Chordal Rings Based on Symmetric Odd-Radix Number Systems

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Abstract

An n-node network, with nodes numbered from $-\lfloor n/2 \rfloor$ *to* $\lceil n/2 \rceil - 1$, *is a chordal ring network with the chord lengths* $1 = s_0, s_1, \ldots, s_{k-1}$ $(2 \le s_i < n/2)$ *when each node i* $(-\lfloor n/2 \rfloor \le i < \lceil n/2 \rceil)$ *is connected to each of the 2k nodes* $i \pm s_i$ *mod n* $(0 \le i < k)$ *via an undirected link, where "mod" represents symmetric residues. We study a class of chordal ring networks in which the chord length* s_i *is a power of an odd "radix" r, that is,* $s_i = r^i$, *for* $r \ge 3$. *We show that this class of chordal rings, with their nodes indexed by radix-r numbers using the symmetric digit set* $[-(r-1)/2, (r-1)/2]$ *are easy to analyze and offer a number of advantages in terms of static network parameters and dynamic performance in many application contexts.*

1. Introduction

Chordal rings have been studied for many years as the interconnection architecture for parallel and distributed systems [Arde81], [Berm95], [Hwan01]. One attractive feature of chordal rings is that they have Hamiltonian cycles built-in and readily visible, whereas for other networks, researchers go to great lengths to establish Hamiltonicity. Other features include symmetry, ease of routing, and robustness. Although the latter advantages are not unique to chordal rings, not many networks offer all these desirable properties simultaneously. On the negative side, determination of topological parameters can be difficult. Even for chordal rings with a single skip link type (degree 4), known as double-loop networks, the problem is nontrivial in general, and it is not yet completely solved [Chen05].

In this paper, we study a class of chordal ring networks in which the chord length s_i is a power of an odd "radix" r, that is, $s_i = r^i$, for $r \ge 3$. We show that this class of chordal rings, with nodes indexed by radix-r signed integers using the symmetric digit set $[-(r-1)/2, (r-1)/2]$ are easy to analyze and offer a number of advantages.

Following some background and basic definitions in Section 2, we study routing in our class of chordal rings, and derive their diameters, in Section 3. We deal with a number of other topological properties in Section 4 and with robustness attributes in Section 5. Our conclusions appear in Section 6.

2. Basic Definitions and Properties

An n-node network, with nodes numbered $-\lfloor n/2 \rfloor$ to $\lceil n/2 \rceil - 1$, is a chordal ring network with chord lengths $1 = s_0, s_1, \ldots, s_{k-1}$ $(s_i < n/2)$ when each node i $(-\lfloor n/2 \rfloor \le i < \lceil n/2 \rceil)$ is connected to each of the $2k$ nodes $i \pm s_i$ $(0 \le i < k)$ via an undirected link; all node-index expressions in this paper are evaluated modulo n, using symmetric residues. Our focus will be on a class of chordal ring networks in which the chord length s_i is a power of an odd "radix" r, that is, $s_i = r^i$, for $r \ge 3$. We index nodes of the chordal ring $CR(n; r, \ldots, r^{k-1})$ by k-digit radix-r numbers using the symmetric digit set $[-(r-1)/2, (r-1)/2]$. We restrict the number of nodes to the maximal value r^k in deriving some of our results. Other values of n do not create insurmountable difficulties, but they do lead to needless clutter in presenting the basic ideas.

Figure 1 depicts a 16-node chordal ring with a single chord length 5, designated as $CR(16; 5)$, where the first parameter is the number of nodes and the parameters following the semicolon are the skip distances besides the mandatory $s_0 = 1$. Nodes 0 to 7 and −8 to −1 can be numbered in the 2-digit symmetric radix-5 number system as 0 0 to 1 2 and $\bar{2}$ 2 to 0 $\bar{1}$, respectively. The node label 1 2, for example, is indicative of a path from node 0 0 to node 7; the path consists of one chord of length 5 and two ring links, with the three traversed in any desired order (a total of three paths). The path thus obtained is a shortest path, leading to a simple and elegant shortest-path routing algorithm that is inherently fault-tolerant when the shortest path is not of length 1, and it leads to only two extra hops in the latter case.

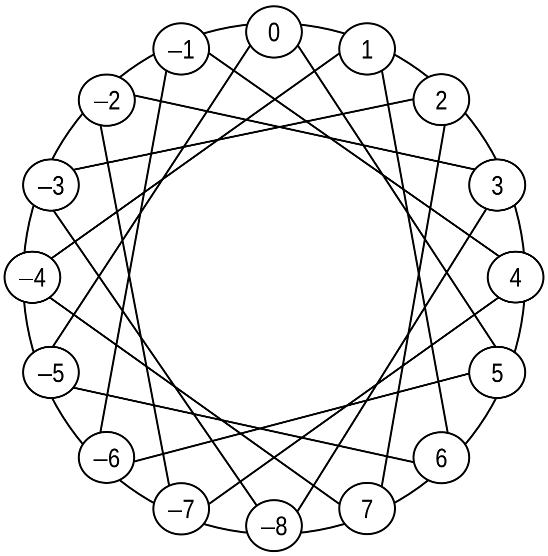

Fig. 1. The chordal ring network $CR(16; 5)$ with 16 nodes and chord length 5.

The following four properties of $CR(n; r, \ldots, r^{k-1})$, $2r^{k-1} < n \le r^k$, are easily established.

Property 1: When the n nodes are indexed from $-\lfloor n/2 \rfloor$ to $\lceil n/2 \rceil - 1$, each node has a unique label in the radix-r number system with the symmetric digit set $[-(r-1)/2, (r-1)/2]$.

Property 2: For $n = 2^q$, the n links corresponding to each chord length r^i constitute a distinct spanning cycle. So, the network contains k edge-disjoint Hamiltonian cycles.

Property 3: For $n = r^k$, the n links corresponding to each chord length r^i form r^i disjoint cycles, each of length r^{k-i}.

Property 4: Between two nodes with symmetric radix-r indices $x_{k-1} \ldots x_1x_0$ and $y_{k-1} \ldots y_1y_0$, there exists a path of length no greater than $|x_{k-1} - y_{k-1}| + \ldots + |x_1 - y_1| + |x_0 - y_0|$.

3. Routing and Diameter

Because of node symmetry, we only need to devise a method to route a message from node 0 to node $x = x_{k-1} \ldots x_1x_0$. To route from an arbitrary node u to another node v, we represent the smaller (in magnitude) of $v - u$ and $u - v$ as $x = x_{k-1} \ldots x_1x_0$ and use a set of links as if routing from node 0 to node x. The following property indicates that shortest-path routing to node x of $CR(n; r, \ldots, r^{k-1})$ can be achieved by using $x_{k-1} \ldots x_1x_0$ as a routing tag.

Property 5: To route from node 0 to node $x = x_{k-1} \ldots x_1x_0$ along a shortest path, simply attach x to the message as a routing tag $t_{k-1} \ldots t_1t_0$. In each hop, from an intermediate node v, pick any nonzero digit in the routing tag t. Let this nonzero digit be t_i. If $t_i > 0$, route to node $v + r^i$ and decrement t_i. If $t_i < 0$, route to node $v - r^i$ and increment t_i. If every digit is 0, the message is at its destination.

Note that greedy routing is a special case of the algorithm described in Property 5 and corresponds to picking the leftmost nonzero t_i (longest skip that does not lead past the destination) in each hop. As a corollary, greedy routing also leads to the selection of a shortest path.

Based on Property 5, the diameter of $CR(n; r, \ldots, r^{k-1})$ is equal to the weight, or sum of absolute values of digits, in a node index with maximum weight. One such maximum-weight node index is of the form $x_{k-1}aa \ldots a$, where $a = (r-1)/2$ and x_{k-1} has the smallest possible (most negative) value of $-\lceil (n - r^{k-1}) / (2r^{k-1}) \rceil$. This leads to the following result.

Property 6: The diameter of $CR(n; r, \ldots, r^{k-1})$ is $D = (k-1)(r-1)/2 + \lceil (n - r^{k-1}) / (2r^{k-1}) \rceil$. In the special case of $n = r^k$, the formula reduces to $D = k(r-1)/2 = (r-1)(\log_r n)/2$.

Figure 2 depicts a representation of the chordal ring network $CR(16; 5)$ of Fig. 1 as a subset of points on the infinite grid $G_{16,5}$ [Chen05]. The parallelogram shown in Fig. 2, or its "digitized" version, tessellates the plane and allows the visualization and derivation of network diameter as the Manhattan or grid distance from a node inside the parallelogram to the closest of its four corners, with the interior node chosen to maximize this distance.

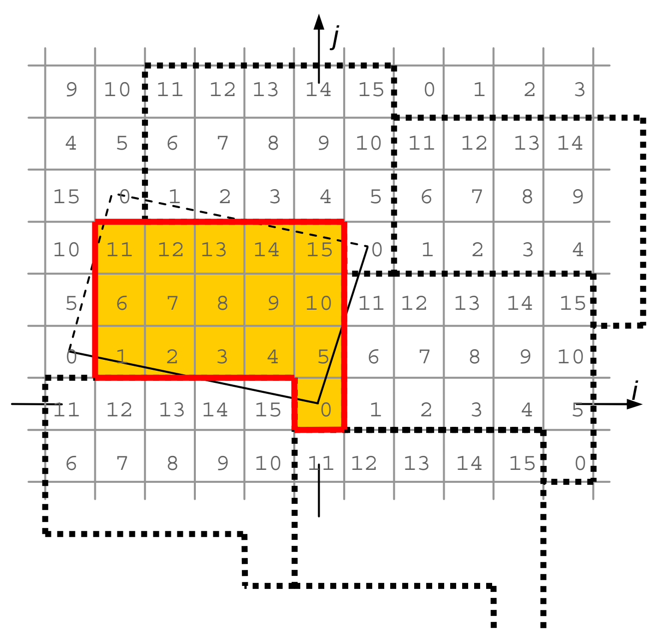

Fig. 2. Part of the infinite grid $G_{16,5}$ associated with the chordal ring $CR(16; 5)$.

Of practical interest is the choice of the odd radix r that would minimize the diameter. To obtain this optimal radix, we write the diameter in the approximate form $D \cong (r-1)(\ln n / \ln r)/2$ for all n (the formula is exact only for $n = r^k$). Differentiating D with respect to r and equating the result with 0 yields the optimality condition $\ln r = (r-1)/r$. This is satisfied, approximately, for $r = 3$. With this optimal choice, $d \cong 1.26 \log_2 n$ and $D \cong 0.63 \log_2 n$. We see that a diameter which is better than the diameter of an n-node hypercube is achieved, but at a greater cost in terms of node degree. More on this comparison will be offered later.

The optimality of $r = 3$ is not surprising. The idea for these networks came to the author as he was looking at a mathematical puzzle dealing with weighing. Suppose that you have a balance and want to choose an optimal set of 4 fixed weights that would allow you the widest possible range of measurement in increments of 1 gram. The solution is 1, 2, 4, 8 (offering a range of 1-15 grams), if weights must be placed on one side and the material or items to be weighed on the other. If, however, fixed weights can go on both sides, the optimal becomes 1, 3, 9, 27, offering a much wider range of weights (1-40 grams). The placement of fixed weights for any desired weight x is derived from the symmetric radix-3 representation of x using the digit set $\{\bar{1}, 0, 1\}$; for example, $x = 14 = (1\ \bar{1}\ \bar{1}\ \bar{1})_{\text{three}}$ requires that the 27-gram weight go on one side and the three others on the side of the material/items being weighed. The corresponding notion in chordal rings is traversing some links backwards along the shortest path.

4. Other Topological Properties

One of the important topological parameters of a network is its average internode distance, because it determines the expected performance in routing under light traffic conditions. Given that the number of steps in the shortest path from node 0 to node x equals the weight of x as a symmetric radix-r number, the following result follows by finding the average radix-r symmetric digit value and multiplying it by k.

Property 7: The average internode distance Δ of $CR(n; r, \ldots, r^{k-1})$, with $n = r^k$ nodes, is $k[1 + 2 + \ldots + (r-1)/2]/r = kr/2 - k/(2r) = (1 + 1/r)D/2$, which is slightly more than half the diameter.

Another important topological property is the bisection width B, an indicator of communication performance under heavy random traffic. The parameter B is quite difficult to obtain for an arbitrary interconnection network. The upper bound $B \le 2(r^k - 1)/(r-1)$ on the bisection width is easily established by noting that it corresponds to cuts on the diametrically opposite sides of a ring drawing of the network (see, e.g., Fig. 1). The lower bound $(n - 1/n)\, r/(r-1)$ can be established by embedding the complete graph K_n into our network and noting the maximum congestion of the embedding based on a balanced distribution of paths, that is, dividing the $n(n-1)/2$ paths of average length $\Delta = (k/2)(r - 1/r)$ over the kn available links equally.

Property 8: The bisection width B of the chordal ring $CR(n; r, \ldots, r^{k-1})$, with $n = r^k$ nodes, is between the lower bound $(n - 1/n)/(r - 1/r)$ and the upper bound $2(n-1)/(r-1)$. In particular, for any fixed radix r, we have $B = O(n)$, with the coefficient of the leading term being in the approximate range of $[1/r, 2/r]$.

So, based on Property 8, we know the bisection width of $CR(r^k; r, \ldots, r^{k-1})$ to within a factor of about 2. For radix $r = 3$ that minimizes the diameter, the bisection width B, which is in the approximate range $[3n/8, n]$, can be seen to be quite comparable to that of an n-node hypercube. For $r = 5$, as derived in the following paragraph, the approximate range of B is $[5n/24, n/2]$, somewhat lower, but still not far from that of a hypercube of comparable size (having $B = n/2$).

One way to take the network cost into account in determining the best radix is to minimize the degree-diameter product $dD = (r-1)(\ln^2 n / \ln^2 r)$. Differentiating dD with respect to r and equating the result with 0 yields the condition $\ln r = 2(r-1)/r$. This condition is satisfied, approximately, for $r = 5$. With this optimal choice, we have $d = D \cong 0.86 \log_2 n$ and $dD \cong 0.74(\log_2 n)^2$. These compare favorably with the respective parameters of the n-node hypercube.

Chordal rings $CR(n; r, \ldots, r^{k-1})$, based on our radix-$r$ construction, are efficient with regard to VLSI layout. In fact, the example in Fig. 2 indicates that the VLSI layouts of these networks are quite similar to those of kD tori. The same number of wraparound links are needed as in tori, although the rules for the connectivity of the wraparound links are different. This difference, however, does not affect the area requirement. The same folding techniques can also be used to remove the need for long wires between neighboring nodes in VLSI layout.

5. Algorithms and Fault Tolerance

Inspection of Fig. 2 indicates that there are often multiple node- and edge-disjoint shortest paths between a given pair of nodes in $CR(n; r, \ldots, r^{k-1})$. For example, from node 0 to node 7, with the node index 1 2), we have the paths 5 + 1 + 1 (through intermediate nodes 5 and 6) and 1 + 1 + 5 (via 1 and 2).

Of course, network robustness does not require that alternate shortest paths exist in all cases. It suffices that in the unlikely event of failures, some near-shortest path be available between any pair of nodes. Our chordal rings are robust in this latter sense.

Using a proof method similar to that used in establishing the connectivity and fault diameter of *k*-ary *n*-cubes [Day97], or *r*-ary *k*-cubes with our notation, we can derive the corresponding results for our chordal ring networks. These are stated as Properties 9 and 10 below.

Property 9: An arbitrary pair of nodes, *u* and *v*, in $CR(r^k; r, \ldots, r^{k-1})$ are connected by $2k$ node/edge-disjoint paths, giving our chordal rings the maximum possible connectivity of $2k$.

We conjecture that a fairly small upper bound on the difference between the length of the longest of these alternate paths and the shortest path between the same two nodes can be derived, but have been unable to establish this bound thus far.

Property 10: The fault diameter of the chordal ring $CR(r^k; r, \ldots, r^{k-1})$, that is, the diameter of the surviving part of the network with $2k - 1$ worst-case faults (guaranteed to leave the network connected), is no greater than $D + 1$.

A fault-tolerant routing algorithm for the chordal ring $CR(n; r, \ldots, r^{k-1})$ can be readily devised. Figure 3 illustrates the availability of several shortest paths between some pairs of nodes that can be exploited for efficient fault-tolerant routing. Details of three versions of our algorithm (assuming global knowledge about faults and their locations, global knowledge about number of faults but not their locations, and only local knowledge) will be reported in the near future.

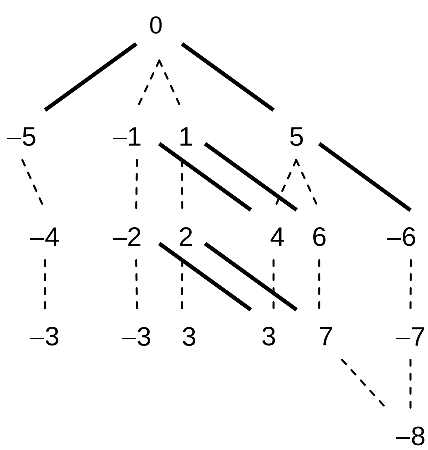

Fig. 3. A graphical depiction of some shortest paths from node 0 to other nodes in $CR(16; 5)$. Solid and dotted lines represent chords and ring links, respectively.

6. Conclusion

We have introduced a class of chordal ring networks and showed them to possess interesting properties with respect to static parameters and dynamic performance under fault-free and faulty conditions. Further research is needed to generalize some of our results that pertain only to particular network sizes to arbitrary *n* in an attempt to improve system scalability. Determining the exact bisection width, obtaining additional results on fault tolerance (including proving or disproving some of our conjectures), and devising emulation schemes for other networks are also desirable. Constructing periodically regular chordal rings [Parh99], in which any node *v* has only one chord of length $r^{k-1-v \bmod (k-1)}$, is also of some interest in order to reduce node degree while preserving some of the desirable topological and algorithmic properties.

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