

Projective geometry- 2D

Acknowledgements

Marc Pollefeys: for allowing the use of his excellent slides on this topic

<http://www.cs.unc.edu/~marc/mvg/>

Richard Hartley and Andrew Zisserman, "[Multiple View Geometry in Computer Vision](#)"

Homogeneous coordinates

Homogeneous representation of lines

$$ax + by + c = 0 \quad (a, b, c)^T$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \quad (a, b, c)^T \sim k(a, b, c)^T$$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3 - (0,0,0)^T$ forms $\mathbf{P}^2$

Homogeneous representation of points

$$x = (x, y)^T \text{ on } l = (a, b, c)^T \text{ if and only if } ax + by + c = 0$$

$$(x, y, 1)(a, b, c)^T = (x, y, 1)l = 0 \quad (x, y, 1)^T \sim k(x, y, 1)^T, \forall k \neq 0$$

The point x lies on the line l if and only if $x^T l = l^T x = 0$

Homogeneous coordinates $(x_1, x_2, x_3)^T$ but only 2DOF

Inhomogeneous coordinates $(x, y)^T$

Points from lines and vice-versa

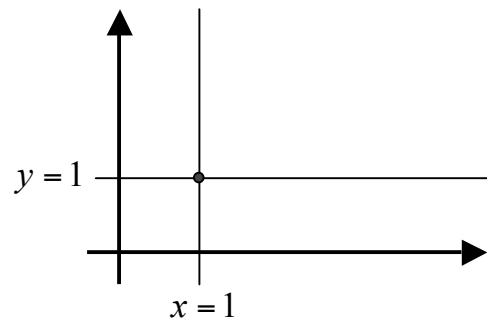
Intersections of lines

The intersection of two lines l and l' is $x = l \times l'$

Line joining two points

The line through two points x and x' is $l = x \times x'$

Example

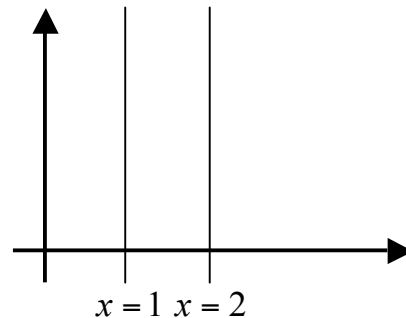


Ideal points and the line at infinity

Intersections of parallel lines

$$l = (a, b, c)^T \text{ and } l' = (a, b, c')^T \quad l \times l' = (b, -a, 0)^T$$

Example



Ideal points $(x_1, x_2, 0)^T$ Note that this set lies on a single line,

Line at infinity $l_\infty = (0, 0, 1)^T$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup l_\infty$$

Note that in $\mathbf{P}^2$ there is no distinction between ideal points and others

Summary

The set of ideal points lies on the *line at infinity*, $\mathbf{l}_\infty = (0, 0, 1)^T$

$\mathbf{l} = (a, b, c)^T$ intersects the line at infinity in the ideal point $(b, -a, 0)^T$

A line $\mathbf{l}' = (a, b, c')^T$ parallel to $\mathbf{l}$ also intersects $\mathbf{l}_\infty$ in the same ideal point, irrespective of the value of c' .

In inhomogeneous notation, $(b, -a)^T$ is a vector tangent to the line.

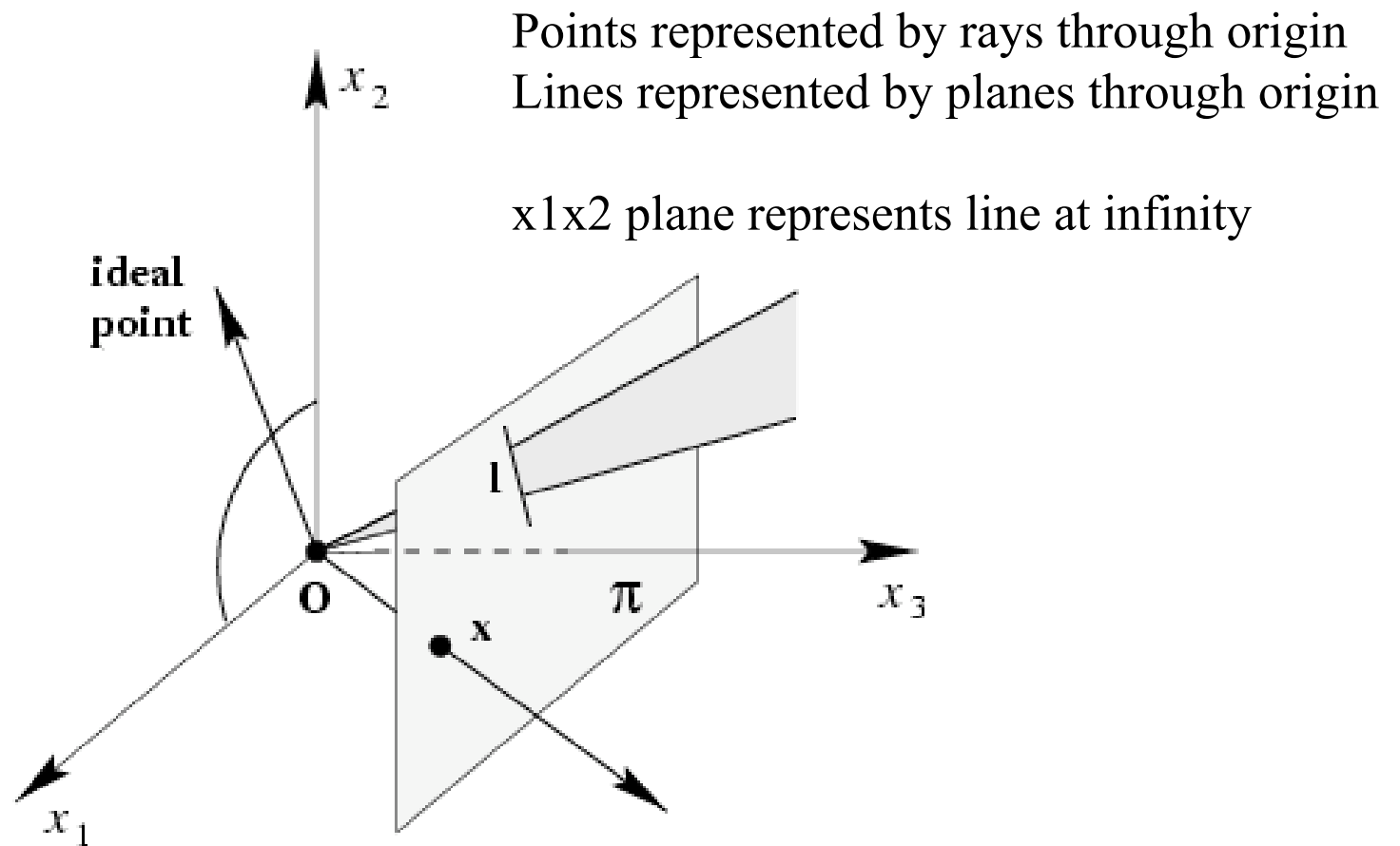
It is orthogonal to (a, b) -- the line normal.

Thus it represents the line direction.

As the line's direction varies, the ideal point $(b, -a)^T$ varies over $\mathbf{l}_\infty$.

--> line at infinity can be thought of as the set of directions of lines in the plane.

A model for the projective plane



exactly one line through two points
exactly one point at intersection of two lines

Duality

$$\begin{array}{ccc} x & \longleftrightarrow & l \\ x^T l = 0 & \longleftrightarrow & l^T x = 0 \\ x = l \times l' & \longleftrightarrow & l = x \times x' \end{array}$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3, y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\boxed{\mathbf{x}^T \mathbf{C} \mathbf{x} = 0} \text{ with } \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\text{5DOF: } \{a:b:c:d:e:f\}$$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

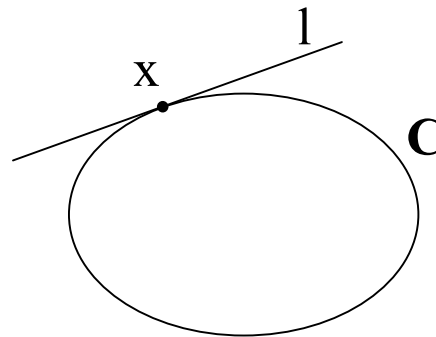
$$\begin{pmatrix} x_i^2 & x_iy_i & y_i^2 & x_i & y_i & f \end{pmatrix} \mathbf{c} = 0 \quad \mathbf{c} = (a, b, c, d, e, f)^T$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to C at point x on C is given by $l=Cx$

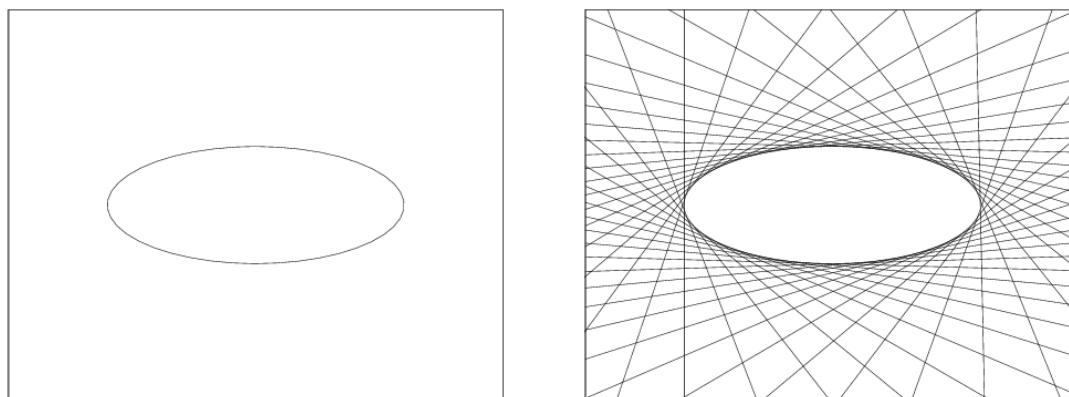


Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{1}^T \mathbf{C}^* \mathbf{1} = 0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^* = \mathbf{C}^{-1}$

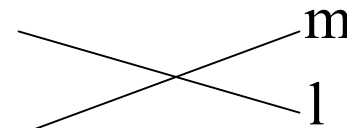
Dual conics = line conics = conic envelopes



Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

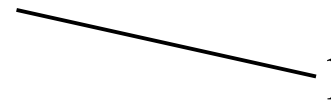
e.g. two lines (rank 2)



$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathbf{l}\mathbf{l}^T$$



Degenerate line conics: 2 points (rank 2), double point (rank 1)

Note that for degenerate conics $(\mathbf{C}^*)^* \neq \mathbf{C}$

Projective transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3×3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector $\mathbf{x}$ it is true that $h(\mathbf{x}) = \mathbf{H}\mathbf{x}$

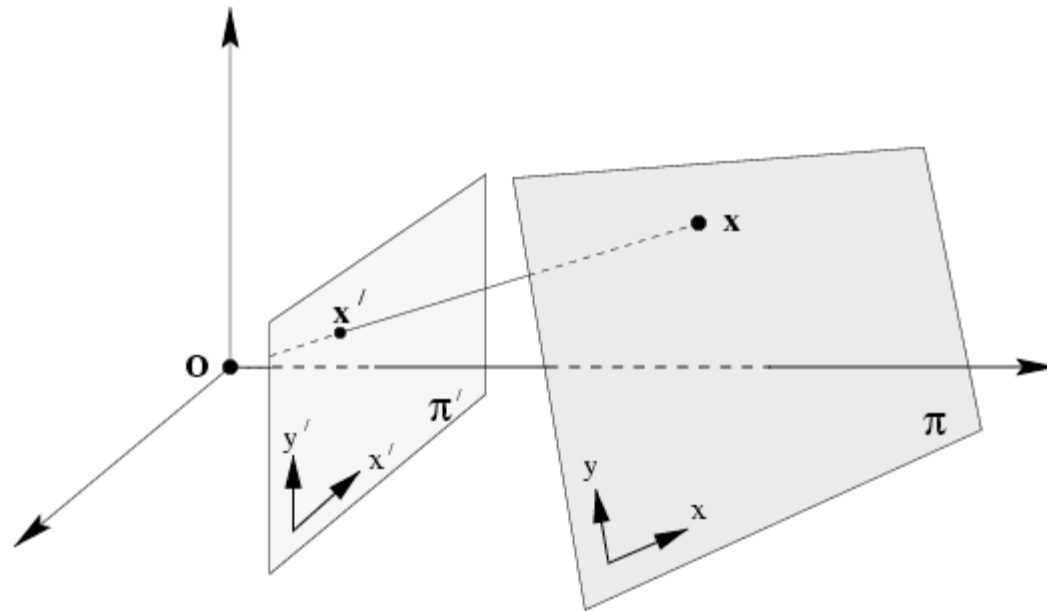
Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H} \mathbf{x}$$

8DOF

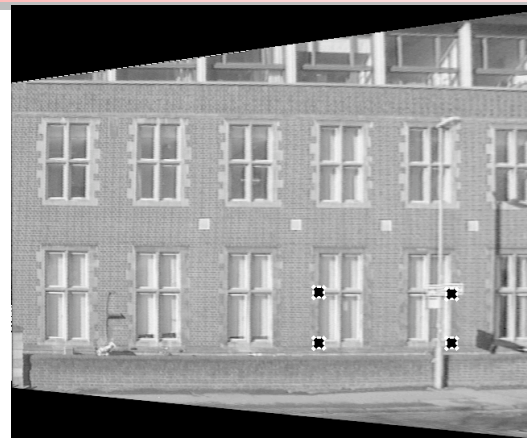
projectivity=collineation=projective transformation=homography

Mapping between planes



central projection may be expressed by $x' = Hx$
(application of theorem)

Removing projective distortion



select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$\begin{aligned} x'(h_{31}x + h_{32}y + h_{33}) &= h_{11}x + h_{12}y + h_{13} \\ y'(h_{31}x + h_{32}y + h_{33}) &= h_{21}x + h_{22}y + h_{23} \end{aligned} \quad \text{(linear in } h_{ij}\text{)}$$

(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary,
better ways to compute (see later)

Transformation of lines and conics

For a point transformation

$$\mathbf{x}' = \mathbf{H} \mathbf{x}$$

Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-\top} \mathbf{l}$$

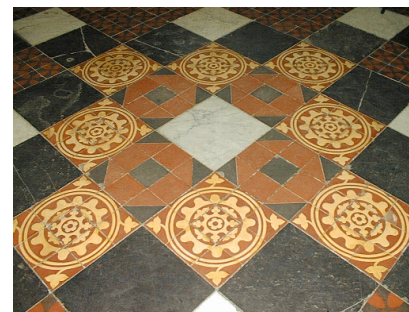
Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\top} \mathbf{C} \mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^* = \mathbf{H} \mathbf{C}^* \mathbf{H}^{\top}$$

Distortions under center projection



Similarity: squares imaged as squares.

Affine: parallel lines remain parallel; circles become ellipses.

Projective: Parallel lines converge.

Class I: Isometries

(iso=same, *metric*=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon \cos \theta & -\sin \theta & t_x \\ \varepsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$

orientation reversing: $\varepsilon = -1$

$$\mathbf{x}' = \mathbf{H}_E \mathbf{x} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{x} \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad (\text{isometry} + \text{scale})$$

$$\mathbf{x}' = \mathbf{H}_s \mathbf{x} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ 0^\top & 1 \end{bmatrix} \mathbf{x} \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

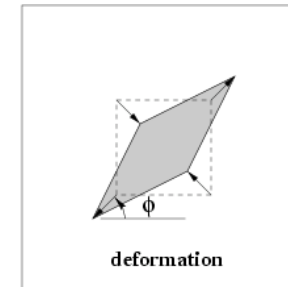
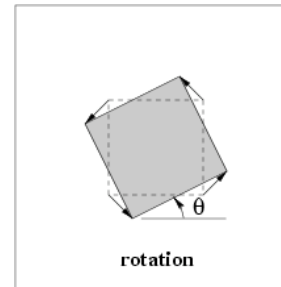
also known as *equi-form* (shape preserving)

metric structure = structure up to similarity (in literature)

Invariants: ratios of length, angle, ratios of areas,
parallel lines

Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



$$\mathbf{x}' = \mathbf{H}_A \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{x}$$

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths,
ratios of areas

Class VI: Projective transformations

$$\mathbf{x}' = \mathbf{H}_P \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \mathbf{x} \quad \mathbf{v} = (v_1, v_2)^\top$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line
(ratio of ratio)

Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ 0^\top & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ v^\top & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,
allows to observe vanishing points, horizon.

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$$

decomposition unique (if chosen $s > 0$) $\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathbf{t}\mathbf{v}^\top$

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

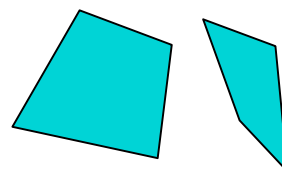
$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2 \cos 45^\circ & -2 \sin 45^\circ & 1.0 \\ 2 \sin 45^\circ & 2 \cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Overview transformations

Projective
8dof

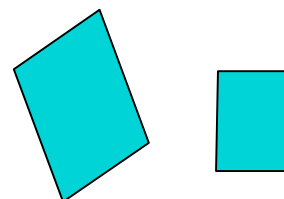
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity,
order of contact (intersection,
tangency, inflection, etc.),
cross ratio

Affine
6dof

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

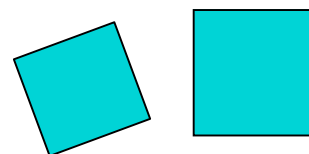


Parallellism, ratio of areas,
ratio of lengths on parallel
lines (e.g midpoints), linear
combinations of vectors
(centroids).

The line at infinity l_∞

Similarity
4dof

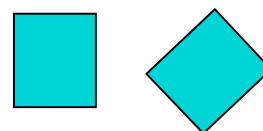
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.