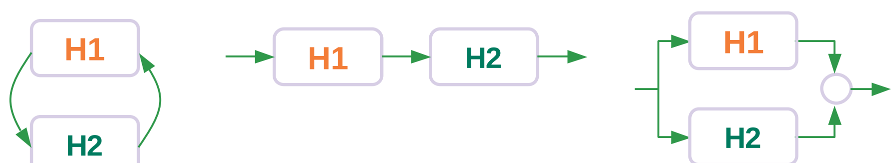
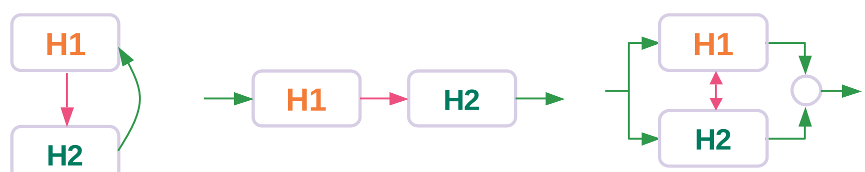


For interconnections through continuous signals, existing tools can be extended to hybrid systems (small gain, passivity, integral quadratic constrains, ISS, etc.) [6,7]



For mixed interconnections new tools need to be developed…

→ continuous signal → discrete signal

Conclusion

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application areas

- Congestion control in data networks
- Vision-based control
- Adaptive control

analytical tools

- Impact maps
- Lyapunov tools
- Interconnection of systems

Switched systems are ubiquitous and of significant practical application

A unified theory of switched systems is barely starting to become available

References

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Background surveys & tutorials:

[S1] D. Liberzon, A. S. Morse, Basic problems in stability and design of switched systems, In *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59-70, Oct. 1999.

[S2] J. Hespanha. Chapter Stabilization Through Hybrid Control. In *Encyclopedia of Life Support Systems*, 2002. To appear.

[S3] J. Hespanha. Tutorial on Supervisory Control. Lecture Notes for the workshop *Control using Logic and Switching* for the 40th Conf. on Decision and Contr., Orlando, Florida, Dec. 2001.

Papers referenced specifically in this talk:

[1] S. Bohacek, J. Hespanha, J. Lee, K. Obraczka. Analysis of a TCP hybrid model. In *Proc. of the 39th Annual Allerton Conference on Communication, Control, and Computing*, Oct. 2001.

[2] P. Peleties P., R. A. DeCarlo. Asymptotic stability of *m*-switched systems using Lyapunov-like functions. In *Proc. of the 1991 Amer. Contr. Conf.*, pp. 1679–1684, 1991.

[3] M. S. Branicky. *Studies in Hybrid Systems: Modeling, Analysis, and Control*. Ph.D. thesis, MIT, Cambridge, MA, 1995.

[4] J. Hespanha. Extending LaSalle's Invariance Principle to Linear Switched Systems. In *Proc. of the 40th Conf. on Decision and Contr.*, Dec. 2001.

[5] J. Hespanha, A. S. Morse. Certainty Equivalence Implies Detectability. *Syst. & Contr. Lett.*, 36(1):1-13, Jan. 1999.

[6] M. Zefran, F. Bullo, M. Stein, "A notion of passivity for hybrid systems." In *Proc. of the 40th Conf. on Decision and Contr.*, Dec. 2001.

[7] J. Hespanha. Root-Mean-Square Gains of Switched Linear Systems. To appear in *Trans. of Autom. Contr.*

See also http://www.ece.ucsb.edu/~hespanha/published.html