Game Theory
Lecture #6

Focus of Lecture:

- Cost Sharing Games
- Core
- Minimum Spanning Tree Games

1 The Cost Sharing Problem

The previous lectures focused on two interesting problems pertaining to the analysis and design of sociotechnical systems. The first problem we focused on social choice, where the goal was to establish a social choice mechanism that satisfied five desired Axioms. Here, we illustrated a negative result by Kenneth Arrow which demonstrated that there does not exist a social choice mechanism that satisfies this objective. The second problem we focused on was the stable matching problem, where our goal was to identify an efficient matching system between two distinct groups. Here, we illustrated a series of positive results demonstrating that (i) a stable matching always exists and (ii) the Gale-Shapley algorithm will always find a stable matching. The sharp contrast between these two sets of results highlights that a system designer should not take anything for granted when working with societal systems. That is, a system operator must approach the analysis and design of such systems in axiomatic way, rigorously arguing about the resulting conclusions from a first principles perspective.

This lecture will focus on a third problem domain pertaining to cost sharing. In any cost (or revenue) sharing problem there is a collection of individuals that engage in a joint venture accompanied by an associated cost (or profit). How should the costs associated with this joint venture get dispersed to the individuals? Note that this choice will directly impact the behavior of individuals who seek to meet their demands at the least possible cost. The goal here is to establish a cost sharing protocol that promotes the most efficient structures to emerge, where efficiency is measured by the cumulative cost incurred to meet the demands of all. The following examples shed some light on the challenges associated with this.

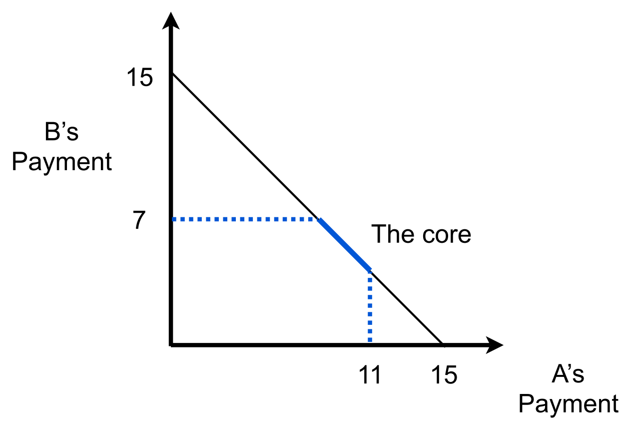
Example 1.1 (A Cost Sharing Problem Among Two Towns) *Suppose two nearby towns, which we will refer to as A and B respectively, decide to construct a joint facility to serve the needs of both towns, e.g., community center, water distribution center, or recreational facility. Each town is required to provide this facility to their community and can proceed on an independent or collective basis. The opportunity costs associated with the three possible ventures are as follows:*

- *Town A can build its own facility for \$11 million*

- *Town B can build its own facility for \$ 7 million*
- *A joint facility serving Town A and B costs \$15 million*

Clearly, having the towns pursue a joint facility is the most fiscally responsible option (\$18 million vs \$15 million). However, pursuing this option requires that the towns find an equitable division of the \$15 million costs.

What division of costs would incentivize each of the towns to engage in this joint venture? One possible division of costs is to divide the costs equally, i.e., each town A and B pays \$7.5 million. However, note that Town B would not engage in such a joint venture as Town B can build its own facility for only \$7 million. Hence, any division of costs that incentive this joint venture must ensure that each town pays less than the opportunity cost. The following figure highlight all costs divisions that incentive both Town A and B to pursue a joint venture.



We will refer to the set of all cost divisions that incentivize the towns to pursue the fiscally advantageous joint venture as the *core*. In the above example, a division of costs is in the core if it satisfies the following three conditions: (i) the costs to towns A and B equals \$15 million; (ii) the cost to town A $\leq$ \$11 million; and (iii) the cost to town B $\leq$ \$7 million. Here, the strategic behavior of the towns imposes the constraints (ii) and (iii).

How do the constraints highlighted above generalize for more than two towns? Are there systematic approaches for deriving specific cost shares that are in the core? The following three town example starts to shed light on the challenges associated with these questions.

Example 1.2 (A Cost Sharing Problem Among Three Towns) *Consider adding a third town C to Example 1.1. Once again, each town is required to provide this facility to their community and can proceed alone or in partnership with either or both of the other towns. The costs associated with the possible ventures are as follows:*

- *Town A can build its own facility for \$11 million*
- *Town B can build its own facility for \$7 million*

- *Town C can build its own facility for \$8 million*
- *Towns A+B can build a joint facility for \$15 million*
- *Towns A+C can build a joint facility for \$14 million*
- *Towns B+C can build a joint facility for \$13 million*
- *Towns A+B+C can build a joint facility for \$20 million*

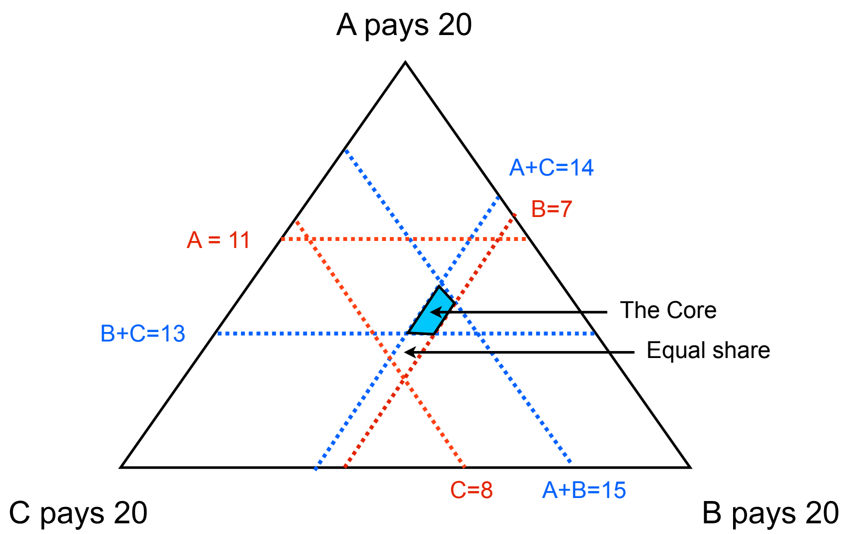
While less obvious than in the previous example, having the towns pursue a joint facility (\$20 million) is the most fiscally responsible option. However, pursuing this option requires that the three towns find an equitable division of the \$20 million cost.

What division of costs would incentivize each of the towns to engage in this joint venture? One possible division of costs is to divide the costs equally, i.e., each town pays \$20/3 million. At first glance, it would appear that this division costs would be in the core as \$20/3 mission is less than each town's individual operating costs, e.g., town B prefers to pay \$20/3 million over its individual opportunity costs of \$7 million. However, alternative arrangements could also be pursued where only two of the three towns pursue a joint venture, e.g., towns B and C could collectively engage in a joint venture. Note that if costs are divided equally, towns B and C would collectively pay a combined \$40/3 million; however, towns B and C could rather pursue their own joint venture which would incur a total cost of \$13 million $<$ \$40/3 million. Hence, an equal division of costs is not in the core!

A division of costs that incentivizes the three towns to pursue a joint ventures, i.e., is in the core, must satisfy the following:

- *The costs for towns A, B, and C totals \$20 million*
- *The costs for town A $\leq$ \$11 million*
- *The costs for town B $\leq$ \$7 million*
- *The costs for town C $\leq$ \$8 million*
- *The cumulative costs for towns A and B $\leq$ \$15 million*
- *The cumulative costs for towns A and C $\leq$ \$14 million*
- *The cumulative costs for towns B and C $\leq$ \$13 million*

The following figure highlights these constraints and the resulting core for this three town cost sharing problem. To understand this plot, consider the line connecting the A and C vertices. This line identifies all possible cost shares where town B pays \$0. All parallel lines to this (A,C) line represents divisions where B pays a constant amount, and the amount B pays increases as we get closer to the B vertex.



2 Model

The model pertaining to any cost sharing problem involves a player set $N = \{1, 2, \ldots, |N|\}$ and a function $c : 2^N \to \mathbb{R}$ that defines the opportunity costs for any set of players $S \subseteq N$.[1] For the three town example given above, we have $N = \{A, B, C\}$ and $C(\{A\}) = 11$, $c(\{B\}) = 7$, $c(\{C\}) = 8$, $c(\{A, B\} = 15$, $c(\{A, C\}) = 13$, $c(\{B, C\}) = 10$, and $c(\{A, B, C\}) = 20$. The goal of any cost sharing problem is to define a cost sharing rule that promotes efficient outcomes. A cost sharing rule is defined as follows:

Definition 2.1 (Cost Sharing Rule) *A cost sharing rule is a function $CS : 2^N \to \mathbb{R}^n$ that allocates the total cost of a venture among the members of a group for every possible group of players $S \subseteq N$, i.e., for any set of players $S \subseteq N$ the cost sharing rule satisfies*

$$\sum_{i \in S} CS(i, S) = c(S)$$

where $CS(i, S)$ represents the cost share of player i in group S. We refer to the cost shares of all players $i \in S$ as an allocation.

The goal of a cost sharing problem is to determine the set of allocations in the core, as was done in the above two examples. The core represents the set of allocations such that no individual, or group of individuals, pays more than its opportunity cost. In the two player example, there were 2 constraints on the allocation needed for the core. In the three player example, there were 6 constraints on the allocation needed for the core. In general, there are $2^{|N|} - 2$ constraints which is exponential in the number of individuals $|N|$. Exhaustively checking all of these constraints is computationally prohibitive and intractable, hence more

[1] Recall that the notation 2^N denotes the *power set of* N, i.e., the set of all subsets of N. For example, if $N = \{A, B, C\}$ then $2^N = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}$.

general arguments need to be developed for ensuring the existence (or lack thereof) with regards to the core. Clearly, any results along this direction are going to intimately hinge on the structure of the opportunity costs c.

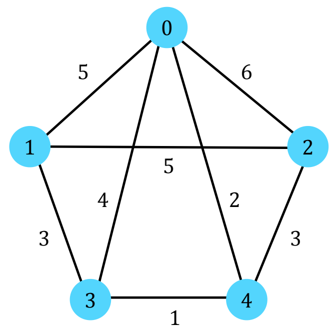
3 Minimum Spanning Tree Games

In general, the core could be either empty or non-empty, and there are advanced results which can be employed to answer this question (for example, the concept of balanced games and the results of Bondareva and Shapley). Instead of introducing these concepts here, we will study a specific class of problems for which the core is always non-empty. To that end, here consider the framework of minimum spanning tree games which focuses on the problem of distributing the infrastructure costs incurred by connecting a set of customers to a common resource. This problem appears in a variety of contexts, such as distributing costs in multicast transmissions.

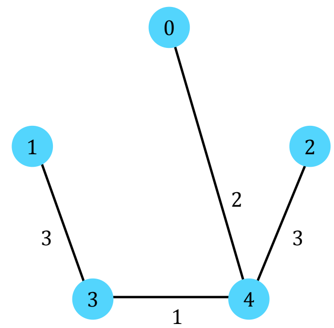
A minimum spanning tree game consists of a finite set of customers N that need to be connected either directly or indirectly to a single supplier, which we denote by 0. We define an interconnection graph with nodes $N^* = N \cup \{0\}$ and directed edges $E = N^* \times N^*$, where each edge $(i,j) \in E$ is associated with a given cost $c_{ij} \geq 0$. Here, we adopt the convention that edge (i,j) is a directed edge pointing from i to j. A spanning tree of the above interconnection graph is defined as a collection of edges $E_N \subseteq E$ such that there is a unique path from every node $i \in N$ to the source $\{0\}$ in the edge set E_N. More formally, from every node $i \in N$ there is a unique sequence of nodes $i = i_0, i_1, ..., i_k = \{0\}$ such that $i_0, \ldots, i_{k-1} \in N$ and $(i_x, i_{x+1}) \in E_N$ for all $x \in \{0, \ldots, k-1\}$. Accordingly, any spanning tree E_N will have exactly $|N|$ edges with a unique edge leaving each node N.

We will call a spanning tree E_N a minimum spanning tree if there does not exist another collection of edges E'_N that form a spanning tree with a lower total cost, i.e., $\sum_{(i,j)\in E_N} c_{ij} \leq \sum_{(i,j)\in E'_N} c_{ij}$ for any spanning tree E'_N. Lastly, we can also define a spanning tree and minimum spanning tree associated with any coalition $S \subseteq N$ in the same fashion, where we restrict attention to nodes in S and edges in $(S \cup \{0\}) \times (S \cup \{0\})$.

An example of a minimum spanning tree is highlighted below. Here, there are four individuals $N = \{1, 2, 3, 4\}$ and the relevant (undirected) edge costs c_{ij} are highlighted in the figure on the left. As an example, $c_{30} = c_{03} = 4$ and $c_{24} = c_{42} = 3$. The edges that are not explicitly highlighted can be thought of as having infinite cost, e.g., $c_{23} = c_{32} = \infty$. Clearly, in any minimum spanning tree game the minimum spanning tree for the full set of individuals N (also referred to as the grand coalition) is the graph that meets all the individual demands at the lowest total cost. However, are there allocations that incentivize this outcome? In other words, is the core guaranteed to be non-empty for any minimum spanning tree game? The following theorem provides the answer to this question.



Minimum Spanning Tree Game



Minimum Spanning Tree

Theorem 3.1 *The core of any minimum spanning tree game is non-empty.*

The proof of this theorem is quite straightforward and amounts to showing that a particular allocation structure is always in the core, hence the core must be non-empty. We construct these cost shares using the structure of the minimum spanning tree for the set N, which we denote by E_N^*. For each individual $i \in N$ there exists a single directed edge $(i, j) \in E_N^*$ where $j \in N \cup \{0\}$. Accordingly, we define the cost share of each individual $i \in N$ as the cost of this outgoing edge (i, j), i.e., $CS(i, N) = c_{ij}$. Accordingly, for the above minimum spanning tree game we have

$$\begin{aligned} CS(1, N) &= 3, \\ CS(2, N) &= 3, \\ CS(3, N) &= 1, \\ CS(4, N) &= 2. \end{aligned}$$

Note that we have $\sum_{i \in N} CS(i, N) = \sum_{(i,j) \in E_N^*} c_{ij}$ as the cost of all edges is accounted for. Further, we denote the latter sum by $c(E_N^*)$ for notational simplicity.

The following proof demonstrates that these cost shares are in the core. Proving this requires showing that the minimum spanning tree for any coalition $S \subseteq N$, denoted by E_S^*, must satisfy

$$\sum_{(i,j) \in E_N^* : i \in S} c_{ij} = \sum_{i \in S} CS(i, N) \leq \sum_{(i,j) \in E_S^*} c_{ij}, \tag{1}$$

meaning that it is the best interest of coalition S to stay in the grand coalition N. We will show this by contradiction. Suppose that

$$\sum_{(i,j) \in E_N^* : i \in S} c_{ij} > \sum_{(i,j) \in E_S^*} c_{ij}, \tag{2}$$

and consider the collection of edges

$$E_N' = E_N^* - \{(i, j) \in E_N^* : i \in S\} \cup E_S^*$$

which entails removing the outgoing edges of each individual $i \in S$ in the graph E_N^* and replacing them with the edges in the minimum spanning tree E_S^*. Given (2), we know that $c(E'_N) < C(E_N^*)$.

We will now complete the proof by arguing that the graph E'_N is in fact a spanning tree for the set N; hence, $c(E'_N) \geq c(E_N^*)$ since E_N^* is the minimum spanning tree for the set N. This provides our contradiction and ensures that (2) can never be satisfied, implying that the proposed cost share is in fact in the core. Showing that the graph E'_N is a spanning tree involves arguing that each node $i \in N$ has a path to the source $\{0\}$ in the tree E'_N. First, note that if $i \in S$ then this is immediate because there is a path to the source $\{0\}$ in the set $E_S^* \subseteq E'_N$. Next, if $i \notin S$ then there are two options: (i) the path to the source in E_N^* did not contain any nodes in S or (ii) the path to the source in E_N^* did contain nodes in S. If (i), then there is still a path from i to $\{0\}$ in the graph E'_N as this is the same path as in E_N^* since none of those edges were removed. Alternatively, if (ii) then there exists a path to some node in $k \in S$. However, since E_S^* is a minimum spanning tree for the nodes S, this means that there exists a path from k to $\{0\}$ in E'_N; hence, there is also a path from i to $\{0\}$ in E'_N. Accordingly, E'_N is a spanning tree for the set N which completes the proof.

4 Conclusions

In this chapter we introduced the problem of cost sharing and demonstrated that finding cost shares in the core (if they exist) is non-trivial. While this lecture focused more on the concept of core and whether or not it is non-empty, next lecture will focus more directly on identifying mechanisms for generating cost shares that are in the core for certain problem instances.

5 Exercises

1. Consider the following cost sharing problem:

   - Player set: $N = \{1, 2, 3\}$
   - Opportunity costs: $c : 2^N \to R$

   $$\begin{array}{rrr} c(\{1\}) = 9, & c(\{2\}) = 8, & c(\{3\}) = 9 \\ c(\{1,2\}) = 14, & c(\{1,3\}) = 15, & c(\{2,3\}) = 13 \\ c(\{1,2,3\}) = 21 & & c(\emptyset) = 0 \end{array}$$

   (a) Identify the core graphically.

   (b) Is the core nonempty? If so, provide an allocation in the core.

2. Consider the example of the minimum spanning tree game considered in this lecture. For each coalition $S \subseteq \{1, 2, 3, 4\}$ determine the minimum spanning tree and resulting

cost. Furthermore, define the proposed allocation (i.e., cost shares) for each coalition S. Note that for each coalition S the accompanying minimum spanning tree will have exactly $|S|$ edges.

3. A scientist has been invited for consultation at three distant cities. In addition to her consultation fees, she expects travel compensation. But since these three cities are relatively close together, travel expenses can be greatly reduced if she accommodates them all in one trip. The problem is how to decide how the travel expenses should be split among her hosts in the three cities. The one-way travel expenses among these three cities, A, B, and C, and her home base, H, are given as follows:

   - Between H and A, cost = 7
   - Between H and B, cost = 9
   - Between H and C, cost = 6
   - Between A and B, cost = 2
   - Between A and C, cost = 4
   - Between B and C, cost = 4

   Assume that the value of the visit is the same for each of the hosts, say 20 units each. Set the problem up as a three-player cost-sharing game by specifying the opportunity cost function.