Game Theory
Lecture #2

Focus of Lecture:

- How do you aggregate the opinions of many?
- Social choice setup
- Axioms of social choice

1 Voting and Preference Aggregation Systems

This chapter starts the process of formally illustrating the challenges that can emerge in the analysis and design of societal systems. In particular, we will focus on one important class of societal systems termed voting or preference aggregation systems. The goal of any voting system is to choose society's best option from a given set of alternatives. The core challenge associated with the design of a voting system is that members of society may have vastly different opinions regarding the best option, and given these disparate opinions it is unclear what should constitute "society's best option."

Example 1.1 (Elections) *An election is a canonical example of a voting system. Here, the set of alternatives is the set of candidates and each individual voting member of society has a preference over the candidates. A common voting system is to have each individual cast a vote for a single candidate, presumably their favorite candidate, with the winning candidate being the one with the most votes. Such a voting system is sometimes coupled with a runoff in case no single candidate received a majority of votes. Alternately, another voting system is to have each individual submit a ranking of all candidates (where the process of selecting a winning candidate still remains to be specified).*

The above example highlights some fundamental questions associated with the design and analysis of voting systems. How should one determine the best societal option given the individual rankings of the members of society? What are the desirable characteristics of a best societal option? Are there voting systems that always ensure that this best societal option is chosen? Are there voting systems that encourage individuals to reveal their true preferences? Do common voting systems and procedures, e.g., the electoral college system in the United States or proportional representation used in much of Europe, always ensure that the best societal option is chosen? Why is there not a common voting system across the world? What is the most efficient voting system?

Example 1.2 (Blockchain Governance) *By design, the Bitcoin blockchain has no centralized governing body, so it can be challenging to modify the core code. In this context, the set of alternatives may be a list of code-change proposals put forward by the community.*

Complicated constraints may exists between various proposals (e.g., perhaps proposal A is only feasible if proposal B is implemented first, but B may be incompatible with some other proposal C).

In this setting, how should the Bitcoin community decide between various proposals? Furthermore, how should new proposals that are continually being added to the list as new issues arise be handled? This second example illustrates that these kinds of problems are not simply relegated to politics, but have relevance for computational and cyber-engineering systems as well.

1.1 The Voting Paradox

This section focuses on an illustrative example provided by Marquis de Condorcet in 1785, termed the voting paradox, that sheds light on the complexity associated with voting systems. Suppose there is a fixed monetary budget that can go to only one of three causes: Health, Security, or Education. Furthermore, there are 12 different members of society, each with their individual preferences regarding how the money should be allocated. The preferences of each member of society is aligned with one of three different parties (Left, Middle, or Right), and each party's preferences are shown in the following table:

LEFT (3)	MIDDLE (4)	RIGHT (5)
Health	*Education*	*Security*
Security	*Health*	*Education*
Education	*Security*	*Health*

In this example, three individuals are aligned with LEFT and have preferences of *Health* over *Security* over *Education*, four are aligned with MIDDLE and have preferences of *Education* over *Health* over *Security*, and so on. Given these preferences, should the budget get allocated to *Health*, *Security*, or *Education*?

The following are mechanisms that could be employed to determine the appropriate allocation of the budget. We will see that each of these mechanisms has its own shortcomings.

- *Mechanism #1: Single Vote* – Each individual casts one vote and the alternative with the most votes is chosen as the social choice. If each individual votes according to their preferences above, *Health* will have 3 votes, *Education* will have 4 votes, and *Security* will have 5 votes; hence, the budget will be allocated to *Security*. Does this allocation seem reasonable? Note that more individuals prefer *Health* to *Security*, which suggests that this outcome not be a good option.
- *Mechanism #2: Pairwise Voting* – Consider a pairwise voting system where each individual votes on the preferred alternative between each pair of alternatives, and the outcome of these pairwise results determines the social choice. For example, if we

restrict attention to *Health* and *Security*, then *Health* is preferred over *Security* by a margin of 7 to 5. Furthermore, if we restrict attention to *Security* and *Education*, then *Security* is preferred over *Education* by a margin of 8 to 4. Hence, it seems like a reasonable societal choice is to allocate the budget to *Health*. However, if we restrict attention to *Health* and *Education*, then *Education* is preferred over *Health* by a margin of 9 to 3, which complicates the analysis.

- *Mechanism #3: Group Voting* – Consider a weighted voting system where each group (Left, Middle, Right) casts one vote with a weight equal to its population size, namely $(3, 4, 5)$, respectively. Given these votes, the alternative with the highest total weight would be chosen. Would any group favor this mechanism? Note that if Right selected *Security*, then both Left and Middle would prefer to choose *Health* to ensure that *Health* is chosen over *Security*.

- *Mechanism #4: Dictatorship* – In this setting, a single group has the unilateral authority to select the budget allocation.

2 Social Choice

The takeaway from the voting paradox is that determining an optimal social choice is challenging. In fact, even the question of what is the optimal social choice is unclear. In this section we will bring in a formal model, termed *social choice*, to study this voting question in a more general context. The elements of this model are as follows:

- Set of individuals: $N = \{1, \ldots, |N|\}$
- Set of alternatives: $X = \{x_1, \ldots, x_{|X|}\}$
- Preferences: For each individual $i \in N$ and every pair of alternatives $x, x' \in X$, exactly one of the following is satisfied:
  - $x \succ x'$ (i prefers x to x')
  - $x \prec x'$ (i prefers x' to x, which we will often write as $x' \succ x$)
  - $x \sim x'$ (i views x and x' as equivalent)

  We will often express the relation terms $(\succ, \prec, \sim)$ as $(\succ_i, \prec_i, \sim_i)$ to highlight the dependence on individual i. The preferences of the individuals contains a list of pairwise comparisons. For compactness, we sometimes will express the preferences of individual i by a function q_i where $q_i : X \times X \to \{\succ, \prec, \sim\}$ defines these pairwise preferences.

The goal in this social choice problem is to derive a social choice function $SC(\cdot)$ of the form:

$$q_N = SC(q_1, \ldots, q_{|N|}), \tag{1}$$

which takes in the preferences of the individuals and aggregates them into a single preference profile of the form $q_N : X \times X \to \{\succ, \prec, \sim\}$. Note that the outcome of this social choice function is a *complete set* of pairwise comparisons, as opposed to the selection of a single alternative.

Example 2.1 (The Voting Paradox) *Let us revisit the aforementioned mechanisms in the context of social choice functions. First, any individual i in* LEFT *has a preference that satisfies: Health $\succ_i$ Security, Health $\succ_i$ Education, and Security $\succ_i$ Education. Similarly, any individual j in* MIDDLE *has a preference that satisfies: Education $\succ_j$ Health, Education $\succ_j$ Security, and Health $\succ_j$ Security. The social choice q_N for each of the four mechanisms is described as follows:*

- *Mechanism #1: Single Vote – This mechanism, as originally stated, does not produce a social choice function because only a single option is chosen, whereas a social choice function requires a full ranking. One can use the total tally of votes to produce a social choice.*
- *Mechanism #2: Pairwise Voting – This mechanism results in a social choice Security $\succ_N$ Education, Education $\succ_N$ Health and Health $\succ_N$ Security.*
- *Mechanism #3: Group Voting – This mechanism, as originally stated, does not produce a social choice function because only a single option is chosen. As before, a social choice function can be constructed by using the total tally of votes.*
- *Mechanism #4: Dictatorship – Assuming that the dictating group is* RIGHT*, this mechanism results in a social choice function Security $\succ_N$ Education, Education $\succ_N$ Health, and Security $\succ_N$ Health.*

2.1 What do we want a social choice function to do?

In this section we shift our attention away from a case-by-case analysis of various specific social choice mechanisms. Rather, we will specify a set of desired properties (which we will call "Axioms") that we want our social choice mechanism to satisfy and then discuss whether or not it is possible to construct such a social choice mechanism exists. Recall that a social choice function is a function of the form $q_N = SC(q_1, \ldots, q_{|N|})$, hence it defines a social choice q_N for *any collection* of individual preferences $(q_1, \ldots, q_{|N|})$. The pairwise voting mechanism and dictatorship mechanism are two distinct mechanisms of this form. Likewise the modified (i.e., extended to the total tally) single vote and group voting are social choice mechanisms according to this definition.

The first axiom restricts attention to the case where the domain (individual preferences) and range (societal preferences) are reasonable.

Axiom # 1 *Reasonable domain and range*

Our first axiom imposes a reasonable constraint on the class of preferences that our mechanism considers. In particular, we will require that both the individual preferences and societal preferences satisfy *completeness* and *transitivity*. That is, an acceptable preference q must satisfy the following:

- Completeness: For any pair of alternatives $x, x' \in X$, $q(x, x') \in \{\succ, \sim, \prec\}$. Furthermore, (i) if $x = x'$, then $x \sim x'$ and (ii) if $x \succ x'$ then $x' \prec x$.
- Transitivity: For any collection of alternatives $x, x', x'' \in X$, if $x \succ x'$ and either $x' \succ x''$ or $x' \sim x''$, then $x \succ x''$.

The completeness property ensures that the pairwise preference relations defined for all preferences pairs are consistent with one another. That is, there is always indifference between the same alternative, i.e., $x \sim x$, and if x is preferred to x', i.e., $x \succ x'$, then we must have $x' \prec x$. The transitivity property ensures that a preference profile can be expressed by a given ranking. In particular, it disqualifies preferences obtained from the pairwise voting mechanism defined above (e.g., *Security* $\succ$ *Education*, *Education* $\succ$ *Health* and *Health* $\succ$ *Security*). Consequently, we will sometimes express a preference profile as a ranking in the form of a stacked vector, as in:

$$\begin{bmatrix} x \\ x' \\ x'' \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ x', x'' \end{bmatrix}$$

with the understanding that the higher the position, the higher the ranking. For example, the preference for the ranking on the left would be $x \succ x'$, $x \succ x''$, and $x' \succ x''$. Similarly, the preference for the ranking on the left would be $x \succ x'$, $x \succ x''$, and $x' \sim x''$. *This axiom requires that all preferences, both the individual preferences and societal preferences, satisfy completeness and transitivity.*

Axiom #2 *Positive Association*

Our second axiom focuses on establishing a degree of consistency in the derived societal choice for varying individual preference profiles. At a high level, we will say that a social choice function exhibits positive association if improvements in the individual preferences of a given alternative do not degrade the societal preference of that particular alternative.

To that end, consider any two sets of individual preference profiles $q = (q_1, \dots, q_{|N|})$ and $q' = (q'_1, \dots, q'_{|N|})$ and any two alternatives x and y in X. We will say that "x is more preferred to y in q' compared to q" if the following three conditions are satisfied:

(i) For all $i \in N$, if $x \succ_i y$ in q, then $x \succ_i y$ in q' (i.e., if any individual i strictly prefers x to y in q, then individual i strictly prefers x to y in q').

(ii) For all $i \in N$, if $x \sim_i y$ in q, then either $x \succ_i y$ or $x \sim_i y$ in q' (i.e., if any individual i is indifferent between x and y in q, then individual i is either indifferent or strictly prefers x to y in q').

(iii) There exists a $j \in N$ such that either: (a) $x \succ_j y$ in q' but either $x \prec_j y$ or $x \sim_j y$ in q or (b) $x \sim_j y$ in q' but $x \prec_j y$ in q (i.e., at least one individual improved the relative standing of x over y in q').

Now consider the societal preferences $q_N = SC(q)$ and $q'_N = SC(q')$. We will say that a social choice function $SC(\cdot)$ exhibits positive association if for any individual preference profiles q and q' and any pair of alternative x and y, where x is more preferred to y in q' compared to q, one of the following two conditions must be satisfied:

(i) If $x \succ_N y$ in $q_N = SC(q)$, then $x \succ_N y$ in $q'_N = SC(q')$.

(ii) If $x \sim_N y$ in $q_N = SC(q)$, then $x \succ_N y$ or $x \sim_N y$ in $q'_N = SC(q')$.

That is, the relative preference of the alternatives y when compared to x can only improve in the societal choice $q'_N = SC(q')$ when compared to $q_N = SC(q)$.

For convenience, let us denote individual preference profiles by a matrix with each column corresponding to an individual's ranking. For example, in case of 4 individuals and two choices, preference profiles can be represented as:

$$q = \begin{bmatrix} x & x & y & y \\ y & y & x & x \end{bmatrix} \text{ and } q' = \begin{bmatrix} x & x & x & y \\ y & y & y & x \end{bmatrix}$$

Axiom 2 seeks to prohibit situations like the following:

$$SC\left(\begin{bmatrix} x & x & y & y \\ y & y & x & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } SC\left(\begin{bmatrix} x & x & x & y \\ y & y & y & x \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix},$$

as the only difference between the two profiles is that $x \succ_3 y$ in q and $y \succ_3 x$. However, even though the relative preference of x improved relative to y in q compared to q', the societal preference was the opposite.

Axiom #3 *Unanimous Decision*

Our third axiom imposes the constraint that if every individual $i \in N$ prefers alternative x to y, then the societal preference should also prefer x to y. More formally, given a preference profile $q = (q_1, \ldots, q_{|N|})$, if $x \succ_i y$ for every individual $i \in N$, then the social choice $q_N = SC(q)$ must also satisfy $x \succ_N y$. Note that in the absence of unanimity, i.e., if not all individuals prefer x to y, then this Axiom does not impose any restriction on the resulting social preference.

Example 2.2 *Suppose the preference profile q is of the form*

$$q = \begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix}.$$

Suppose the social choice mechanism $SC(\cdot)$ satisfies Axiom #3: Unanimous Decision. Then the viable societal preferences $q_N = SC(q)$ associated with this mechanism are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad or \quad \begin{bmatrix} y \\ x \\ z \end{bmatrix} \quad or \quad \begin{bmatrix} x, y \\ z \end{bmatrix},$$

as there is unanimity in the ranking $x \succ_i z$ for all i.

Axiom # 4 *Independence of irrelevant alternatives*

Our fourth axiom essentially states that pairwise rankings between two alternatives, e.g., x and y, are not impacted by the relative position of a third ("irrelevant") alternative, z.

More formally, let q and q' be any two preference profiles. Suppose that the relative preference between x and y is the same for each individual $i \in N$ in the preference profiles q and q', i.e., for all i:

(i) if $x \succ_i y$ in q, then $x \succ_i y$ in q';

(ii) if $x \sim_i y$ in q, then $x \sim_i y$ in q';

(iii) if $y \succ_i x$ in q, then $y \succ_i x$ in q'.

Then the relative preference between x and y in the resulting societal preferences $q_N = SC(q)$ and $q'_N = SC(q')$ must be the same, i.e.,

(i) if $x \succ_N y$ in q_N, then $x \succ_N y$ in q'_N;

(ii) if $x \sim_N y$ in q_N, then $x \sim_N y$ in q'_N;

(iii) if $y \succ_N x$ in q_N, then $y \succ_N x$ in q'_N.

Example 2.3 *Consider two preference profiles of the form*

$$q = \begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix} \quad and \quad q' = \begin{bmatrix} z & x & x & x & z \\ x & z & y & z & y \\ y & y & z & y & x \end{bmatrix}$$

Observe that alternative z is an irrelevant alternative with regards to the relative preferences between x and y. Accordingly, if the social choice function $SC(\cdot)$ satisfies Axiom #4: Independence of Irrelevant Alternative, then the relative preference between x and y in the resulting societal preferences $q_N = SC(q)$ and $q'_N = SC(q')$ should be the same, i.e., (i) if $x \succ_N y$ in q_N, then $x \succ_N y$ in q'_N; (ii) if $x \sim_N y$ in q_N, then $x \sim_N y$ in q'_N; and (iii) if $y \succ_N x$ in q_N, then $y \succ_N x$ in q'_N.

Axiom # 5 *Non-Dictatorship*

Our final axiom deals with the notion of a dictatorship. We will say that a social choice function has a *dictator* if the resulting social choice is always aligned with the preference of some individual $i \in N$. That is, a social choice function has a dictator if there exists an individual $i \in N$ such that for any preference profile $q = (q_1, \ldots, q_{|N|})$, the societal choice $q_N = SC(q)$ satisfies the following for any pair of alternatives x and y: (i) $x \succ_N y$ if and only if $x \succ_i y$; (ii) $x \sim_N y$ if and only if $x \sim_i y$; and (iii) $y \succ_N x$ if and only if $y \succ_i x$. This particular axiom implies that there is no dictator in a society of at least three individuals. That is, there is no individual $i \in N$ whose opinion decides all issues, regardless of the opinions of others.

3 Conclusions

In this lecture we specified the problem of social choice and began our exploration of various social choice mechanisms. We observed that each social choice mechanism exhibited properties that were not necessarily desirable. Accordingly, we identified five Axioms (or properties) that any reasonable social choice mechanism should possess. In the next lecture we will focus on the design of mechanisms to meet these five Axioms.

4 Questions

We begin by reviewing 3 different voting rules and investigating their various benefits and drawbacks. First, recall the voting paradox example:

Left (3)	Middle (4)	Right (5)
Health	*Education*	*Security*
Security	*Health*	*Education*
Education	*Security*	*Health*

In each of the following mechanisms, an individual submits a full ranking of alternatives (i.e., not just a vote for a single issue).

Mechanism 4.1 (Plurality Voting) *The topic with the most 1st-place rankings wins; every voter's 2nd and 3rd choice is ignored. In the above example, Plurality voting would give Security the win, since Security has 5 first-place rankings, but Health and Education have only 3 and 4, respectively. As stated, plurality voting only selects a winner. The complete tally can be used to define a complete social choice function.*

Note: Plurality voting is used in most elections in the United States.

Mechanism 4.2 (Instant Runoff) *First, add up each topic's 1st-place rankings. If any topic has more than 50% of the vote based on that count, then that topic wins. If no topic has more than 50% of the vote based on 1st-place rankings, then we eliminate the lowest-ranked topic and "virtually" re-run the election. In the above example, no topic has over 50% of the votes, but Health had the fewest 1st-place rankings, so Health is eliminated. With Health removed, the rankings are now:*

	Left (3)	Middle (4)	Right (5)
1st Preference	*Security*	*Education*	*Security*
2nd Preference	*Education*	*Security*	*Education*

Now Security has 8 votes, which is more than 50%, so Security wins. To produce a complete social choice function, we can restart the process with the winning topic successively removed.

Note: the Instant Runoff method is used in many elections in Australia and several other places in the world.

Mechanism 4.3 (Copeland's Method (simplified)) *The winner is the topic who wins the most hypothetical 2-person contests against other candidates. It turns out that this method does not always produce a unique winner! In the above example, we need to count how many individuals rank Security above Health, how many individuals rank Health above Education, how many individuals rank Education above Security, and so on. First, just looking at Security and Health, we have*

	Left (3)	Middle (4)	Right (5)
1st Preference	*Health*		*Security*
2nd Preference	*Security*	*Health*	
3rd Preference		*Security*	*Health*

Health beats Security 7 times, and Security beats Health 5 times. Continuing with the other pairings (without re-drawing the table each time), we have that Health beats Education 3 times; Education beats Health 9 times; and Security beats Education 8 times; while Education beats Security 4 times. Health has a total of 10 wins, and Security and Education each have 13 wins, so in this case Copeland's method failed to choose a unique winner because Security and Education tied. Nonetheless, these pairwise comparisons can be used to produce a complete social choice function, i.e., not just the winner(s).

Plurality Voting, Instant Runoff, and Copeland's Method are used in the following problems.

1. Consider the following fictional situation (inspired by 2016's U.S. Republican Presidential Primaries). There are 3 candidates: (A), (B), and (C) and 9 voters. The voters' preferences are follows:

	4 voters	3 voters	2 voters
1st Preference	A	C	B
2nd Preference	B	B	C
3rd Preference	C	A	A

4 voters think (A) is better than (B) and that (B) is better than (C); 3 voters think (C) is better than (B), and (B) is better than (A), and so forth.

(a) Which candidate wins under Plurality Voting?

(b) Which candidate wins under Instant Runoff?

(c) Does Copeland's Method choose a winner in this case? If so, which candidate?

(d) What are the benefits of Instant Runoff over Plurality?

(e) Copeland's Method is known as a *Condorcet Method*, which means that if it chooses a single winner, that winner would beat *every* other candidate in a head-to-head race. Verify that this is true with the (A)-(B)-(C) example.

(f) *(bonus):* In the Security-Education-Health example, Copeland's Method gave a tie. What would be a good tie-breaking method for that example?

(g) *(bonus):* Several other voting methods are listed at `https://en.wikipedia.org/wiki/Condorcet_method#Single-method_systems`. Choose one and perform it for the Security-Education-Health example.

2. A social choice function for a given preference profile is:

$$SC\left(\begin{bmatrix} x & z & x \\ y & x & y \\ z & y & z \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

Go through each of the five axioms. Which axioms are not satisfied here?