Last note set: Introduction to transmission lines

1. Transmission lines are a linear system - superposition can be used
2. Wave equation permits forward and reverse wave propagation on lines

$$V(x,t) = V^{+}(t - x/v) + V^{-}(t + x/v)$$

$$I(x,t) = \frac{1}{Z_0}[V^{+}(t - x/v) - V^{-}(t + x/v)] + I_{DC}$$

$$v = \frac{1}{\sqrt{LC}} \quad \textit{phase velocity}$$

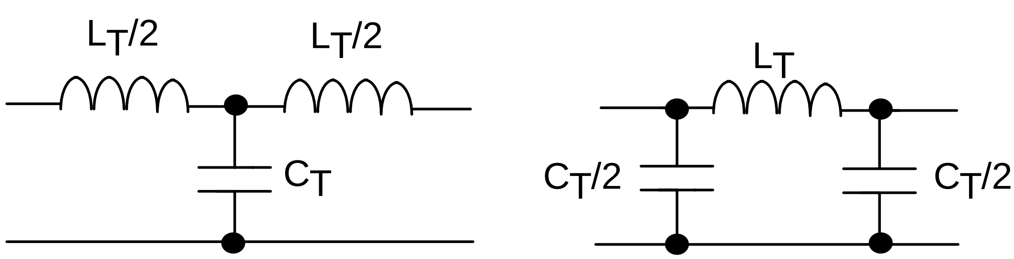
$$Z_0 = \sqrt{L/C} \quad \textit{characteristic impedance}$$

(L, C per unit length)

3. Reflections occur at discontinuities in impedance. A reflection coefficient Γ can be defined.

$$\Gamma = \frac{V^{-}}{V^{+}} = \frac{R_L/Z_0 - 1}{R_L/Z_0 + 1}$$

4. Transmission line can be represented by an equivalent circuit:



$L_T = Z_0\tau$ $\qquad C_T = \tau/Z_0$ $\qquad \tau$ = length/v = ***time of flight***

Goals: Understand frequency domain analysis of transmission lines

- phasor notation
- position dependence - phase constant β
- reflections
- movement of reference plane
- impedance and Γ variation with position
- Smith Chart

Transmission Lines in the Frequency Domain and the Smith Chart

Time domain analysis is intellectually clearer, the picture being forward and reverse waves propagating, reflecting, and re-reflecting. This analysis becomes intractable as soon as we introduce reactive impedances as multiple convolutions will be required for time-domain reflection analysis.

So, we will analyze in the frequency domain instead. Frequency domain analysis of transmission lines is a classical approach to this problem.

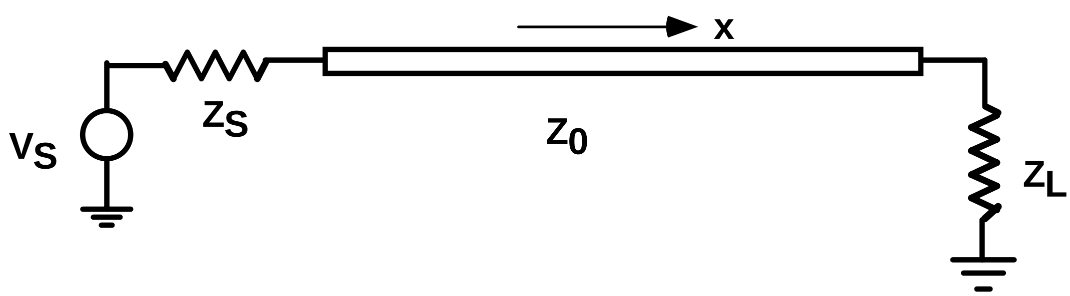


FIG. 1. Transmission line with impedance Z_o connects the source with Thevenin equivalent generator impedance Z_s to a load Z_L.

$$\boldsymbol{V_s} = \mathrm{Re}\left\{\boldsymbol{V_o e^{j\omega t}}\right\}$$
$$\boldsymbol{V_o} = |\boldsymbol{V_o}|\boldsymbol{e^{j\phi_o}}$$
so,
$$\boldsymbol{v_s(t)} = |\boldsymbol{V_o}|\cos(\boldsymbol{\omega t + \phi_o})$$

On a transmission line at position *x*, waves travel in time as $x \pm vt$ where velocity *v* is the phase velocity. Equivalently, at a time *t*, waves vary with position *x* according to $t \pm x/v$.

Thus, we can represent voltage on the line, v(x,t) as:

$$\cos(\omega t + \phi_o) \longrightarrow \cos[\omega(t \pm x/v) + \phi_o]$$
$$= \cos[\omega t + \phi_o \pm x(\omega/v)]$$
$$= \cos[\omega t + \phi_o \pm \beta x]$$

where β is the *phase (or propagation) constant* $\beta = \omega/v = 2\pi/\lambda$. β converts distance to radians. Here, λ is the wavelength. The sketches below illustrate the concept. Here, the function cos ($\omega t - \beta x$) is plotted.

1. Let's suppose that <u>$t = t_0$</u> so that the wave appears frozen in time on the x axis. If distance $x_2 - x_1 = \lambda$ as shown, the corresponding phase change over this distance is 2π.

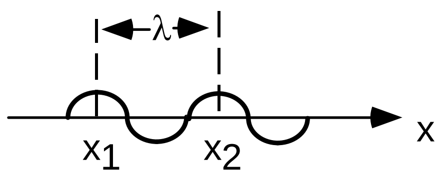


FIG. 2A.

2. Now, if we plot this same cosine function as a function of βx, we see that the wavelength is equal to a phase $\beta x = 2\pi$ since $\lambda = 2\pi/\beta$.

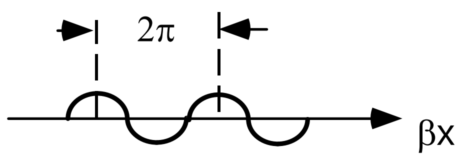


FIG. 2B.

3. Also, we can set <u>$x = 0$</u> and observe the wave function in the time domain for increasing time. In this case, one period requires a time interval $T = 2\pi/\omega = 1/f$ as shown.

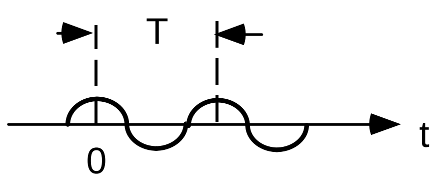


FIG. 2C.

4. Finally, you might ask why the cos ($\omega t - \beta x$) wave is the forward (to the right) travelling wave direction. To see why, track a point of constant phase with position on the x axis as time progresses from t_1 to t_2.

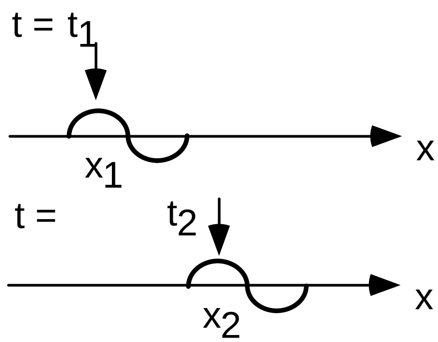


FIG. 3. Plot of wave propagation in the forward direction at times t_1 and t_2.

We can see that $t_2 > t_1$ and $x_2 > x_1$ for a forward direction of travel, and that the cosine function will have the same value at points of constant phase. Therefore,

$$\cos(\omega t_1 - \beta x_1) = \cos(\omega t_2 - \beta x_2) \qquad [1]$$

Thus,

$$\omega t_1 - \omega t_2 = \beta x_1 - \beta x_2 < 0 \qquad [2]$$

from the drawing. A forward wave must have a positive phase velocity.

$$\frac{\omega}{\beta} = v > 0 \qquad [3]$$

From eq. [2],

$$v = \frac{x_2 - x_1}{t_2 - t_1} > 0\,. \qquad [4]$$

x_2 must be greater than x_1, therefore, the wave is travelling in the forward direction for cos ($\omega t - \beta x$).

Of course, these waves can also be described by **complex exponentials**:

$$|V_O| e^{j\omega t} \quad e^{j\phi_o} \quad e^{\pm j\beta x}$$

(sine wave) (phase) (position along line)

The $e^{j\omega t}$ time dependence is always taken as implicit and is frequently omitted when using phasor notation.

$$|V_O| \quad e^{j\phi_o} \quad e^{\pm j\beta x}$$

Voltage and Current on Transmission Lines

Now, we can use this notation to describe the voltage and current on a transmission line at any location on the line.

$$V(x) = V^{+}(x) + V^{-}(x) = V^{+}(0)e^{-j\beta x} + V^{-}(0)\, e^{+j\beta x}$$

$$\begin{aligned} I(x) &= \frac{1}{Z_0}[V^{+}(x) - V^{-}(x)] \\ &= \frac{1}{Z_0}[V^{+}(0)e^{-j\beta x} - V^{-}(0)\, e^{+j\beta x}] \end{aligned}$$

Reflections in the frequency domain:

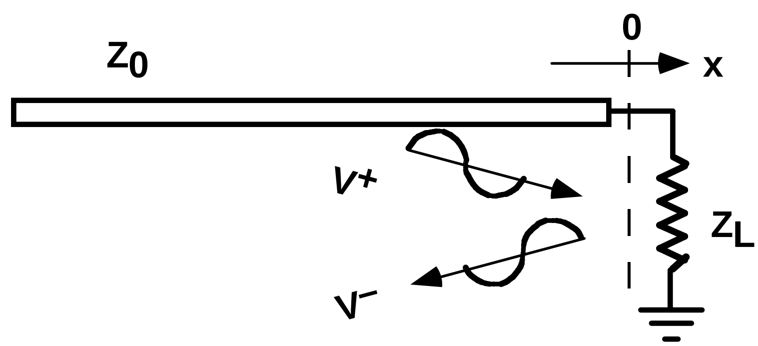


FIG 4. Reflection from a mismatched load impedance Z_L

The forward wave is travelling in the positive x direction and reflects from the load Z_L. We set x=0 at the load end of the transmission line as our reference plane. In frequency domain analysis, we assume that the wave amplitudes are steady state values.

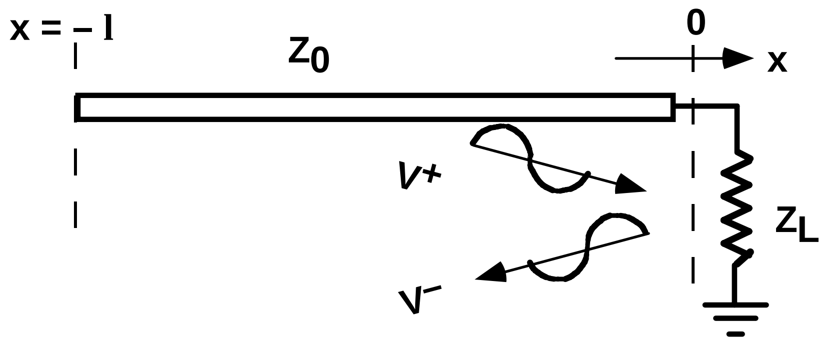
From the definition of reflection coefficient,

$$V^-(0) = V^+(0)\Gamma_L$$

$$\Gamma_L = \frac{z_L - 1}{z_L + 1}$$

where z_L is the normalized load impedance Z_L/Z_0. Γ_L is in general complex.

Movement of reference plane:



Now, we must determine voltage and current on the line as a function of position. This is often referred to as moving the reference plane. Here, we move from x = 0 at the load to $x = -\ell$ at the left end of the drawing.

$$V(x) = V^+(x) + V^-(x) = V^+(x)[1 + \Gamma(x)]$$

where

$$\Gamma(x) = \frac{V^-(x)}{V^+(x)}$$

is the position-dependent reflection coefficient. Substituting for $V^-(x)$ and $V^+(x)$,

$$\Gamma(x) = \frac{V^-(0)e^{j\beta x}}{V^+(0)e^{-j\beta x}} = \Gamma(0)e^{2j\beta x}$$

and

$$V(x) = V^+(0)e^{-j\beta x}[1 + \Gamma(0)e^{2j\beta x}] .$$

From this we can see that the reflection coefficient at position $-\ell$ from the load is given by

$$\Gamma(-l) = \Gamma(0)e^{-2j\beta l}$$

and the reflection coefficient goes through a phase shift of

minus **2π (2)(l/λ)** radians

minus **2βl** radians

minus **360 (2)(l/λ)** degrees

The reflection coefficient changes phase with position

Some observations:

1. Γ(x) is periodic.

$$\Gamma(x) = \Gamma(0)e^{2j\left(\frac{2\pi}{\lambda}\right)x}$$

$4\pi x/\lambda = 2\pi$ whenever $x = n\,\lambda/2$ where n is an integer.

2. V(x) is also periodic.

$$V(x) = V^{+}(0)e^{-j\beta x}\left[1+\Gamma(0)e^{2j\beta x}\right]$$

=1 every nλ   =1 every nλ/2
= -1 every nλ/2   = -1 every nλ/4

<u>**Example:**</u> Let Γ(0) = -1 (short circuit)

x	βx	Γ(x)	V(x)
0	0	-1	0
-λ/4	-π/2	1	$2jV^{+}(0) = 2V^{+}(0) \angle\ 90^{o}$
-λ/2	-π	-1	$0\ \angle 180^{o}$

<u>The result:</u> Standing wave pattern in voltage.

- Voltage maxima and minima are separated by λ/4
- Successive maxima are separated by λ/2

Voltage Standing Wave Ratio (VSWR)

We have seen that in general, there are two waves travelling in opposite directions on a transmission line. We also saw that the variation of voltage along the line at position x due to the sum of these two waves is given by:

$$\boldsymbol{V(x) = V^{+}(0)e^{-j\beta x}[1 + \Gamma(0)e^{2j\beta x}]} .$$

The magnitude is given by:

$$|V(x)| = |V^{+}(0)|\ |1 + \Gamma(0)e^{2j\beta x}|$$

So, we can easily determine the minimum and maximum voltage magnitude that will be found along the transmission line at some position x:

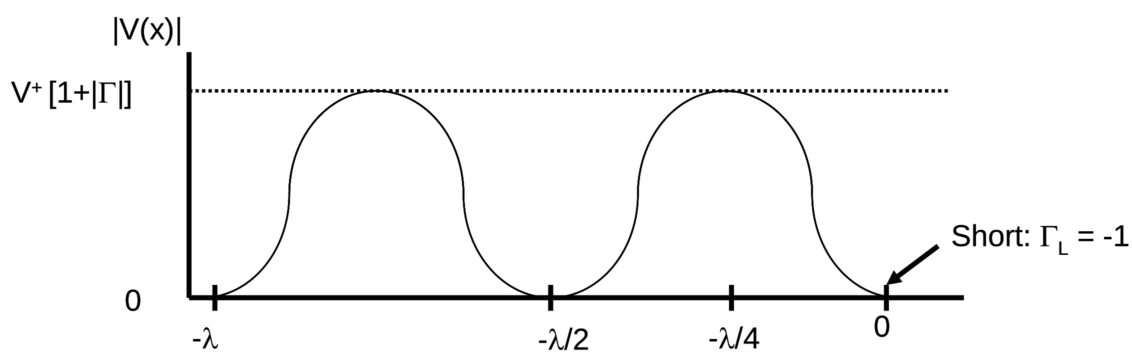
$$|V(x)|_{max} = |V^{+}(0)|\ (1 + |\Gamma(0)|)$$

$$|V(x)|_{min} = |V^{+}(0)|\ (1 - |\Gamma(0)|)$$

Taking the ratio of max to min gives us the VSWR:

$$VSWR = \frac{|V(x)_{max}|}{|V(x)_{min}|} = \frac{1 + |\Gamma(0)|}{1 - |\Gamma(0)|}$$

An open or short circuited line will give us an infinite VSWR because the minimum voltage on the line is zero; $|\Gamma(0)| = 1$ for both cases.



Why is VSWR important? It is often used as a specification for an amplifier.

Impedance vs. Position

Refer to the same picture. The impedance at any point on the line can be found from the current and voltage or equivalently from the reflection coefficient:

$$Z(x) = \frac{V(x)}{I(x)} = \frac{[V^+(x) + V^-(x)]}{\left[(V^+(x) - V^-(x)) \big/ Z_0\right]} = Z_0 \frac{1+\Gamma(x)}{1-\Gamma(x)}$$

The normalized impedance at any point is easily found.

$$z(x) = Z(x)/Z_0 = \frac{1+\Gamma(x)}{1-\Gamma(x)}$$

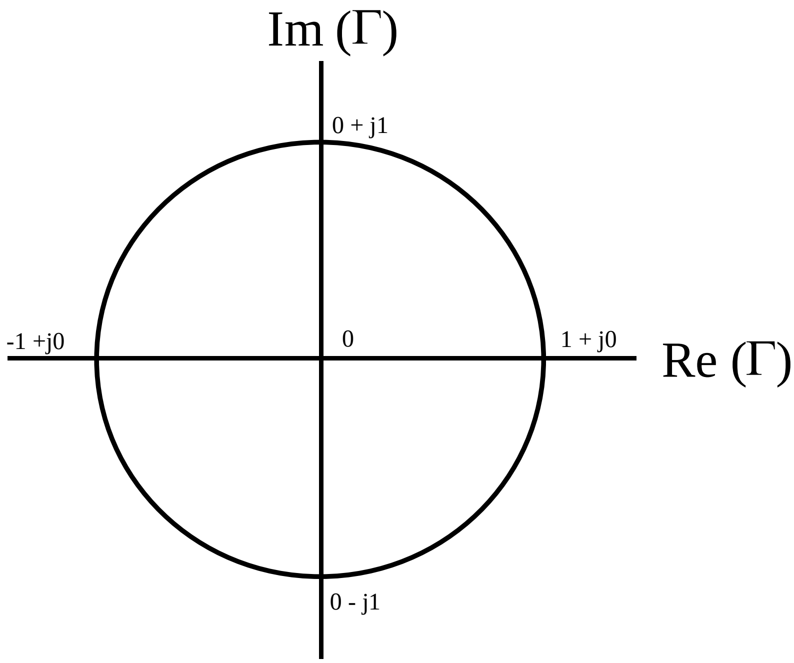
So, impedance depends on the position along the transmission line.
While this is conceptually simple, there is a lot of math involved. This can become tedious. So, we could benefit from a graphical representation - this is called the **Smith Chart** after Philip Smith who invented this convenient graph of transmission line parameters back in the 1930's.

The relationship for the normalized impedance, $z(x)$, is the key to the Smith Chart. The Smith Chart is just a polar plot of reflection coefficient. Impedance is determined by

$$z = \frac{1+\Gamma}{1-\Gamma}$$

It is a one-to-one mapping between complex numbers Γ and z, and is in fact an analytic function and a conformal transformation. You can read about this in math books on complex analysis.

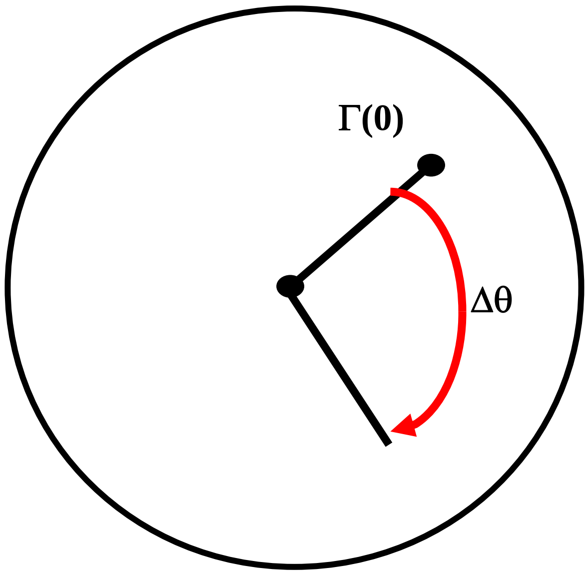
In the two-dimensional plane of Γ, the Γ plane, a reflection coefficient Γ is represented by a point.



As we move away from the load by a distance ℓ on the transmission line, Γ rotates by an angle Δθ where:

$$\Delta\theta = -2\beta\ell = -360^{o} \times \frac{2\ell}{\lambda}$$

One whole rotation is required in the Γ plane for each half-wavelength movement on the line.



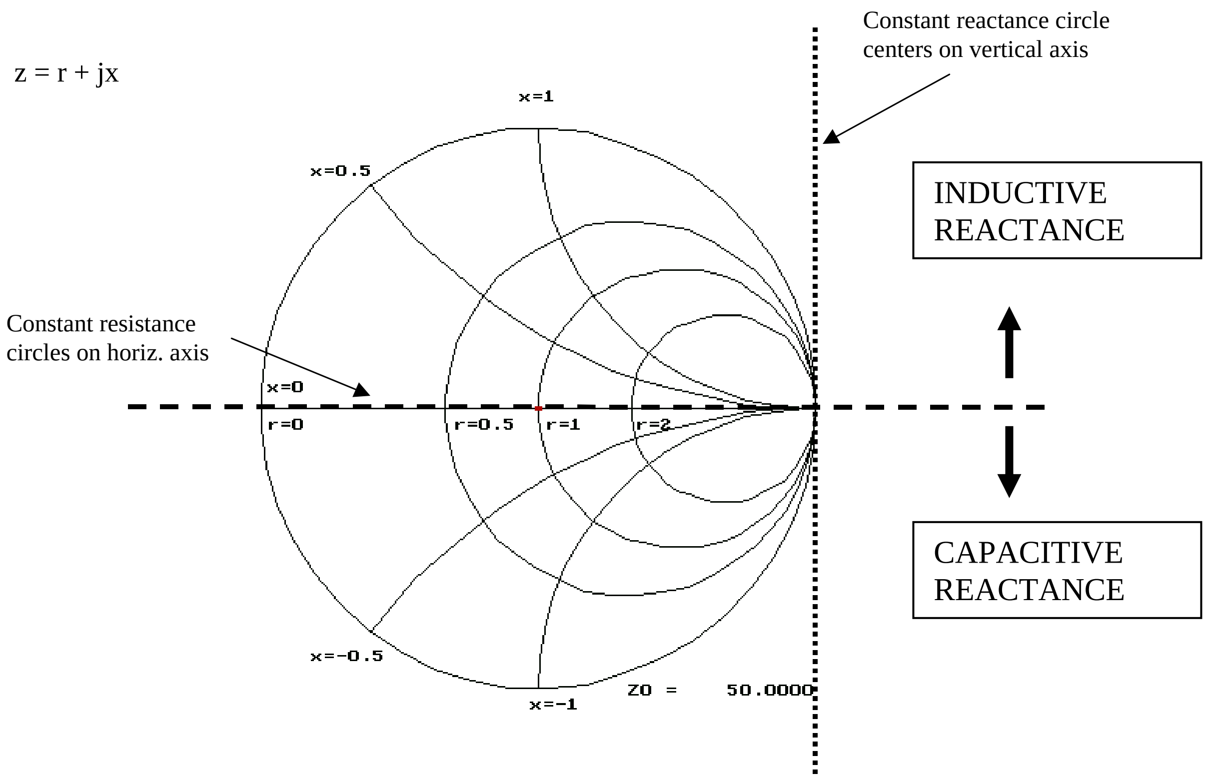
Note the following:

1. $\beta\ell$ is defined as "electrical length" in degrees.
2. The degree scale on the edge of the Smith Chart represents $2\beta\ell$, the angle of the reflection coefficient. Note that this is TWICE the electrical length.
3. Note that phase delay corresponds to a clockwise movement in angle.

Lines of constant resistance can be plotted on the chart through the use of

$$z = \frac{1+\Gamma}{1-\Gamma}$$

These will take the form of circles (See Gonzalez, Sect. 2.2) whose centers are on a line across the center of the chart as shown below.



Note that the units of r are normalized to Z0, in this case 50 ohms. So, the circle labeled r = 0.5 corresponds to the 25 ohm resistance circle in this case. But, Z0 can be any convenient real value. If Z0 were 100 ohms, r=0.5 would represent 50 ohms.

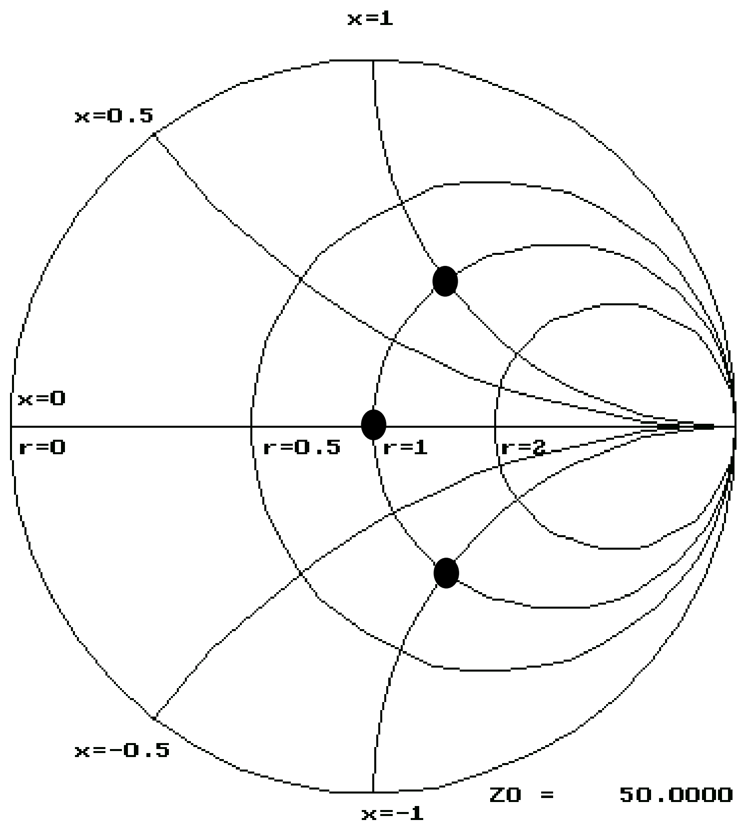
Similarly, the reactances x can be represented by circles. These have their centers on the vertical axis at the right edge of the chart. These are also normalized to Z0. Positive x corresponds to inductive reactance and is above the center line of the chart. Negative x represents capacitive reactance.

Given this mapping onto the Γ plane, we can associate any reflection coefficient (a point on the plane) with an impedance simply by reading the z coordinates of the point.

We can also associate the change of impedance with position with a rotation on the chart. Just rotate the Γ vector clockwise around the chart at the rate of one rotation for every half wavelength of movement on the line. Then read off the impedance directly from the chart.

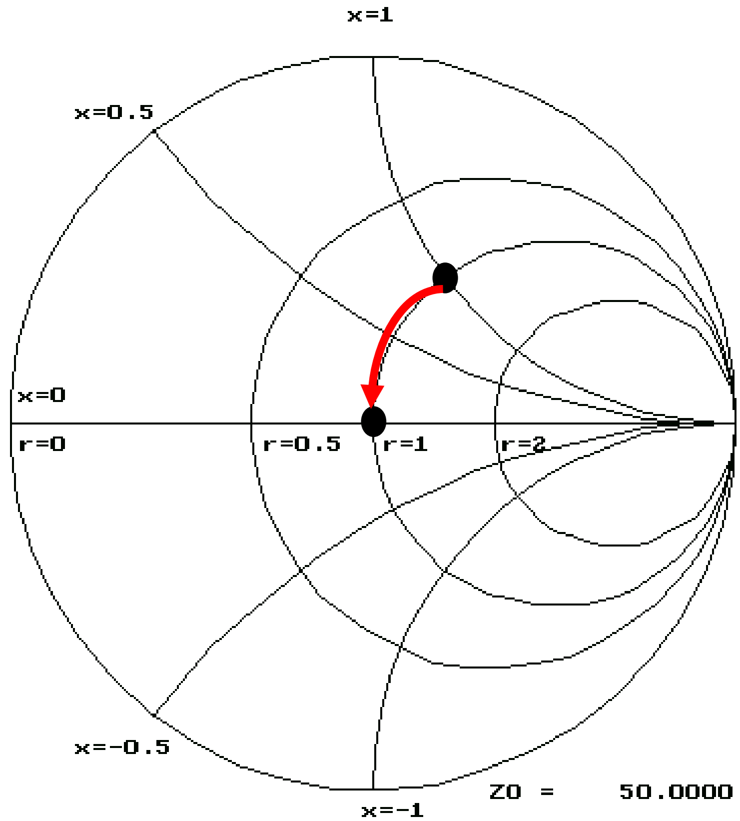
$$\Gamma(x) = \Gamma(0)e^{2j\beta x}$$

Note that all points on this chart represent series equivalent impedances.

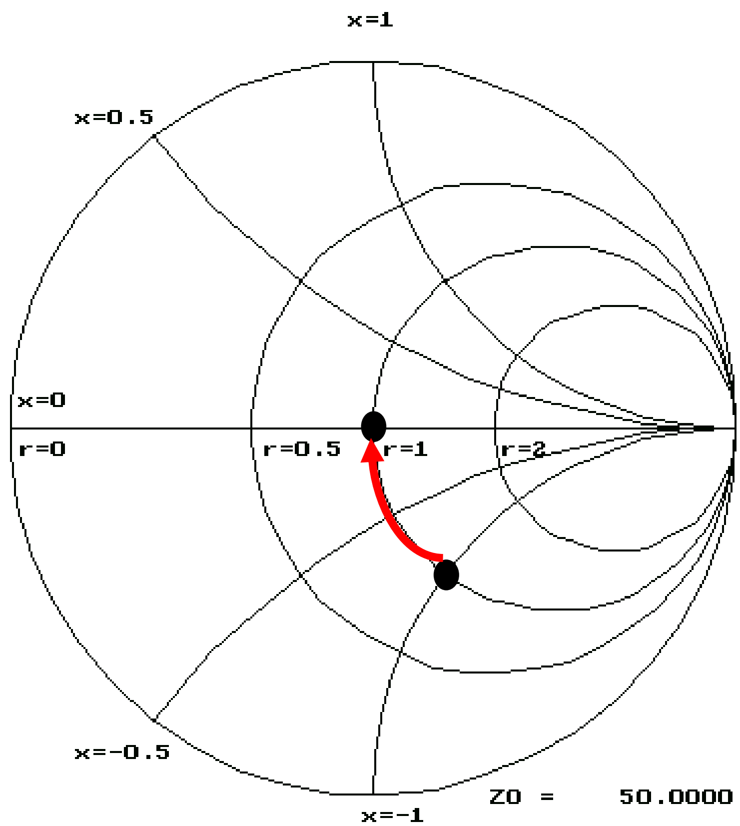


So, for example, z = 1 + j1 represents a series RL network. z = 1 - j1 is a series RC. z = 1 is a point right in the center of the chart.

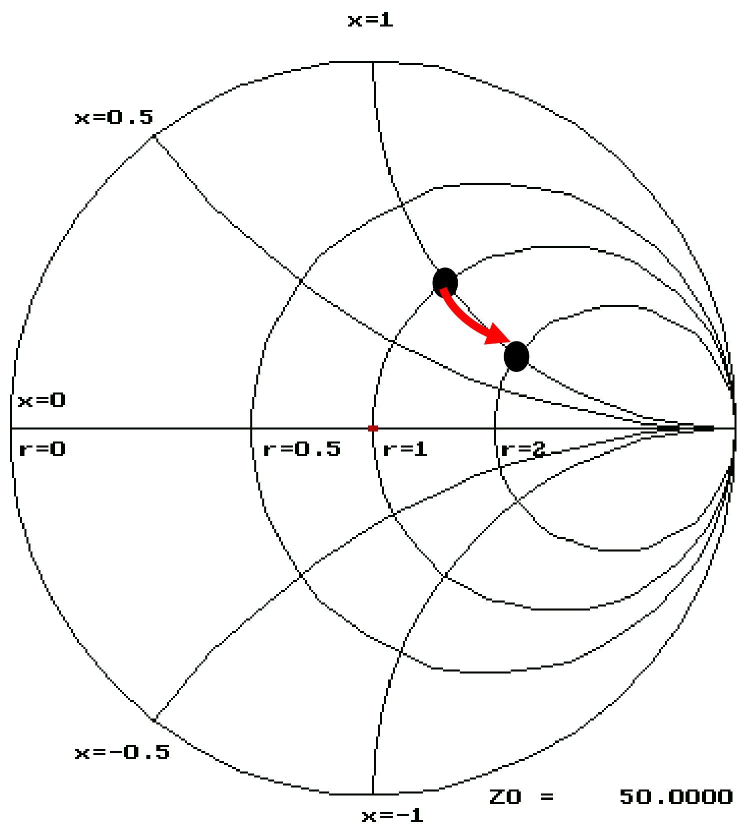
Normalization: Consider the point z = 1 + j1. If Z0 = 50, then Z = 50 +j50 when the impedance is denormalized. If Z0 were 1000 ohms, we would have Z = 1000 + j1000 for the same point. In this way, the chart can be used over an arbitrary range of impedance.



On this chart, we see an impedance 1 +j1 corresponding to the series RL network. If we add a capacitive reactance -j1 in series with this, the point will move along the constant resistance line r = 1 to the center. The reactance has been cancelled.



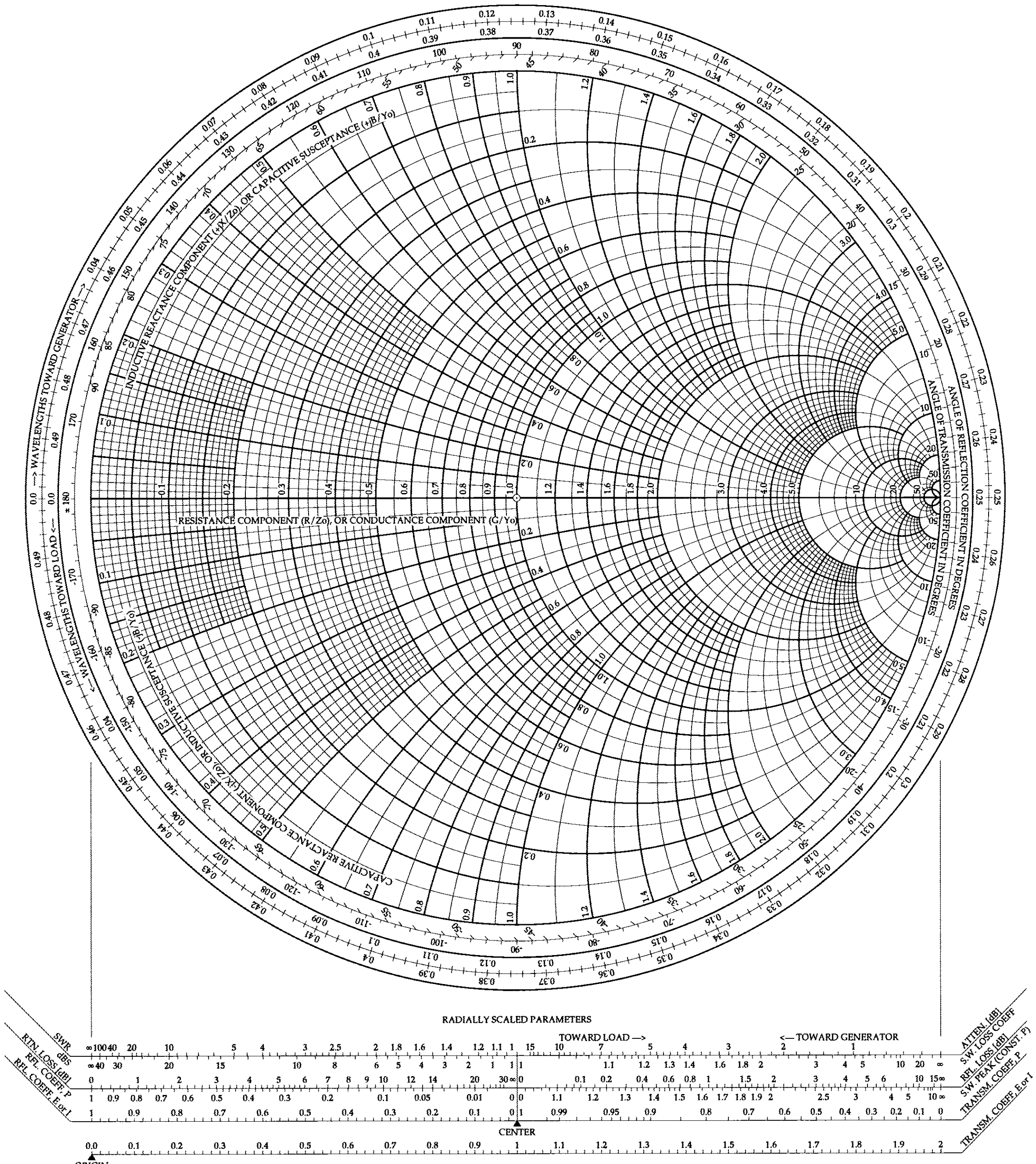
In this example, we have started with a series RC network, z = 1 - j1. By adding inductive reactance + j1 in series, we can again move along the r = 1 circle up to the center of the chart.



Series resistance. The chart can also represent the addition of resistance. This will correspond to movement along a circle of constant reactance. For example, if we begin with z = 1 + j1 again and add a normalized r = 1 to that, we arrive at z = 2 + j1.

The Complete Smith Chart

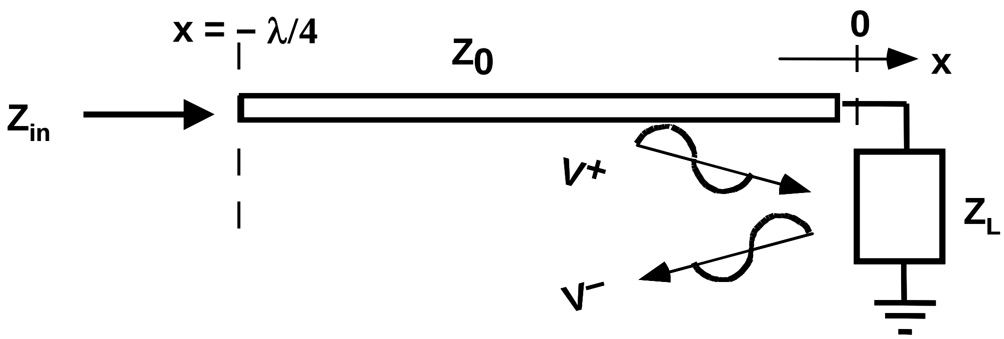
Black Magic Design



Let's look at some interesting examples.

Quarter wavelength transmission line.

How will impedances be transformed by this line?



$$\Gamma(x) = \Gamma(0)e^{2j\beta x}$$

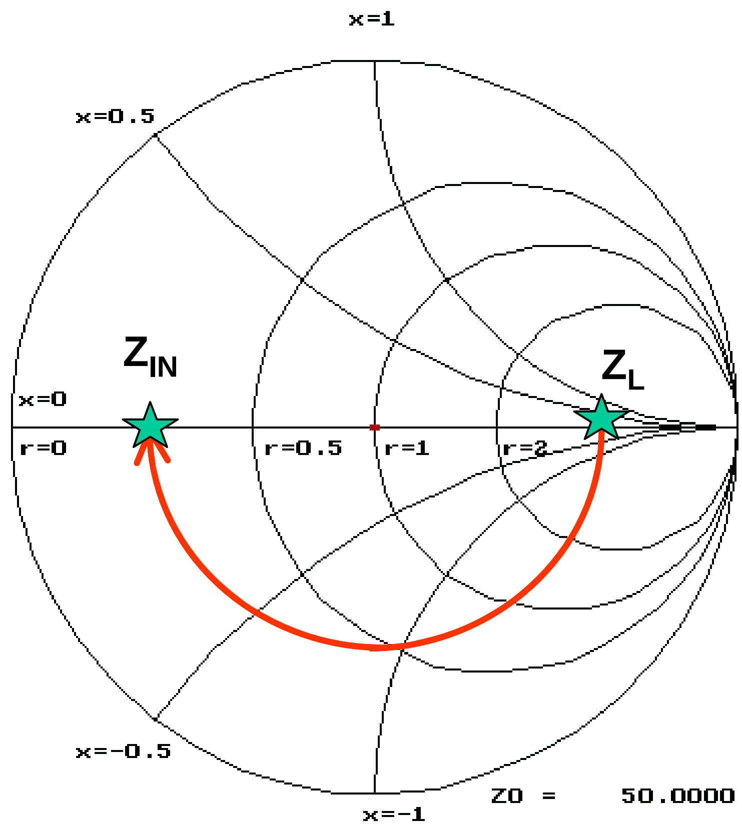
The reflection coefficient retains the same magnitude but changes phase with position. The variable x is defined as shown in the above diagram.

Let x = - λ/4. The propagation constant can be used to calculate the electrical length, βx

$$\beta = \frac{2\pi}{\lambda} \quad \text{so,} \qquad \beta x = -\pi / 2$$

Thus, $$\Gamma(-\lambda/4) = \Gamma(0)\, e^{-j\pi}$$

In other words, the angle of the reflection coefficient is rotated clockwise 180° for a quarter-wave line. (negative angle = clockwise rotation)



What does this do to the impedance?

$$\mathcal{Z}(-\lambda/4) = [1 + \Gamma(-\lambda/4)] / [1 - \Gamma(-\lambda/4)]$$

$$= [1 + \Gamma(0) \angle 180] / [1 - \Gamma(0) \angle 180]$$

1. We rotate 180^o on the Smith chart clockwise for the quarter wave line. This transforms a real Z_L into a real Z_{IN} in the example above.

Ex. $\mathcal{Z}_L = Z_L/Z_o = 200/50 = 4$. Rotating 180^o takes us to $Z_{IN} = 0.25$. This is 12.5 ohms when denormalized.

2. Or, using the above equation for normalized impedance, and Γ_L

$$\Gamma_L = \Gamma(0) = (\mathcal{Z}_L - 1)/(\mathcal{Z}_L + 1) = 3/5$$

$$\mathcal{Z}_{IN} = (1 + 0.6 \angle 180] / [1 - 0.6 \angle 180] = 0.25$$

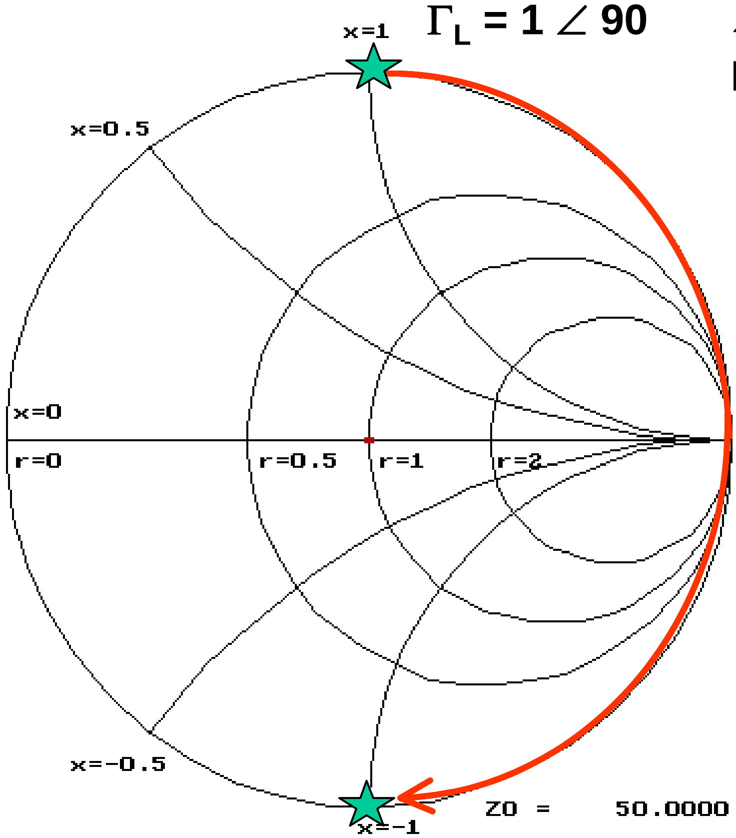
3. You can also use the formula below which is specific for the impedance transformation of a quarter wave line. This obviously is good for either normalized or unnormalized impedances.

$$Z_{IN} = Z_o^2 / Z_L$$

Clearly, the Smith Chart is the easiest to use.

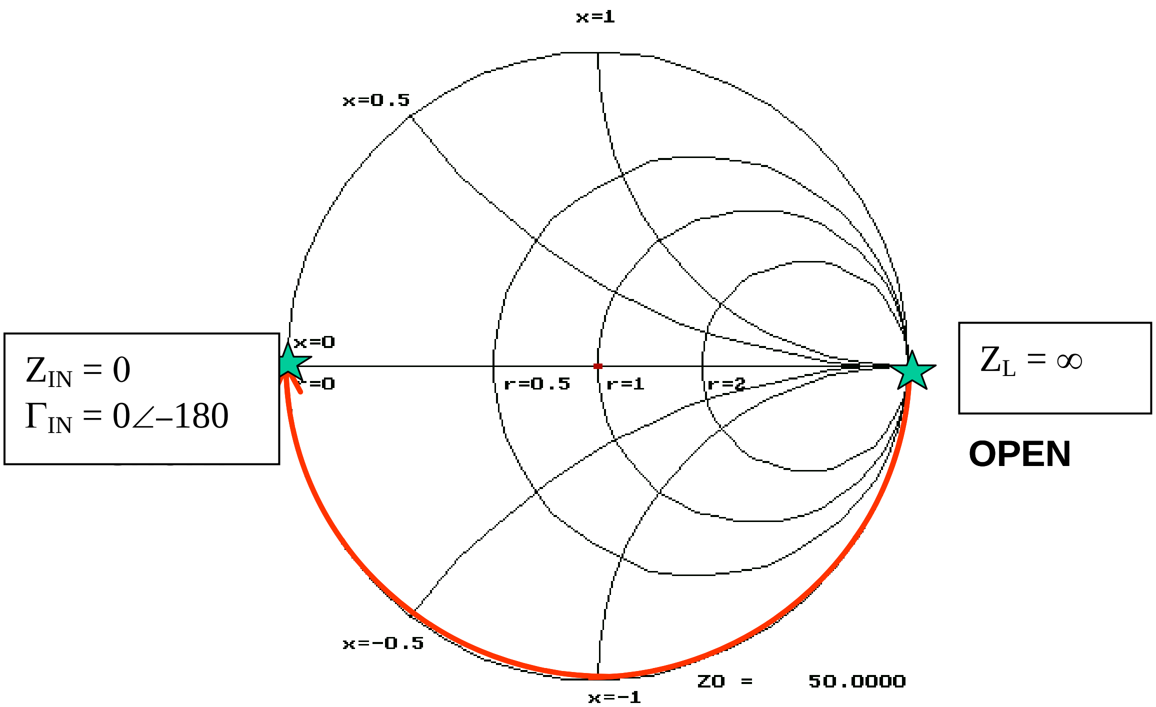
4. The Smith Chart also can be used with complex loads:

$\Gamma_L = 1 \angle 90$ $\mathcal{Z}_L = 0 + j1$
Pure inductor



$\Gamma_{IN} = 1 \angle -90$ $\mathcal{Z}_L = 0 - j1$
Pure capacitor

5. What about an open circuit at the load end?



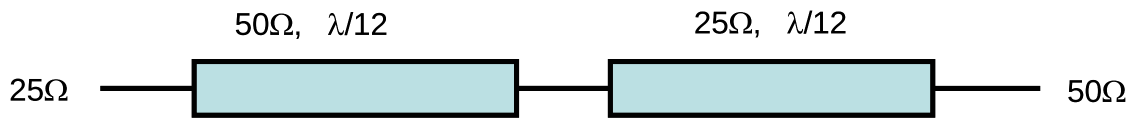
So, an open circuited quarter wave line produces a short circuit (when we ignore loss). Conversely, a shorted quarter wave line produces an open circuit.

Note that the electrical length of the line is a function of frequency, so these transformations will be exact only at the design frequency.

While the λ /4 line seems to do some magic as an impedance transformer, there are also applications in impedance matching networks which we will be investigating soon for lines of other lengths.

On the Zo Smith Chart, the angle of rotation through a transmission line of impedance Zo is always twice the electrical length: $-2\beta x$. Because the reference plane is defined at the load, and x is negative, the angle is positive (Clockwise rotation) when moving away from the load toward the generator.

Exercise: Use the Smith Chart to verify that the transmission line combination below will transform 50 Ω into 25 Ω.



Admittance Chart

We can also use the Smith Chart for admittances.

Impedance: $Z = R + jX$

Normalized Impedance: $\mathcal{Z} = r + jx$

Admittance: $Y = 1/Z = G + jB$

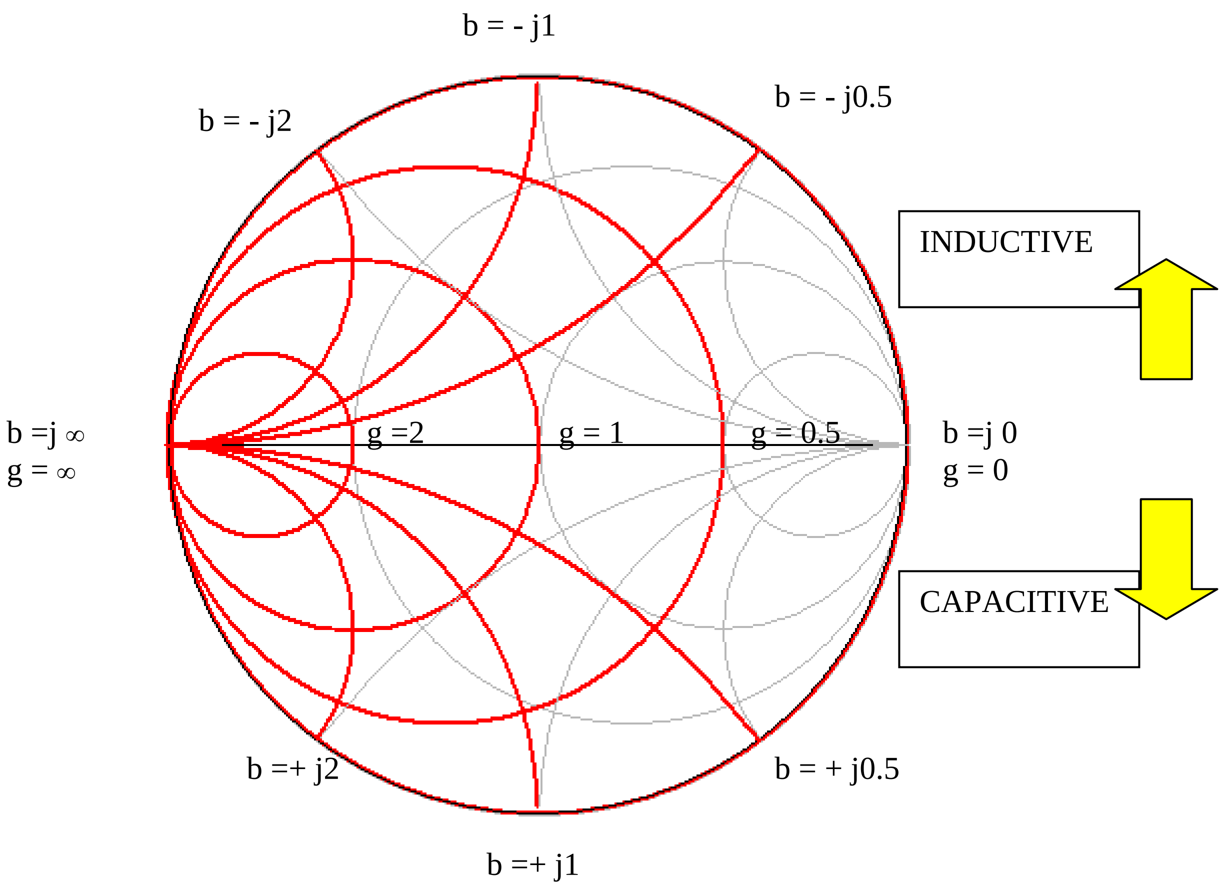
Conductance + j Susceptance

Normalized Admittance: $\mathcal{Y} = g + j\,b$ $\qquad \mathcal{Y} = Y/Yo$

We can plot the values of both r and x and g and b on the Γ plane by rotating the impedance coordinates by 180°. r and x then become constant conductance circles and constant susceptance circles. Since Y = 1/Z, taking 1/Z is equivalent to a 180° rotation in angle.

This is the impedance – admittance chart shown below. The admittance coordinates are the bold red lines.

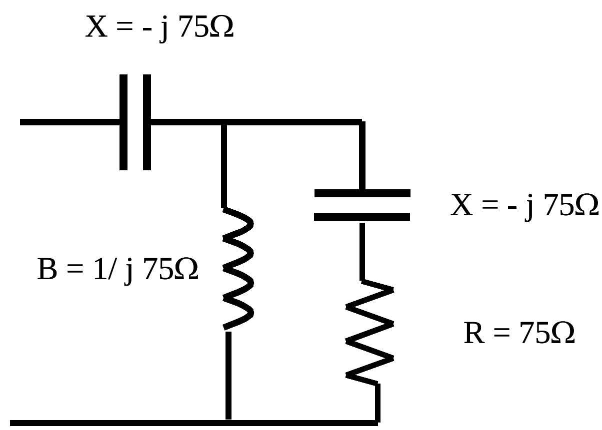
You can see from this that it is possible to use the chart to quickly translate between impedance and admittance. Every point on the chart can be interpreted in both ways.



The impedance Smith chart is convenient for evaluating the effect of adding components in series. The admittance chart is useful for components in parallel.

EXAMPLES

Example. Use the YZ Smith Chart to determine Zin of the circuit below.



1. Normalize the reactances and admittances to Zo and Yo.

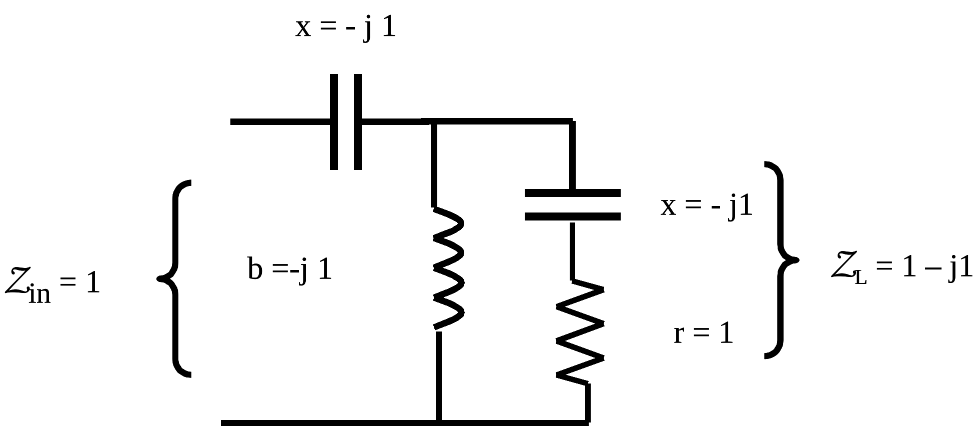
   What would be a good choice for Zo in this case?

2. Starting from the resistor, add capacitive reactance – j 1 to the impedance
3. Using the admittance chart, add susceptance – j 1
4. Going back to impedance, add another – j1.

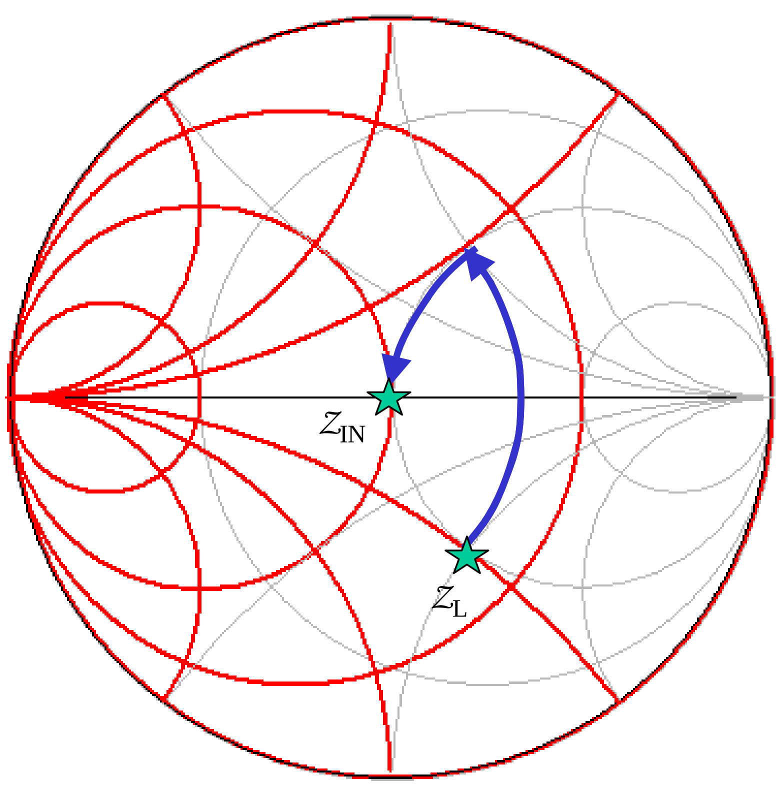
The result is Zin = 75 + j0

An important observation

The path that we follow on the Smith Chart will depend on the direction of the path through the circuit: load to source or source to load.



From our previous example, we proceeded from Z_L to Z_{in} by starting with $Z_L = 1 - j1$, taking into account the shunt admittance of the inductor by moving up on a constant conductance circle to the unit resistance circle, then down to the center through the series



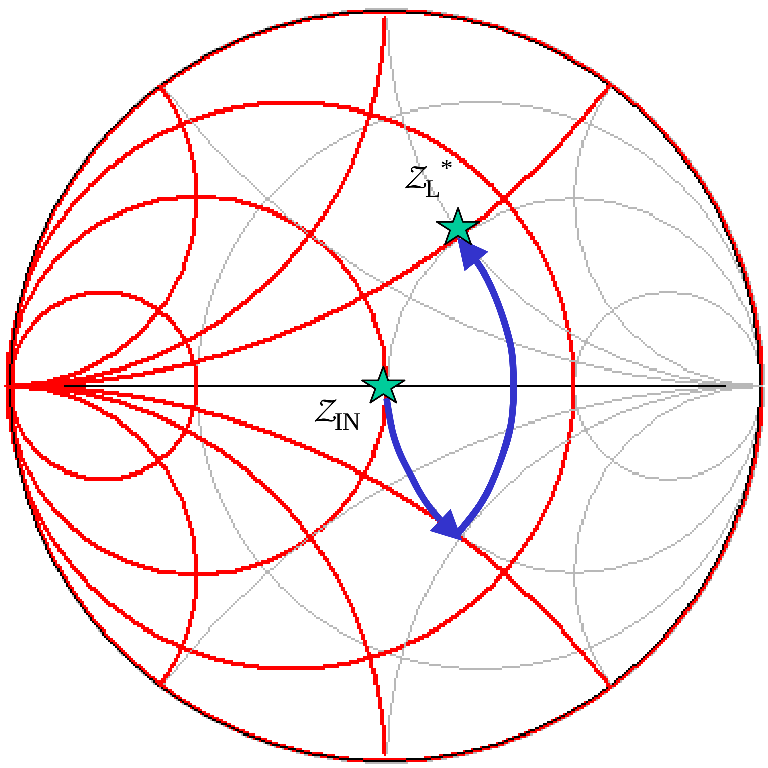
capacitor.

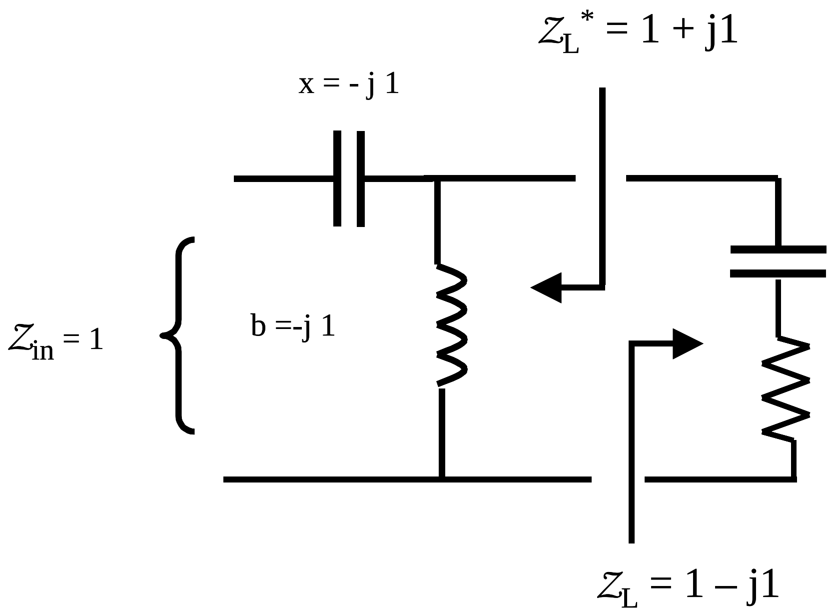
What happens when we go the other way?

1. Start at the center (Z_{IN}). Go down in reactance – j1.
2. Go up in susceptance –j1
3. We are now at Z_L^*

This is just a graphical version of the statement:

Match the load with its complex conjugate impedance or admittance





Beware: It is easy to get confused and design a network that will not match your load if you do not keep in mind this rule.