# Networked Control System Protocols
# Modeling & Analysis
# using Stochastic Impulsive Systems

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engineering

# Talk outline

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**Examples**

- feedback over shared communication network
- estimation using remote sensor

**Analysis tools**

- Stochastic Hybrid Systems driven by renewal processes
- Lyapunov-based analysis of Stochastic Hybrid Systems

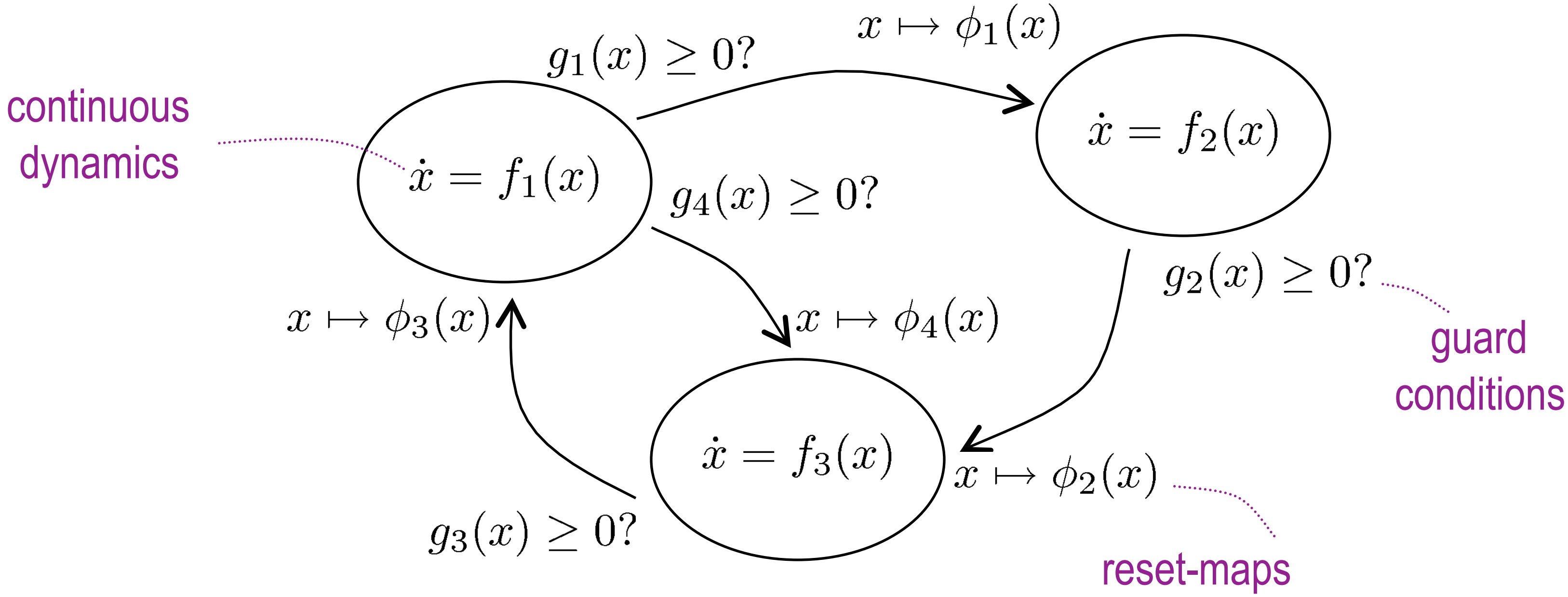
*(ex) students:* D. Antunes (IST), Y. Xu (Advertising.com)

*collaborators:* C. Silvestre (IST)

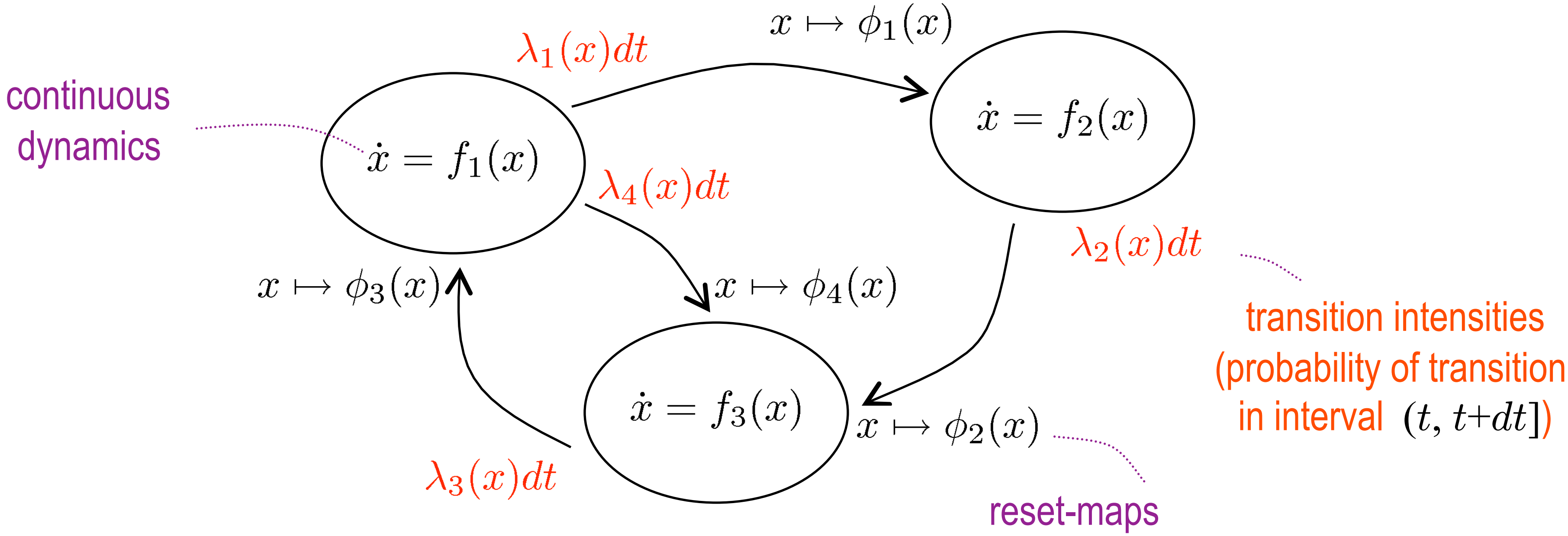
*acknowledgements:* NSF, AFOSR (STTR program)

*disclaimer:* This is an overview, technical details in papers referenced in bottom right corner… http://www.ece.ucsb.edu/~hespanha

# Deterministic Hybrid Systems



# Stochastic Hybrid Systems



# Stochastic Hybrid Systems

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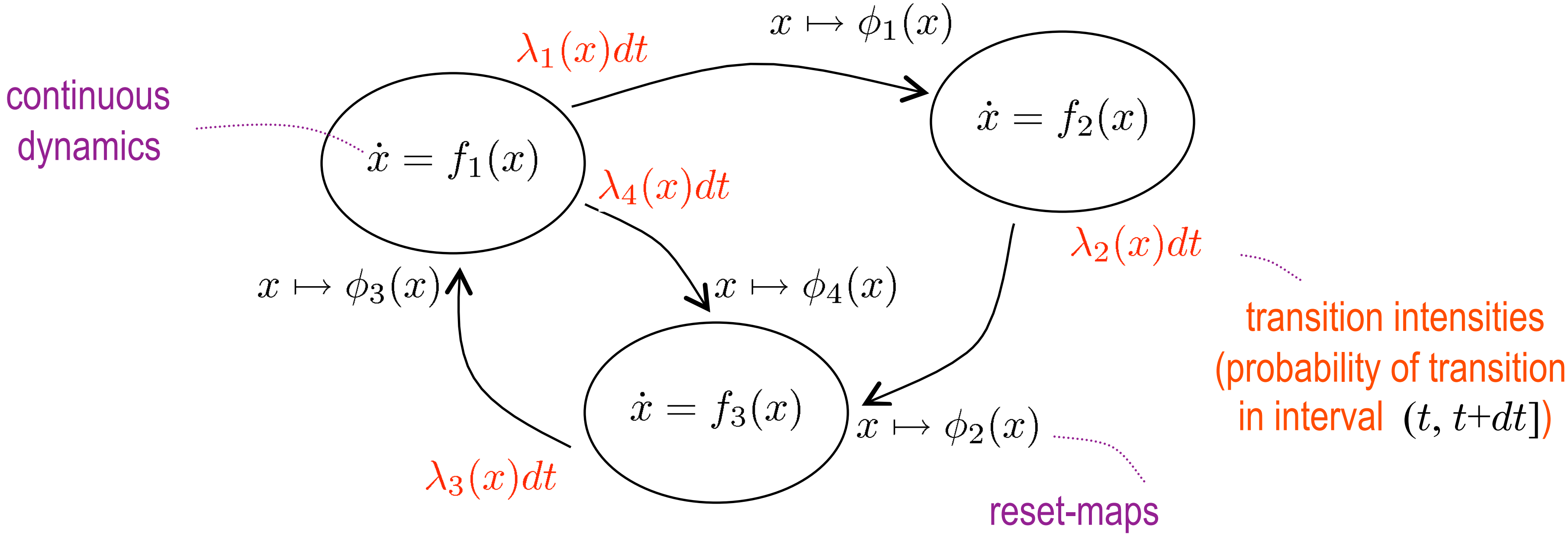
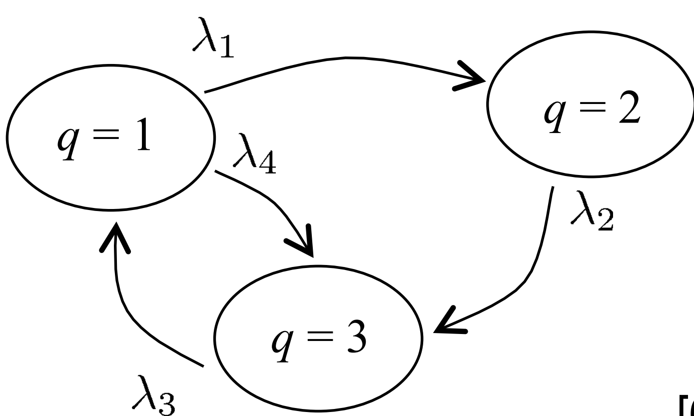



Special case: When all $\lambda_\ell$ are constant

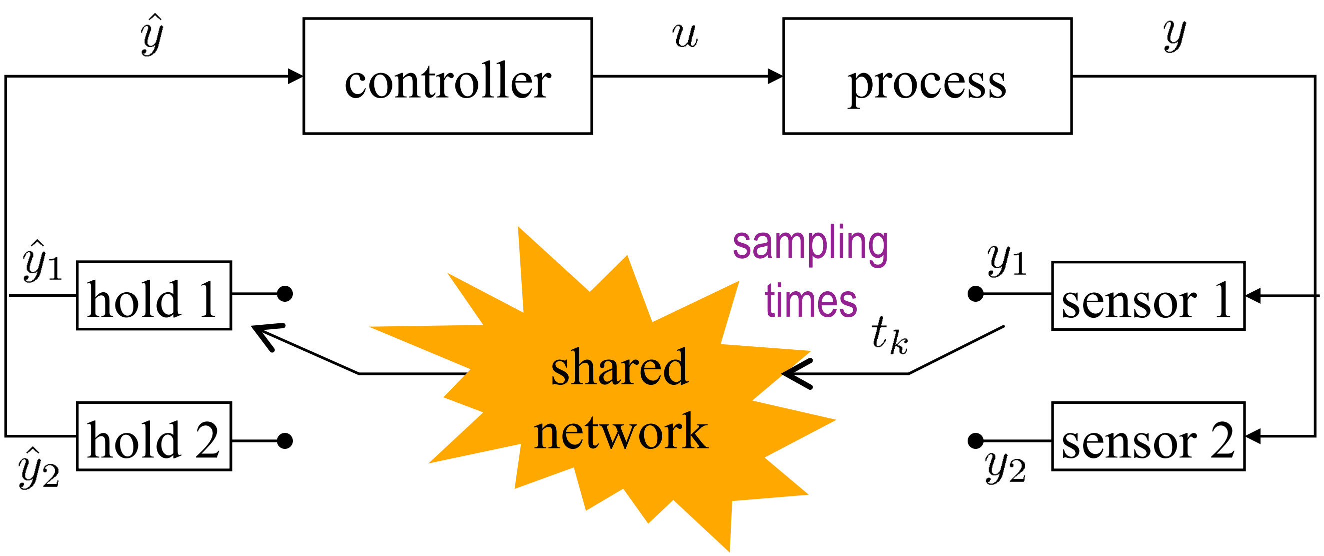
$\Rightarrow$ $x(t)$ is a Markov process &
times between jumps are exponentially distributed




closely related to the so called
*Markovian Jump Systems*
[Costa, Fragoso, Boukas, Loparo, Lee, Dullerud]

# Example I: Networked Control System






process: $\dot{x}_P = A_P x_P + C_P u$

$y = C_P x_P + D_P u$

controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$

$\hat{y} = C_C x_C + D_C \hat{y}$

round-robin network access:

$$\dot{\hat{y}} = 0$$

hold

$$\hat{y}(t_k) = \begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{cases} \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix} & k \text{ odd} \\ \begin{bmatrix} \hat{y}_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix} & k \text{ even} \end{cases}$$

# Example I: Networked Control System

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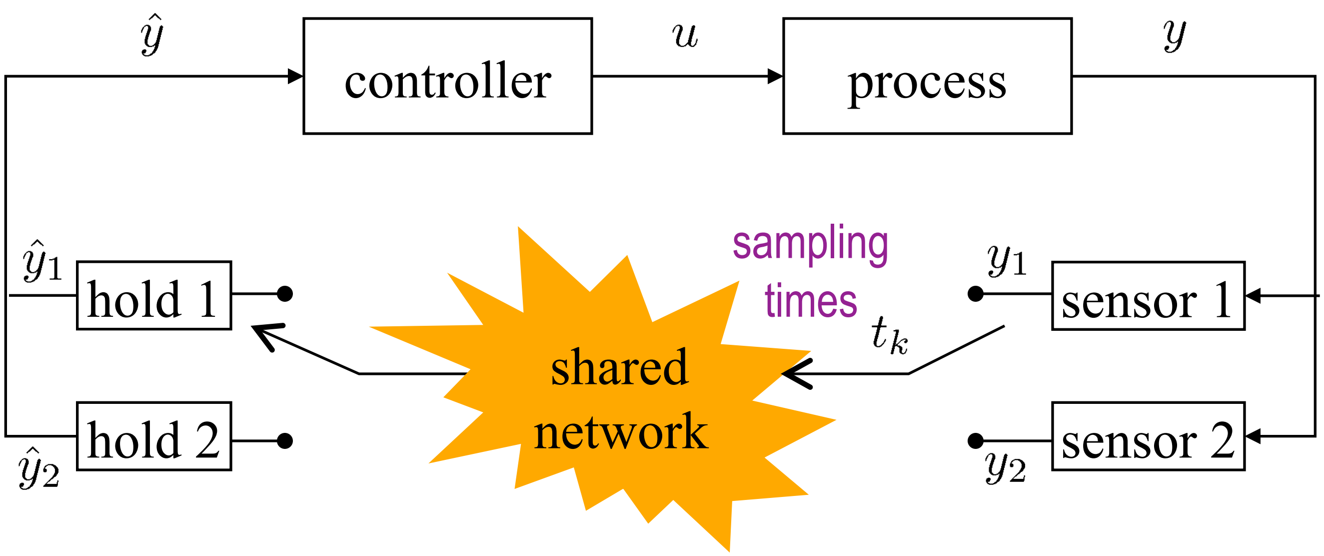


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controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$

$\hat{y} = C_C x_C + D_C \hat{y}$

***What if the network is not available at a sample time $t_k$ ?***

1st wait until network becomes available

2nd send (old) data from original sampling of continuous-time output

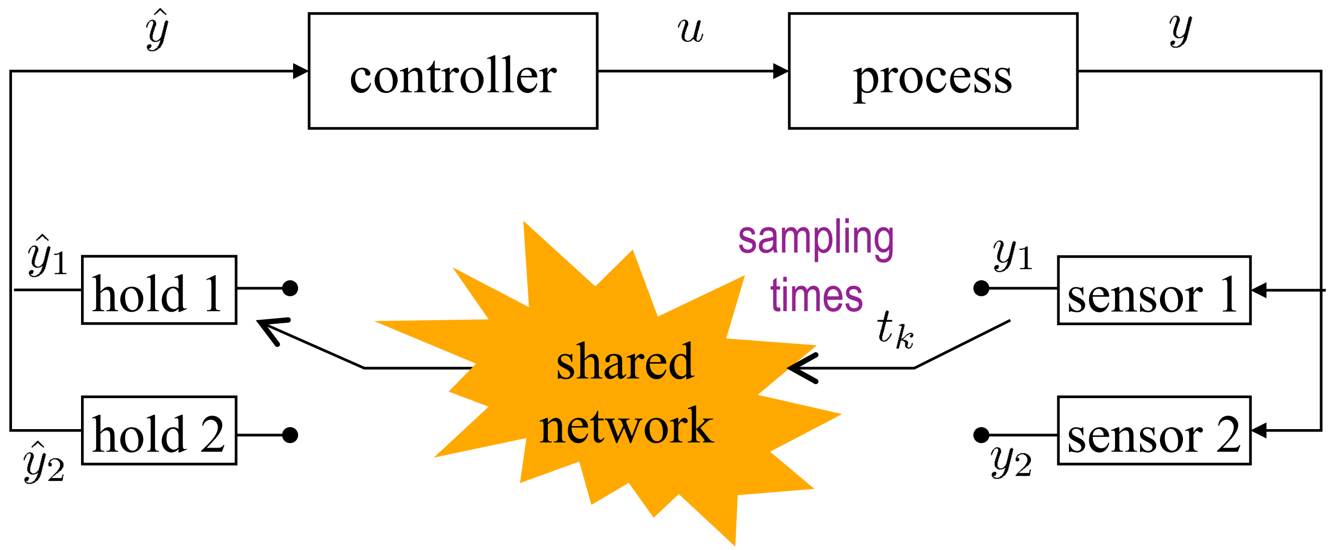
or

2nd send (latest) data from current sampling of continuous-time output

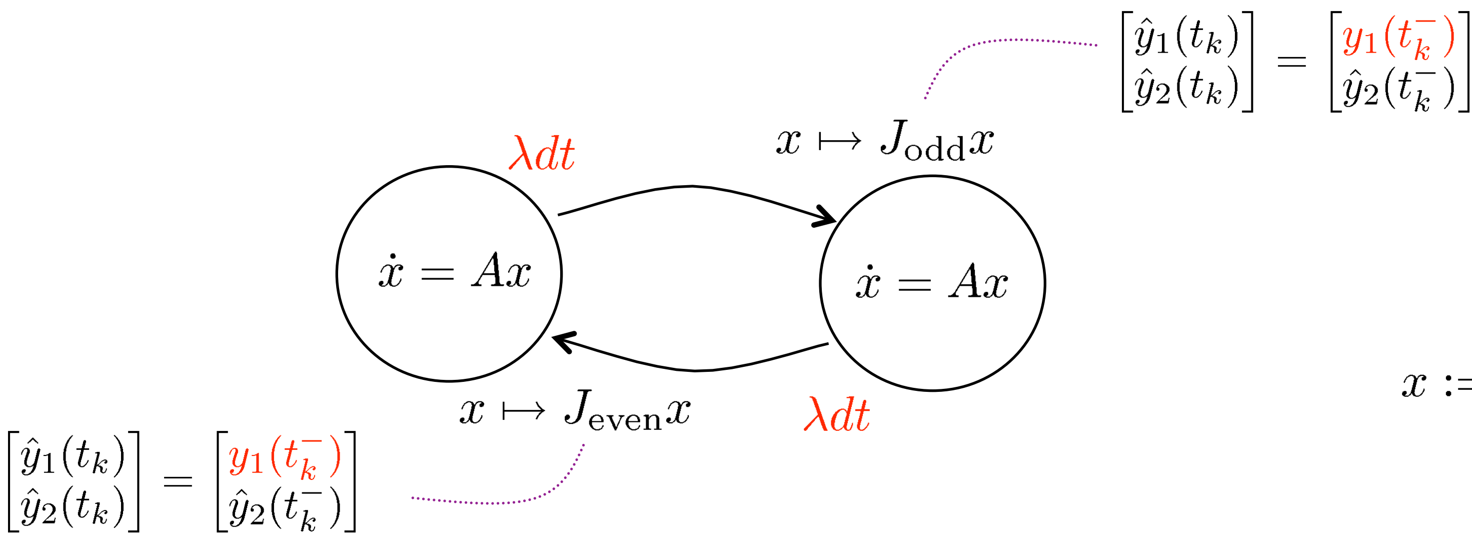
$\Rightarrow$ intersampling times $t_{k+1} - t_k$ typically become random variables

# Example I: Networked Control System






Suppose $t_{k+1} - t_k \sim$ i.i.d., exponentially distributed




$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

# Example I: Networked Control System

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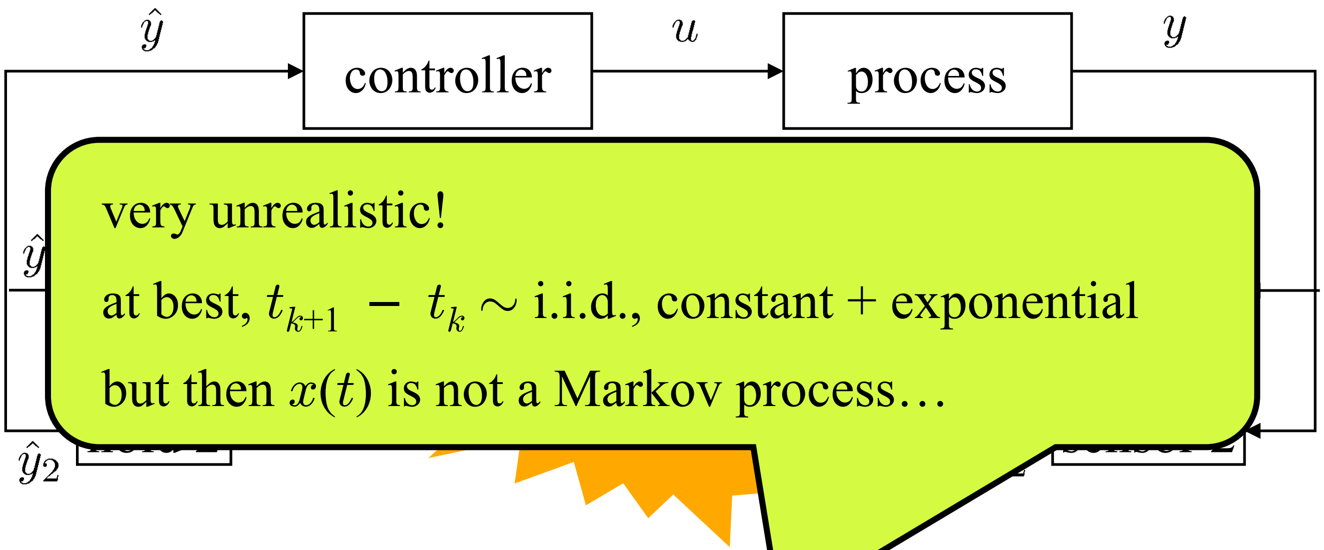


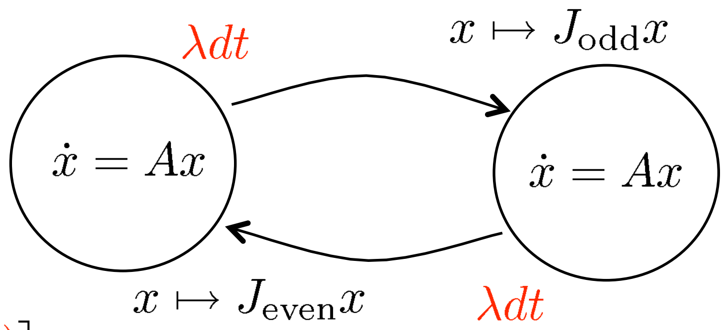

very unrealistic!

at best, $t_{k+1} - t_k \sim$ i.i.d., constant + exponential

but then $x(t)$ is not a Markov process…

Suppose $t_{k+1} - t_k \sim$ i.i.d., exponentially distributed

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix}$$

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# Example I: Networked Control System

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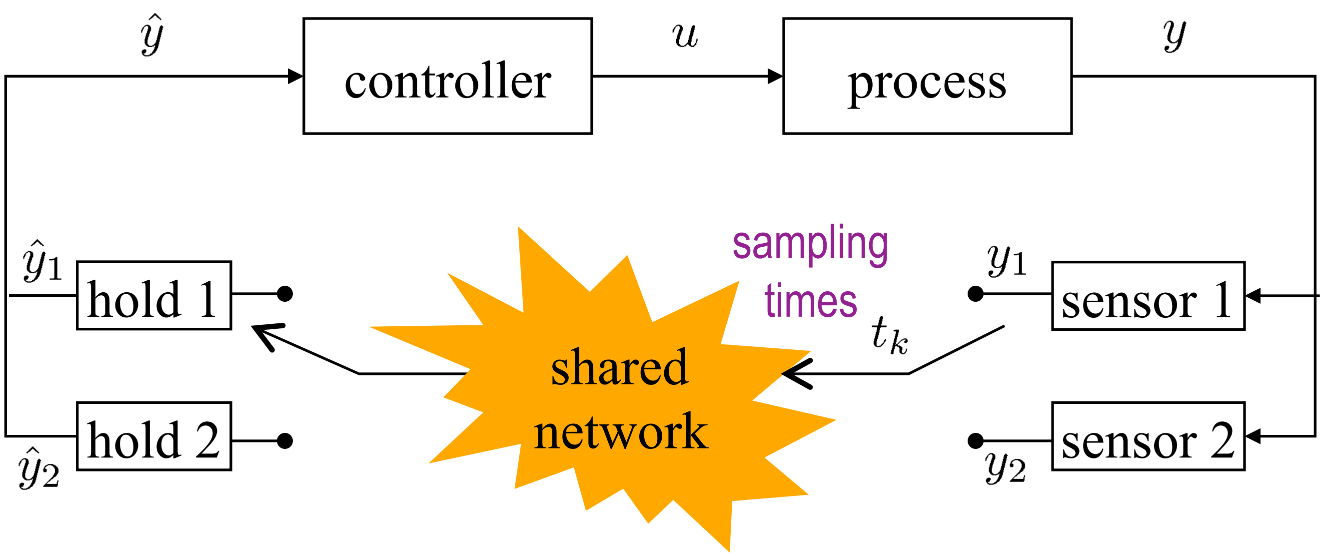
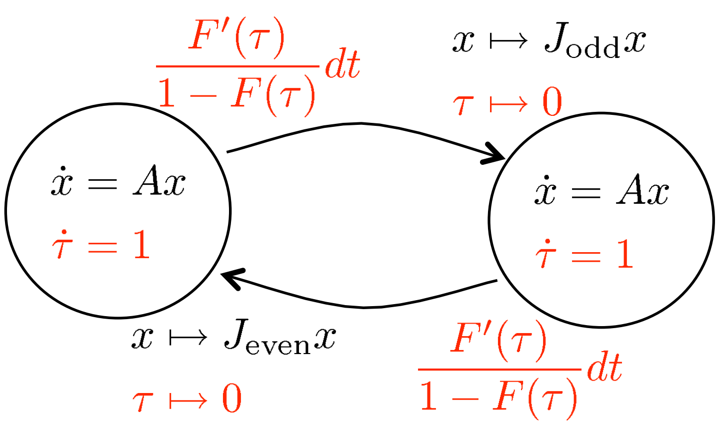
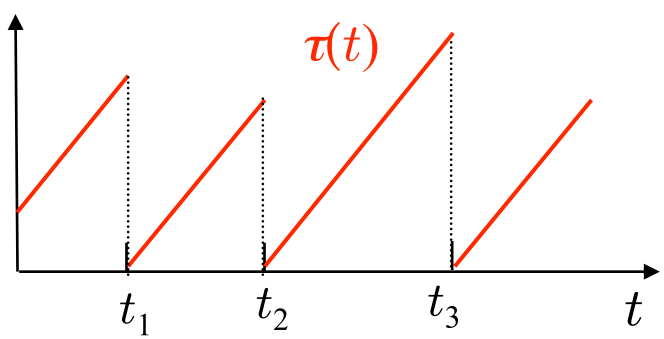



Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$



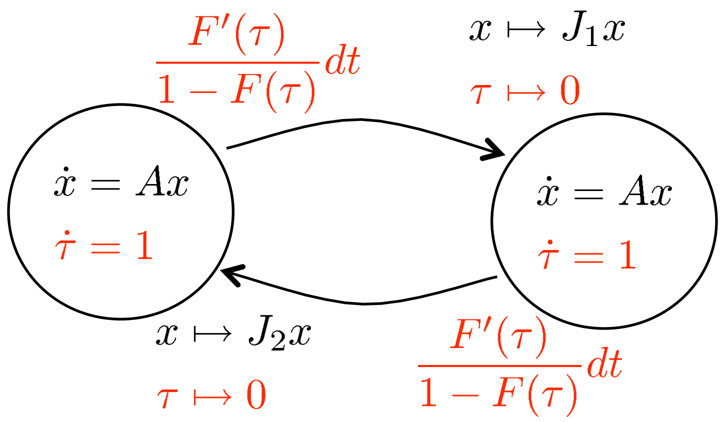




$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

the aggregate state $(x,\tau)$ is a Markov process

# Impulsive Syst. driven by Renewal Proc.






impulsive system ≡ same continuous dynamics for all modes

$N(t)$ ≡ # of jumps before time $t$

renewal process
(iid inter-increment times)

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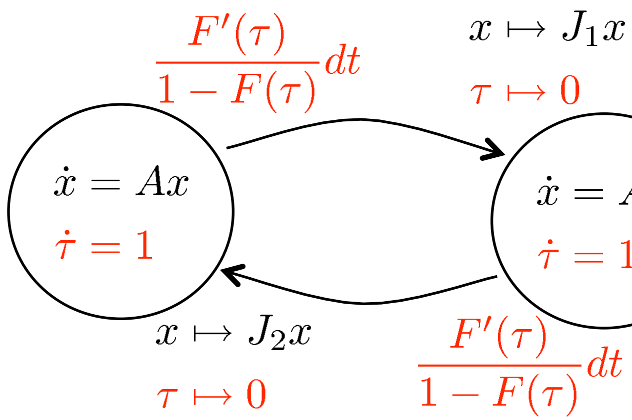


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$N(t)$ ≡ # of jumps before time $t$

renewal process
(iid inter-increment times)

**Theorem:** (for simplicity, conditions stated for equal reset matrices: $J_1 = J_2 \in \mathbb{R}^{n\times n}$)

system is stochastically stable, i.e., $\int_0^\infty \mathrm{E}[\|x(t)\|^2]dt < \infty$

$$\Updownarrow$$

$$\mathrm{E}_{F(T)}\left[\int_0^T e^{A't}e^{At}dt\right] < \infty$$

expected value
w.r.t. inter-jump times

and

LMI on $P_{n\times n}$

$$\exists P > 0 : \mathrm{E}_{F(T)}\left[e^{A'T}J'PJe^{AT}\right] - P < 0$$

or

spectral radius condition on $n^2\times n^2$ matrix

$$\sigma\left(\mathrm{E}_{F(T)}\left[e^{A'T}J' \otimes e^{A'T}J'\right]\right) < 1$$

Kronecker product

[ACC'09]

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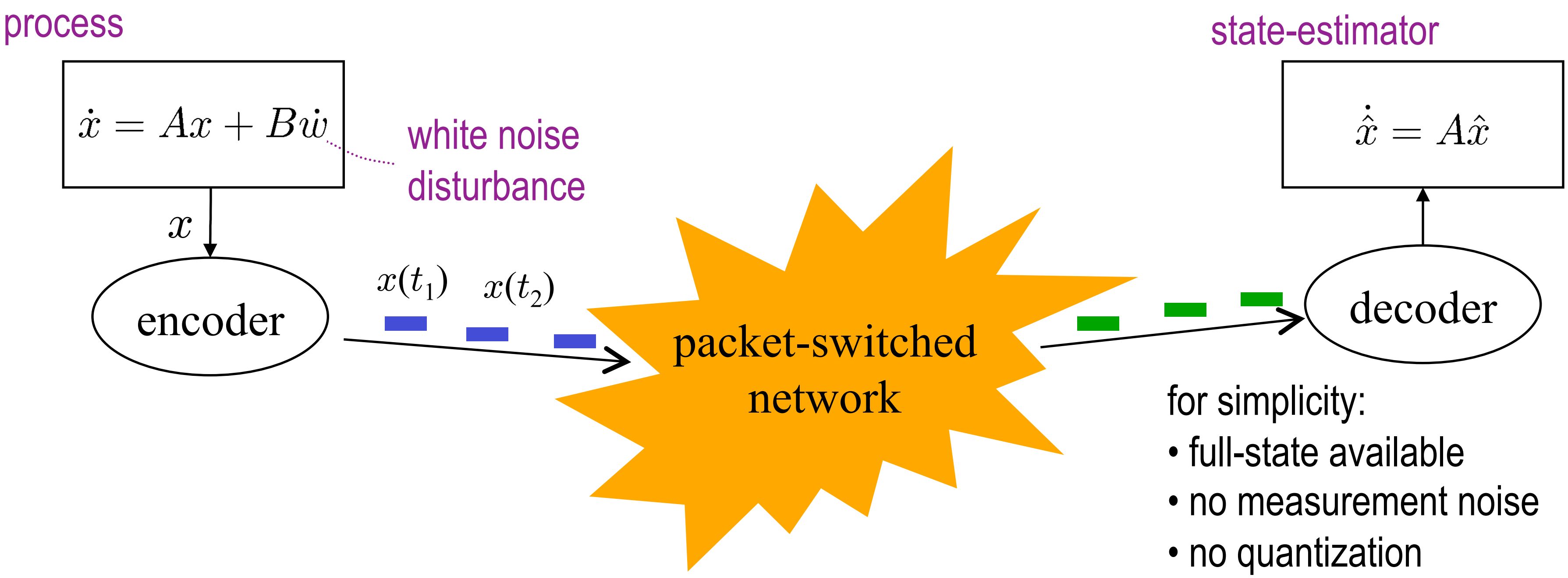
# Example II: Estimation through network



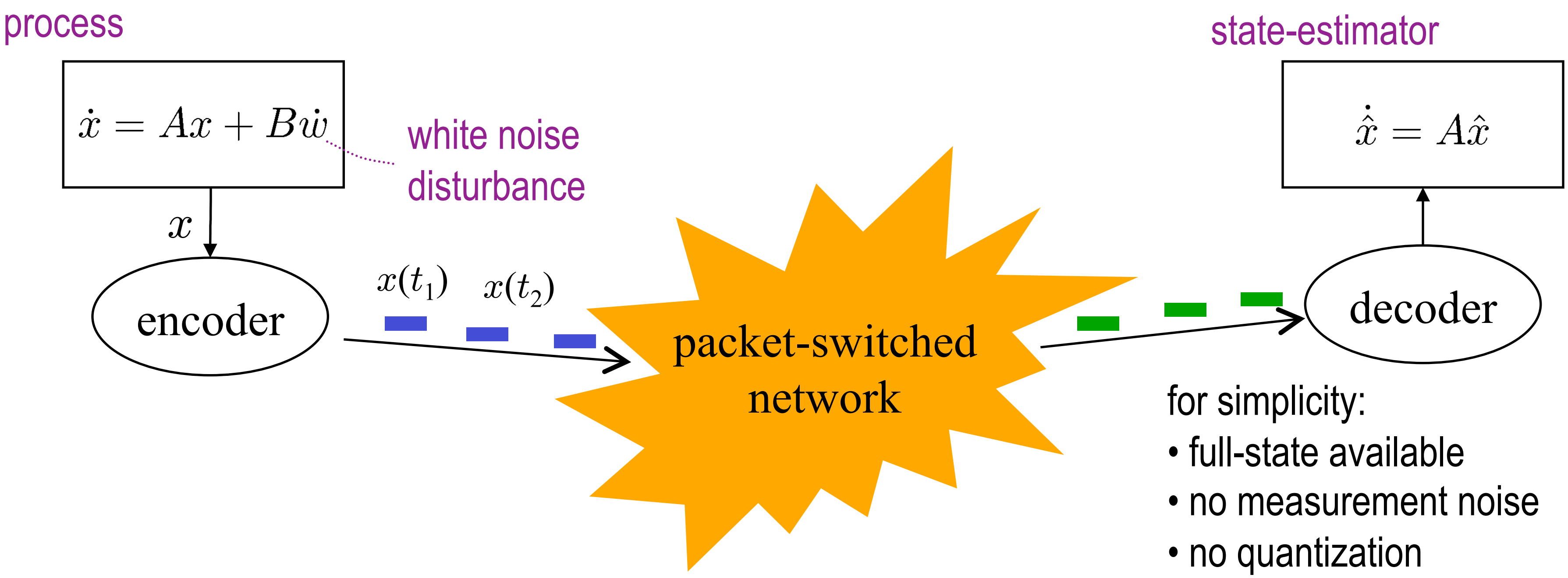

for simplicity:

- full-state available
- no measurement noise
- no quantization
- no transmission delays

encoder logic ≡ determines *when* to send measurements to the network

decoder logic ≡ determines *how* to incorporate received measurements

# Stochastic communication logic




for simplicity:
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encoder logic ≡ determines *when* to send measurements to the network

1. keep track of remote estimate $\hat{x}$
2. send measurements stochastically
3. probability of sending data increases as $\hat{x}$ deviates from $x$

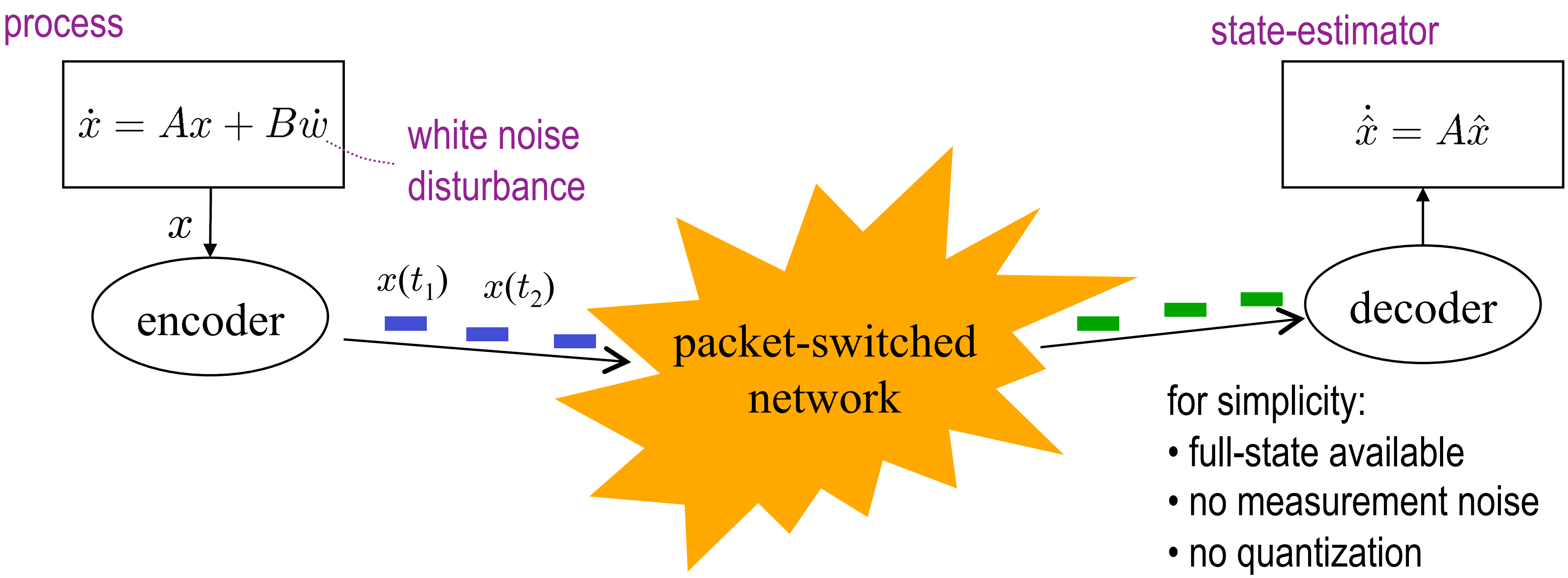
decoder logic ≡ determines *how* to incorporate received measurements

4. upon reception of $x(t_k)$, reset $\hat{x}(t_k)$ to $x(t_k)$

[similar ideas pursued by Astrom, Tilbury, Hristu, Kumar, Basar]

# Example II: Remote estimation






for simplicity:

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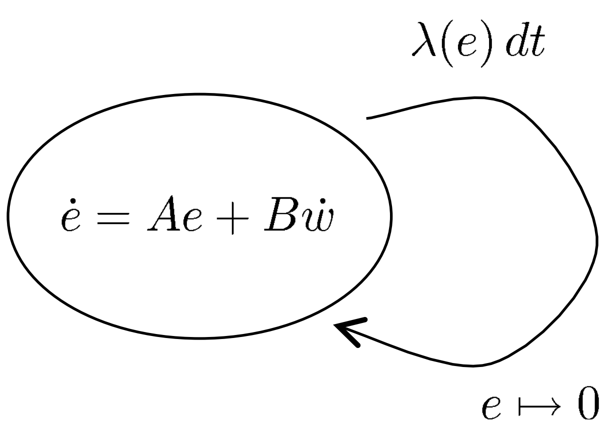
Error dynamics: $e := x - \hat{x}$




prob. of sending data in $[t, t+dt)$ depends on current error $e$

reset error to zero

[CDC'04, CRC Press'06]

# Analysis — Lie derivative

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$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Given scalar-valued function $\psi : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x(t),t) = \frac{\partial \psi}{\partial x} f(x) + \frac{\partial \psi}{\partial t}$$

derivative along solution to ODE

$L_f \psi$
Lie derivative of $\psi$

Basis of "Lyapunov" formal arguments to establish boundedness and stability…

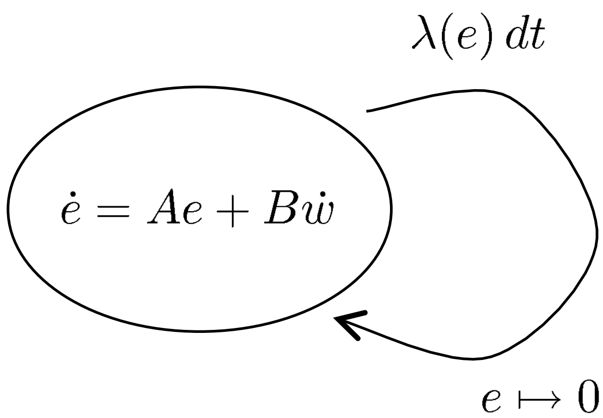
E.g., picking $\psi(x,t) := \|x\|^2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x(t),t) = \frac{\partial \psi}{\partial x} f(x) + \frac{\partial \psi}{\partial t} \leq 0 \quad \Rightarrow \quad \psi(x(t),t) = \|x(t)\|^2 \leq \|x(0)\|^2$$

$\|x(t)\|$ remains bounded along trajectories !

# Generator of a SHS






prob. of sending data in $[t,t+dt)$ depends on current error $e$

reset error to zero

Given scalar-valued function $\psi : Q \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}[\psi(q,x,t)] = \mathrm{E}\Big[(L\psi)(q,x,t)\Big]$$

$x$ & $q$ are discontinuous, but the expected value is differentiable!!!

Dynkin's formula (in differential form)

where

generator for the SHS

$$(L\psi)(e,t) = \frac{\partial \psi}{\partial e}Ae + \frac{\partial \psi}{\partial t}$$ ........ Lie derivative

instantaneous variation

$$+ \Big[\psi(0,t) - \psi(e,t)\Big]\lambda(e)$$ ........ reset term

intensity

$$+ \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2\psi}{\partial e^2}B\right)$$ ........ diffusion term

*Disclaimer*: see *Nonlinear Analysis*'05 for technical assumptions

# Generator of a SHS

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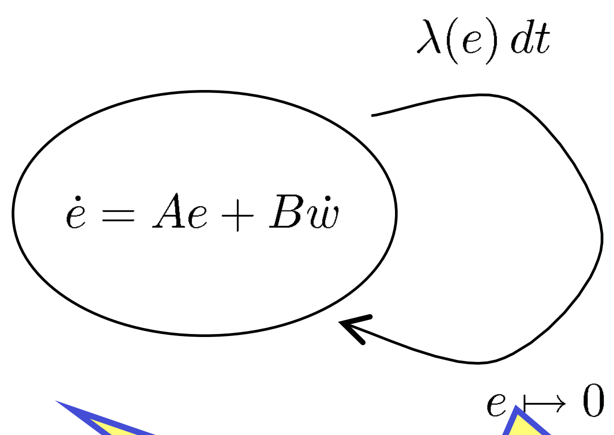


prob. of sending data in $[t, t+dt)$ depends on current error $e$

reset e ... to zero

generalizes to large classes of Stochastic Hybrid Systems

Given scalar-valued fu...

$x$ & $q$ are discontinuous, but the expected value is ...ifferentiable!!!

...'s formula

...differential form)

where

$$(L\psi)(e,t) = \ldots \frac{\ldots}{\partial e} Ae \ldots t$$ Lie derivative

instanta...ous variation

generator for the SHS

$$+ \Big[\psi(0,t) - \psi(e,t)\Big]\lambda(e)$$ reset term

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# Lyapunov-based stability analysis

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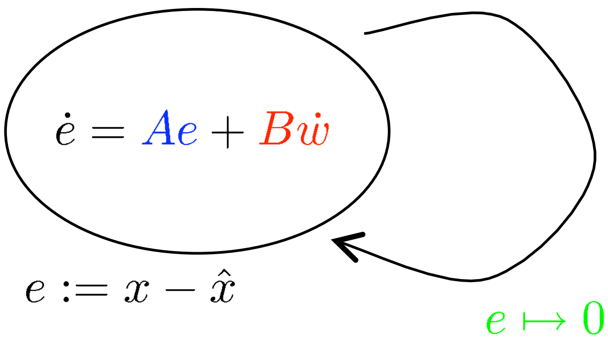
error dynamics
in remote estimation




$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\psi(e)] = \mathrm{E}\Big[(L\psi)(e)\Big]$$ Dynkin's formula

$$(L\psi)(e) = \frac{\partial\psi}{\partial e}Ae + \Big[\psi(0) - \psi(e)\Big]\lambda(e) + \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2\psi}{\partial e^2}B\right)$$

For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $\mathrm{E}[\, e \,] \to 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $\mathrm{E}[\, \| e \|^m \,]$ bounded if and only if $\gamma > m\, \Re[\lambda(A)]$

getting more moments bounded requires higher comm. rates

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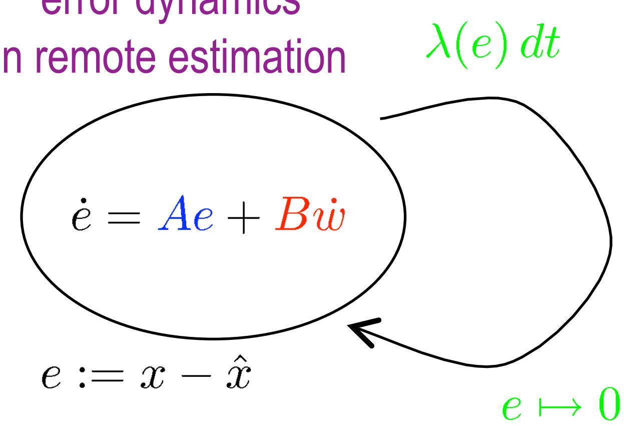


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For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $\mathrm{E}[\,e\,] \to 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $\mathrm{E}[\,\|\,e\,\|^m\,]$ bounded if and only if $\gamma > m\,\Re[\lambda(A)]$

getting more moments bounded requires higher comm. rates

For polynomial rates: $\lambda(e) = (e'\,Q\,e)^k$ $\quad Q > 0,\ k > 0$ (reactive transmissions)

1. $\mathrm{E}[\,e\,] \to 0$ (always)
2. $\mathrm{E}[\,\|\,e\,\|^m\,]$ bounded $\forall\, m$

Moreover, one can achieve the same $\mathrm{E}[\,\|e\|^2\,]$ with less communication than with a constant rate or periodic transmissions…

[CDC'04, Birkhauser'06]

# Conclusions

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1. A simple SHS model that finds use in several areas
   (networked control systems, network traffic modeling, biochemistry)

2. The analysis of SHSs is challenging but there are tools available
   (generator, Lyapunov methods, moment dynamics, truncations)

3. Lots of work to be done:
   - theory
     - ✓ stability/robustness/performance of SHS
   - networked control systems
     - ✓ protocol design to optimize performance & minimize communication resources