Satisfiability

A Lecture in CE Freshman Seminar Series:
Ten Puzzling Problems in Computer Engineering

About This Presentation

This presentation belongs to the lecture series entitled "Ten Puzzling Problems in Computer Engineering," devised for a ten-week, one-unit, freshman seminar course by Behrooz Parhami, Professor of Computer Engineering at University of California, Santa Barbara. The material can be used freely in teaching and other educational settings. Unauthorized uses, including any use for financial gain, are prohibited.

Edition	Released	Revised	Revised	Revised	Revised
First	Apr. 2007	Apr. 2008	Apr. 2009	Apr. 2010	Apr. 2011
		Apr. 2012	Apr. 2015	Apr. 2016	Apr. 2020

Making Change: Ground Rules

For the sake of the puzzles in this presentation:
Coin means one of the pieces 1¢, 5¢, 10¢, 25¢ (no 50¢ or $1 coin)
Bill means one of the denominations $1, $5, $10, $20, $50, $100

Warm-up puzzle: What is the largest sum of coins that you can have in your pocket without being able to make exact change for $1?

Answer: $1.19

Q1: Repeat the warm-up puzzle but this time requiring exact change for $2.

Making Change: Mathematical Formulation

Puzzle: Largest sum of coins not containing exact change for $1.

Formulating a solution: Maximize the sum $M = 25Q + 10D + 5N + P$ subject to $25q + 10d + 5n + p \neq 100$ for any $q \leq Q$, $d \leq D$, $n \leq N$, $p \leq P$

Challenge: You held some bills, all $50 or smaller denominations, and some coins, when a friend asked you if you could change a $100 bill for her. You discovered that you could not make exact change. What is the maximum sum of money that you could have had?

Answer: $140.19

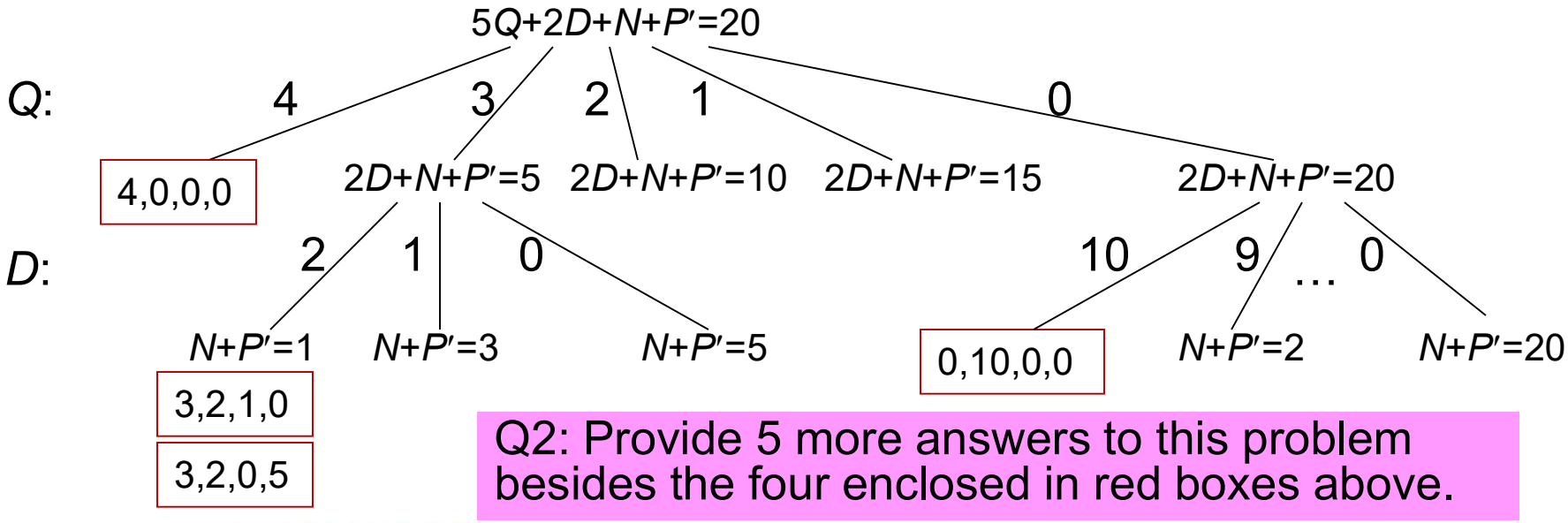
The Many Ways of Making Change

In how many distinct ways can you make change for $1?

Diophantine equation

Count the number of integer solutions to the equation $25Q + 10D + 5N + P = 100$

Because P must be a multiple of 5, we take $P = 5P'$ and rewrite the equation as $5Q + 2D + N + P' = 20$



Q2: Provide 5 more answers to this problem besides the four enclosed in red boxes above.

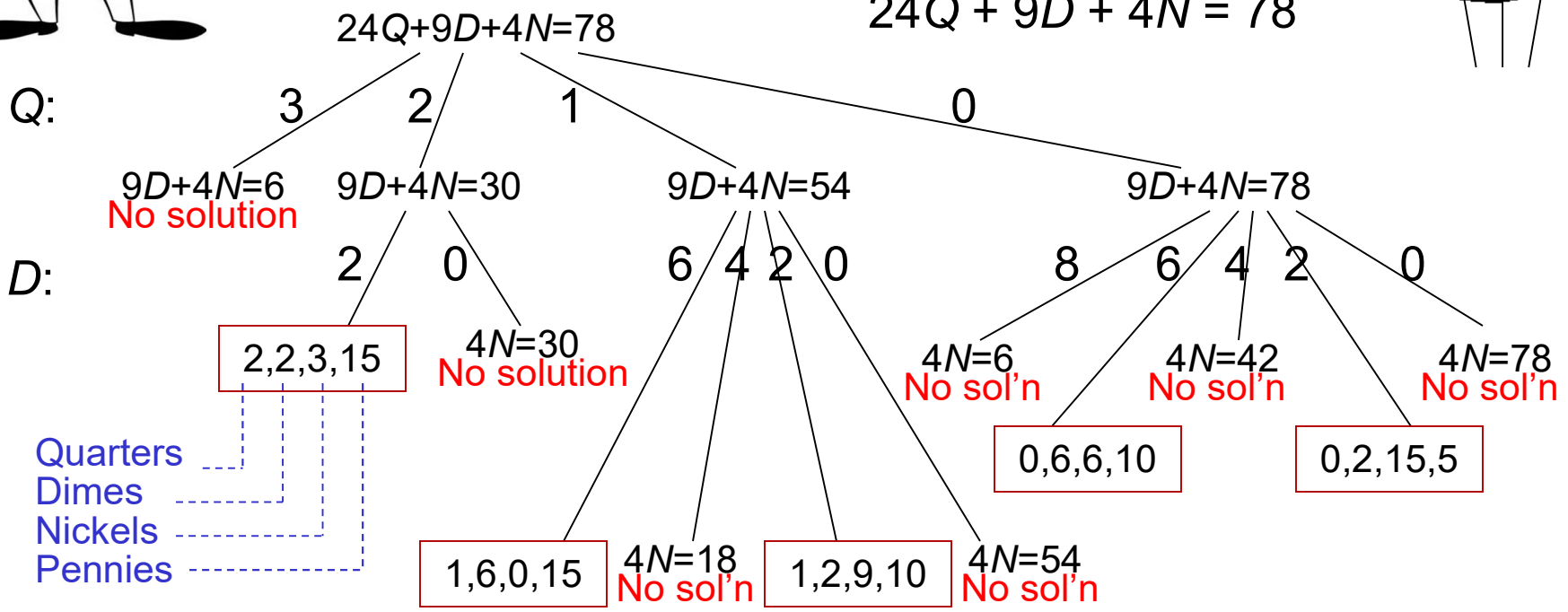
Making Change with Constraints

Can you make change for $1 using exactly 22 coins?

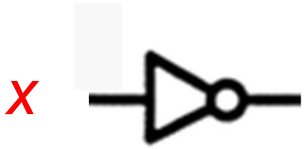
Is there a solution to $25Q + 10D + 5N + P = 100$ that satisfies the constraint $Q + D + N + P = 22$?

$P = 22 - Q - D - N$

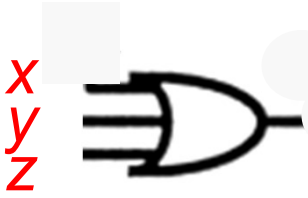
$24Q + 9D + 4N = 78$



Logic Gates as Arithmetic Operators



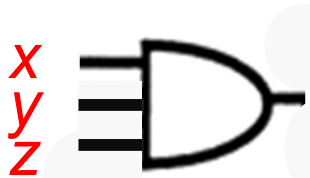
Inverter: Changes 0 to 1 and 1 to 0



$1 - (1 - x)(1 - y)(1 - z)$

Max

Needs one 1 among the inputs to produce a 1 output



xyz

Min

All inputs must be 1s to produce a 1 output

Inputs and outputs of logic gates are in {0, 1}

What Is (Boolean) Satisfiability?

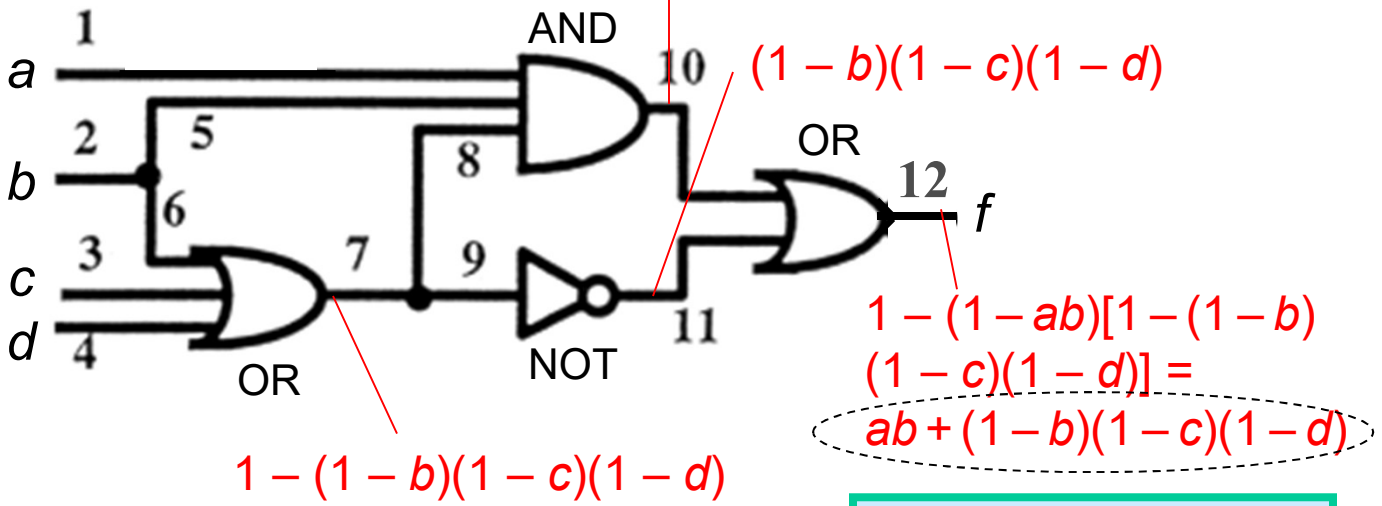
We converted the puzzle:

Can you make change for $1 using exactly 22 coins?

To the equivalent problem:

Is the equation $24Q + 9D + 4N = 78$ satisfiable for Q, D, and N nonnegative integers?

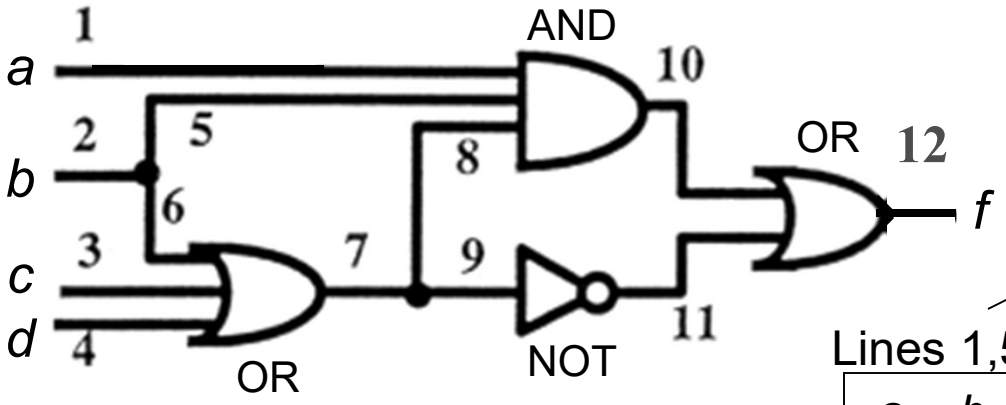
Inputs and outputs of logic circuits are in {0, 1}



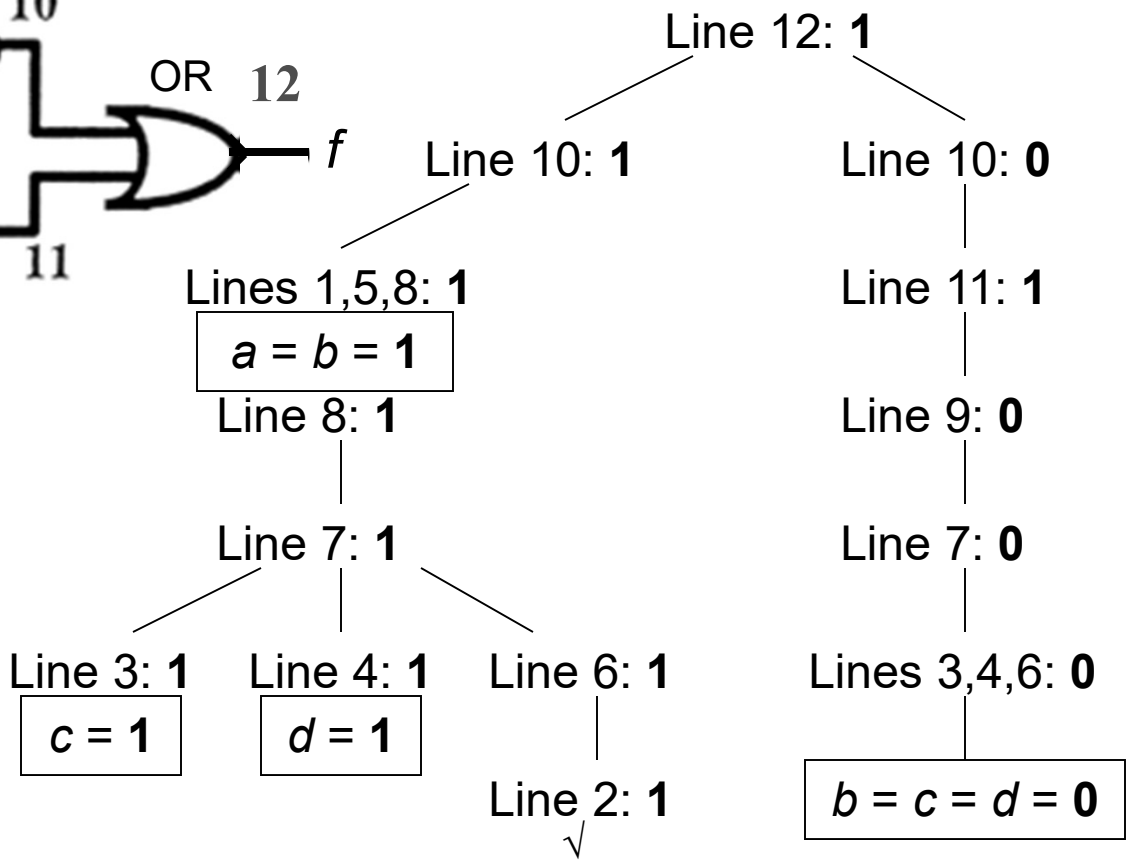
Circuit satisfiability: Can we choose input values so that $f = 1$?

Satisfied (=1) for:
$b = 1$, $a = 1$; or
$b = 0$, $c = d = 0$

Why Is Satisfiability Difficult?



As was the case for our money-changing puzzles, the tree of options quickly expands in size, thus leading to a great deal of work for extracting a solution from the vast array of possibilities



The problem is much easier going the other way: Given the circuit input values, what is its output?

A Restricted Form of Satisfiability

The logic circuit consists of a number of OR gates, each with up to 3 inputs, all feeding an AND gate that produces the output *y*

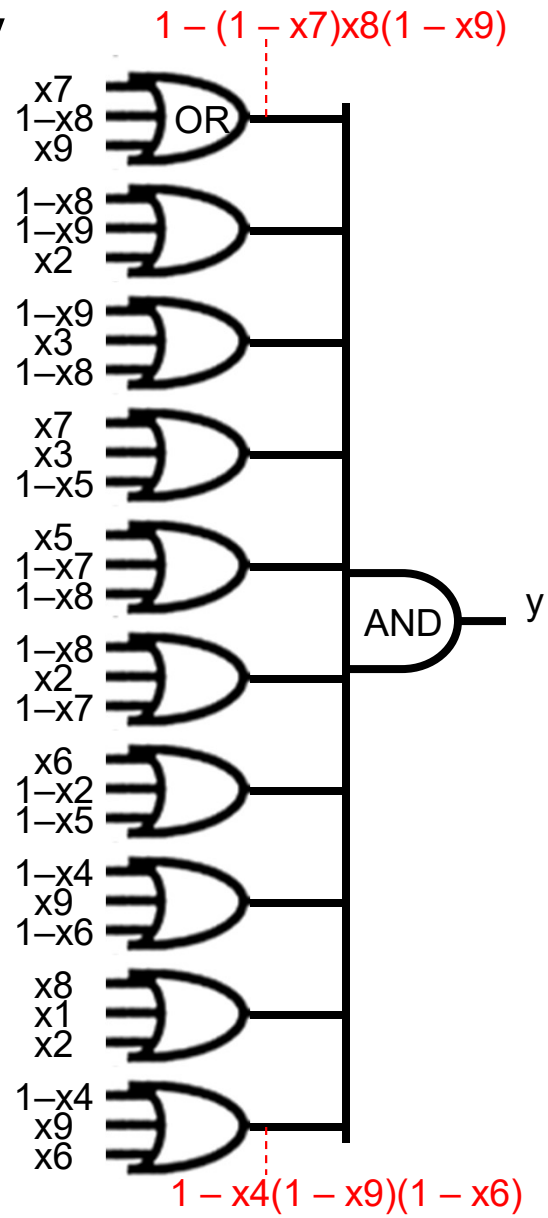
Each circuit input is a variable x_i or $1 - x_i$

Even this restricted version, known as 3SAT, isn't fundamentally simpler than the general version

Example: Does the expression below evaluate to 1 for some set of values in {0, 1} assigned to x1-x9?

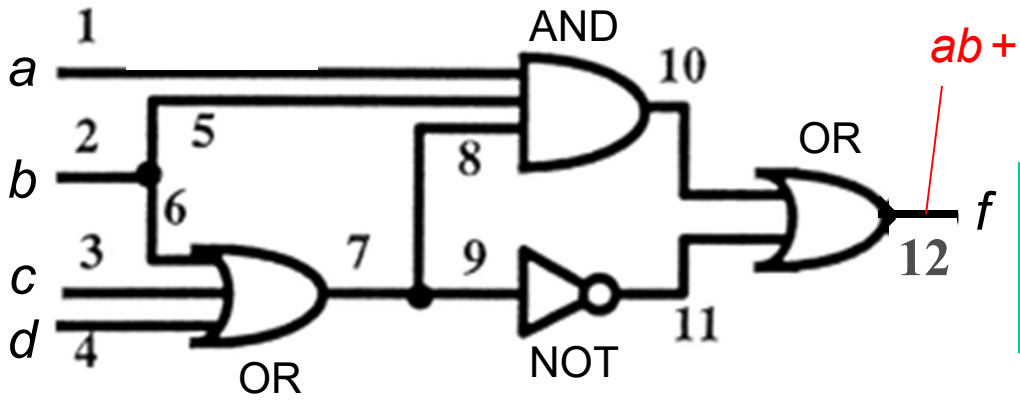
[1 – (1 – x7)x8(1 – x9)] [1 –x8x9(1 – x2)]
[1 – x9(1 – x3)x8] [1 – (1 – x7)(1 – x3)x5]
[1 – (1 – x5)x7x8] [1 – x8(1 – x2)x7]
[1 – (1 – x6)x2x5] [1 – x4(1 – x9)x6]
[1 – (1 – x8)(1 – x1)(1 – x2)] [1 – x4(1 – x9)(1 – x6)]

Answer: Yes, e.g., for x1 = x3 = x6 = x9 = 1 and x8 = 0



Why Is Satisfiability Useful?

Inputs and outputs of logic circuits are in {0, 1}



$ab + (1 - b)(1 - c)(1 - d)$

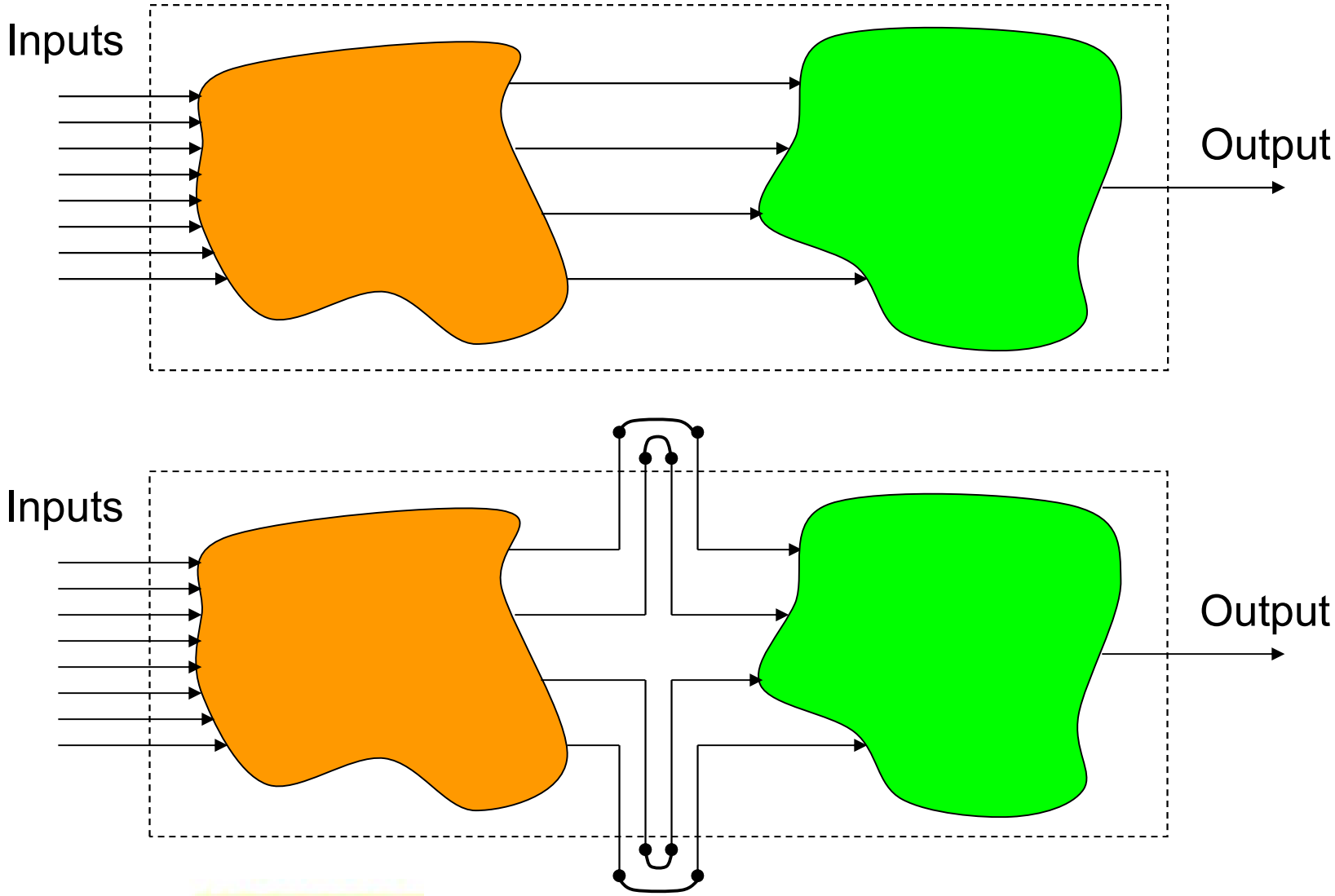
$f = 1$ satisfied for
$b = 1$, $a = 1$; or
$b = 0$, $c = d = 0$

Suppose that the circuit illuminates an emergency light when $f = 1$.
You operate the circuit for weeks and the light never goes on.
Is this an indication that everything is okay, or a sign that the circuit's output has been disconnected from the emergency light?

The satisfiability result that we obtained tells you that you can test the circuit for f stuck-at-0 by applying any of the inputs ($a\ b\ c\ d$) = 1 1 – – or ($a\ b\ c\ d$) = – 0 0 0, for a total of 6 possible test inputs

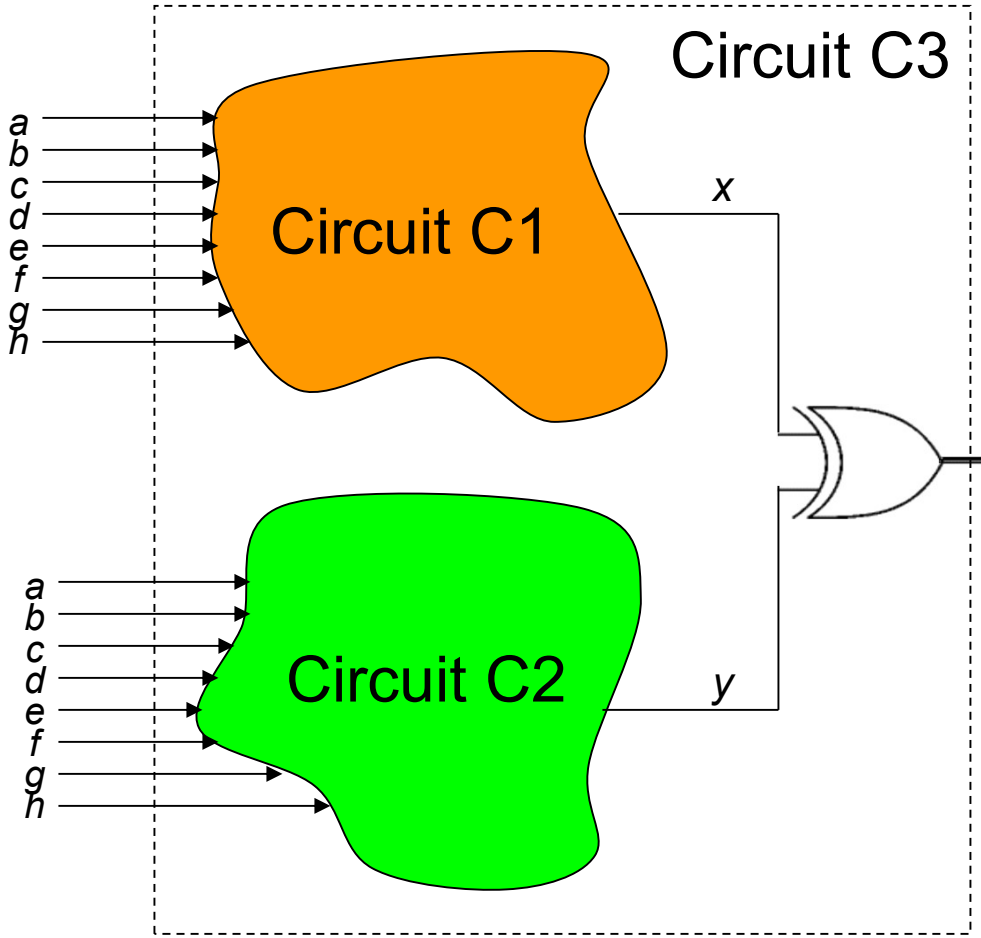
Q3: If the light goes on, how can you test the circuit to determine whether this is due to f being stuck-at-1 rather than a real emergency?

Improving Testability via Testpoint Insertion



Equivalence of Digital Circuits

Question: Are C1 and C2 equivalent (always produce the same output)?



Output of XOR gate is 1 iff its two inputs are unequal

Circuits C1 and C2 are equivalent iff C3 is not satisfiable

The Satisfiability Game

Pick one of the numbers shown in the table (say *x*)

Color all boxes containing *x* green
and all boxes containing –*x* red

You win if every row has a green box

You lose if some row has three red boxes

(a) Pick –8

(b) Pick 6

(c) Pick 3

It is clear that we can win by choosing one number from each of the sets {–4, 9} and {1, 2}

7	–8	9
–8	–9	2
–9	3	–8
7	3	–5
5	–7	–8
–8	2	–7
6	–2	–5
–4	9	–6
8	1	2
–4	9	6

Q4: Play the Satisfiability Game (at average level) on the following Web site until you win a round. Submit a screen-shot of the final screen as proof.

http://www.cril.univ-artois.fr/~roussel/satgame/satgame.php?level=1&lang=eng

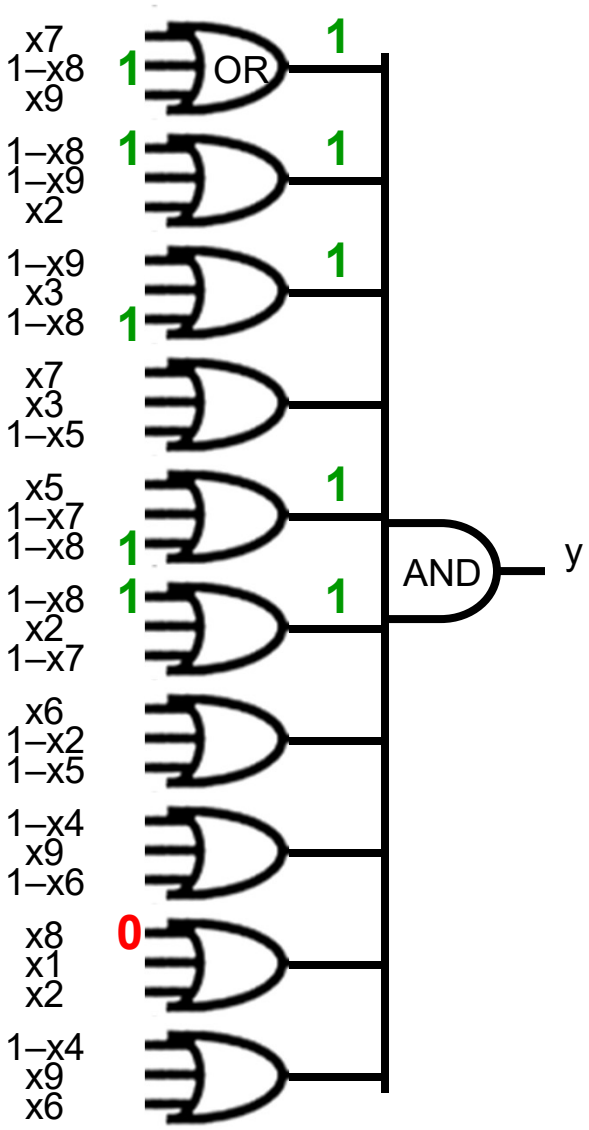
The Satisfiability Game Is the Same as SAT3

Each OR gate corresponds to a row in the table

Having a green box in a row is the same as the associated OR gate having an input 1

Having a red box in a row is the same as the associated OR gate having an input 0

We win if all OR gates have outputs 1;
We lose if some OR gate has output 0 (row of 3 red boxes)



7	–8	9
–8	–9	2
–9	3	–8
7	3	–5
5	–7	–8
–8	2	–7
6	–2	–5
–4	9	–6
8	1	2
–4	9	6

A Possible "Strategy" for SAT3

At each step, pick a number that produces the most green boxes and as few red boxes as possible

Consider the example on the right, where most of the entries are not shown to avoid clutter

(a) Pick 9; Satisfies 3 rows

(b) Pick –7: Satisfies 3 rows

(c) Pick 5: Satisfies 3 rows

This seemingly reasonable strategy does not work because the 3 choices lead to an unsatisfiable row

For any alleged strategy, one can build a specific counterexample that would defeat that strategy

Exercise: Fill in the rest of this SAT game in a way that makes it difficult for a player following the "strategy" to succeed

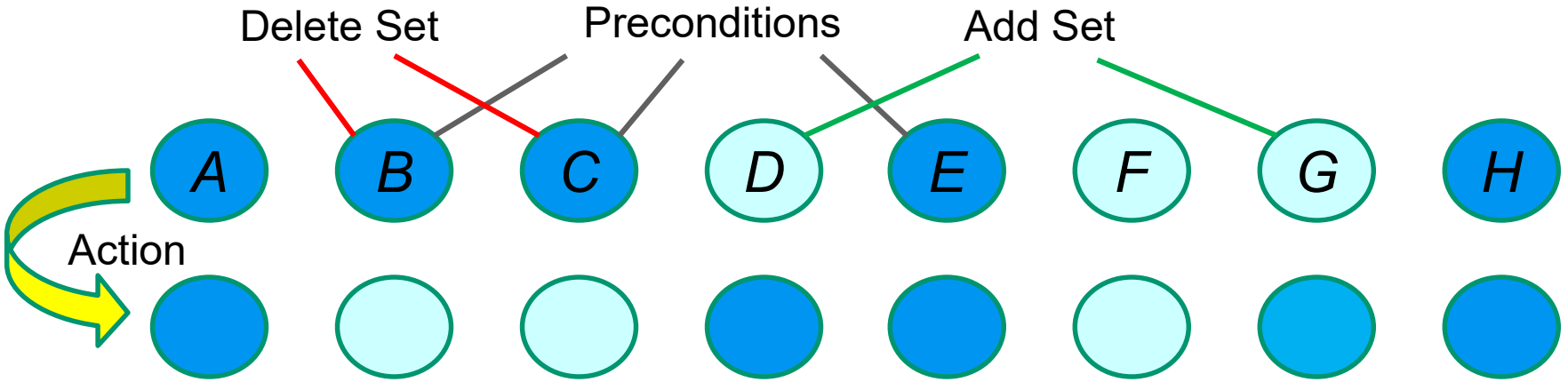
		9
–5	–9	7
	5	
	5	
	–7	
		–7
		–7
	9	
5		
9		

Planning in AI: An Application of Satisfiability

Problem definition: A ***plan*** is a sequence of ***actions*** that leads (with high likelihood) to a desired ***goal***

System of interest has states, characterized by a set of variables, and actions that change the system state

Example with Boolean variables: Each action has three elements.
A set of *preconditions* (variables that must have the value 1)
A set of *adds* (variables that are set to 1 as a result of the action)
A set of *deletes* (variables that are set to 0 as a result of the action)

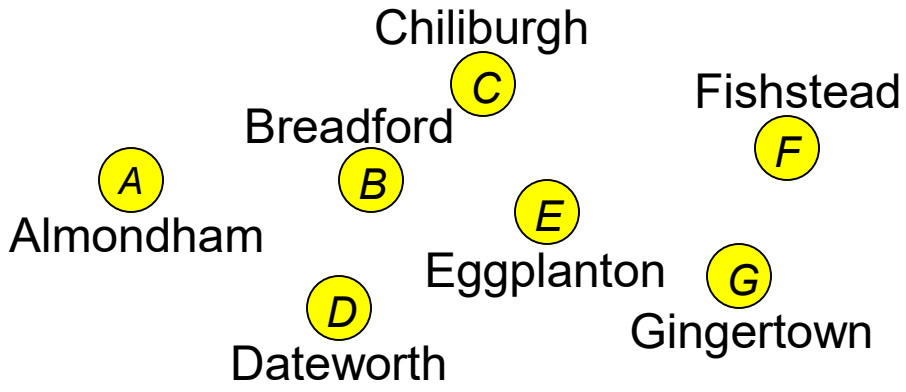


Yet Another Application of Satisfiability

Map labeling: We have a map with n cities which must be labeled. Let's say the label for city X can go above ($X = 1$) or below ($X = 0$) it. When cities are close to each other, we have constraints such as $B = 1$ and $C = 0$ are not allowed simultaneously, which can be written as $B(1 - C) = 0$ or equivalently $1 - B(1 - C) = 1$

Problem: Choose the Boolean variables A-G such that the following equation is satisfied (an instance of 2SAT):

$$[1 - B(1 - C)]\,[1 - D(1 - B)]\,[1 - E(1 - C)]\,[1 - G(1 - F)] = 1$$



Satisfied, e.g., for

$A = D = E = G = 0$

$B = C = F = 1$