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DIAMETER FORMULAS FOR A CLASS OF UNDIRECTED DOUBLE-LOOP NETWORKS

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An n-node network, with nodes numbered 0 to $n - 1$, is an undirected double-loop network with chord lengths 1 and s ($2 \leq s < n/2$) when each node i ($0 \leq i < n$) is connected to each of the four nodes $i \pm 1$ and $i \pm s$ via an undirected link; all node-index expressions are evaluated modulo n. Let $n = qs + r$, where r ($0 \leq r < s$) is the remainder of dividing n by s. Furthermore, let $s = ar + b$, where b ($0 \leq b < r$) is the remainder of dividing s by r. In this paper, we provide closed-form formulas for the diameter of a double-loop network for the case $q > r$ and for a subcase of the case $q \leq r$ when $b \leq aq + 1$. In the complementary subcase of $q \leq r$, when $b > aq + 1$, network diameter can be derived by applying the $O(\log n)$-time algorithm of Zerovnik and Pisanski (*J. Algorithms*, Vol. 14, pp. 226-243, 1993). Obtaining a closed-form formula for diameter of the double-loop network in the latter subcase remains an open problem.

Keywords: Chordal ring, Loop network, Network diameter, Parallel processing, Ring network, Routing distance, Undirected graph.

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1. Introduction

Double-loop networks find applications in the design of communication networks, multimodule memory structures, data alignment in parallel memory systems, and supercomputer organizations. Past research on double-loop network has dealt with both directed networks (e.g., [1-5]) and undirected ones (e.g., [6-13]). Among problems studied in connection with double-loop networks are routing algorithms, determination of graph diameter, and optimal assignment of chord lengths. For more detail on properties of double-loop networks and their applications, we refer the reader to the survey papers [14] and [15].

A double-loop network is characterized by its number n of nodes (network size) and by its two chord lengths. In this paper, we take one chord length to be 1, and denote a double-loop network as $LL(n, 1, s)$, where nodes are numbered 0 to $n - 1$, the chord length s satisfies $2 \leq s < n/2$, and each node i ($0 \leq i < n$) is connected to each of the four nodes $i \pm 1$ and $i \pm s$ by an undirected link; all node-index expressions are evaluated modulo n. Note that the condition on s only excludes the case $s = n/2$ for even n; this is to limit our discussion to networks of uniform node degree 4 and to avoid having to deal with the degenerate degree-3 case. Note also that setting one chord length to 1 is a real restriction in that it excludes some double-loop networks such as $LL(15, 3, 5)$ from consideration. Any double-loop network with one chord length relatively prime to its size n is, however, included. Such a chord length, which defines a Hamiltonian cycle, can be converted to a chord length of 1 via renumbering of the nodes. Formally, a double-loop network $LL(n, 1, s)$ is a graph $G = (V, E)$, where:

$$V = Z_n = \{0, 1, \ldots, n-1\} \tag{1}$$

$$E = \{(i, i \pm 1 \bmod n), (i, i \pm s \bmod n) \mid 0 \leq i < n\} \tag{2}$$

Figure 1 depicts the double-loop network $LL(16, 1, 5)$ as an illustrative example. Section 3 contains two other examples: $LL(19, 1, 0)$ and $LL(39, 1, 17)$. An additional example, $LL(15, 1, 6)$, used in constructing Fig. 2, highlights the fact that n and s need not be relatively prime.

Let $d(i, j)$ and $D(n, 1, s)$ denote the length of the shortest path from node i to node j and the diameter of $LL(n, 1, s)$, respectively. Because $LL(n, 1, s)$ is vertex-symmetric, we have:

$$D(n, 1, s) = max_{i,j}\{d(i, j)\} = max_j\{d(0, j)\} \tag{3}$$

Consider a network size $n = qs + r$, where r ($0 \leq r < s$) is the remainder of dividing n by s. Furthermore, when $q \leq r$, let $s = ar + b$, where b ($0 \leq b < r$) is the remainder of dividing s

by r. Du et al [8] showed the following upper bound for $D(n, 1, s)$:

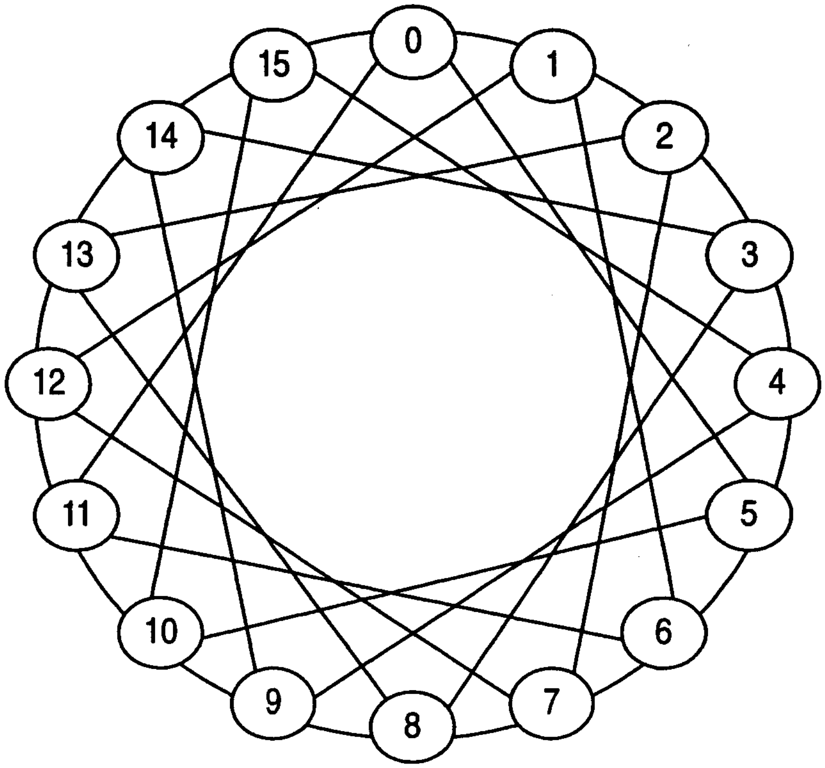

Fig. 1. The double-loop network $LL(16, 1, 5)$ with 16 nodes and chord lengths 1 and 5.

$$D(n, 1, s) \leq max\{q + 1, r - 2, s - r - 1\}$$

In [2], we provided a new algorithm to compute the diameter of directed double-loop networks. Zerovnik and Pisanski [13] gave a corresponding $O(\log n)$-time algorithm to compute the diameter of an n-node undirected double-loop network. Mukhopadhyaya and Sinha [9] discussed an optimal $O(D)$-time routing algorithm for an undirected double-loop network with diameter D, while also enumerating a number of open research problems. One such open problem is to derive an analytical formula for the diameter of $LL(n, 1, s)$.

In this paper, we derive two diameter formulas for undirected double-loop networks for the case $q > r$ and for the subcase of the case $q \leq r$ corresponding to the condition $b \leq aq + 1$ whose components were defined above. The diameter formulas are presented in Section 3, following a review of some relevant known results and derivation of two inequalities relating $D(n, 1, s)$ to the distance between lattice points in a parallelogram in Section 2. A list of key notation is presented in Table 1 for ease of reference.

Table 1. List of key notation.

$\equiv$	Is congruent to
$\lfloor x \rfloor$	Floor function; largest integer that is no greater than x
$\lceil x \rceil$	Ceiling function; smallest integer that is no less than x
Γ to Ω	All upper case Greek letters represent parallelogram regions in the plane
A to X	All upper case Roman (non-italic) letters represent points in the plane
a	Quotient of dividing s by r
b	Remainder of dividing s by r
D	Network diameter
$D(n, 1, s)$	Diameter of an n-node double-loop network with chord lengths 1 and s
$dist(\mathrm{A}, \mathrm{B})$	Distance between points A and B on the plane
$d(i, j)$	Distance between nodes i and j in a network or graph
E	Edge set of a graph
G	Graph
LL	Double-loop network
m_s	Signed integer denoting the number of skip links on a path
m_1	Signed integer denoting the number of ring links on a path
$max\{\}$	Largest value among those in a set or multiset
$min\{\}$	Smallest value among those in a set or multiset
n	Number of nodes in a network or graph
q	Quotient of dividing n by s
r	Remainder of dividing n by s
s	Chord length in a loop network (skip distance)
V	Node set of a graph
Z	The set of all integers
Z_n	The set $\{0, 1, \ldots, n-1\}$

2. Tools and Preliminary Results

The four links incident to node i in $LL(n, 1, s)$, connecting it to nodes $i + s$, $i - s$, $i + 1$, and $i - 1$, are called the forward skip, backward skip, forward ring, and backward ring links, respectively. Along a shortest path from i to j in $LL(n, 1, s)$, skip links of at most one type (forward or backward) and ring links of at most one type are taken. Given that the order of traversing these links makes no difference, we can identify a shortest path from i to j with two signed integers m_s and m_l, where $|m_s|$ is the number of skip links and $|m_l|$ is the number of ring links taken; sign of m_s or m_l indicates the forward (positive) or backward (negative) direction.

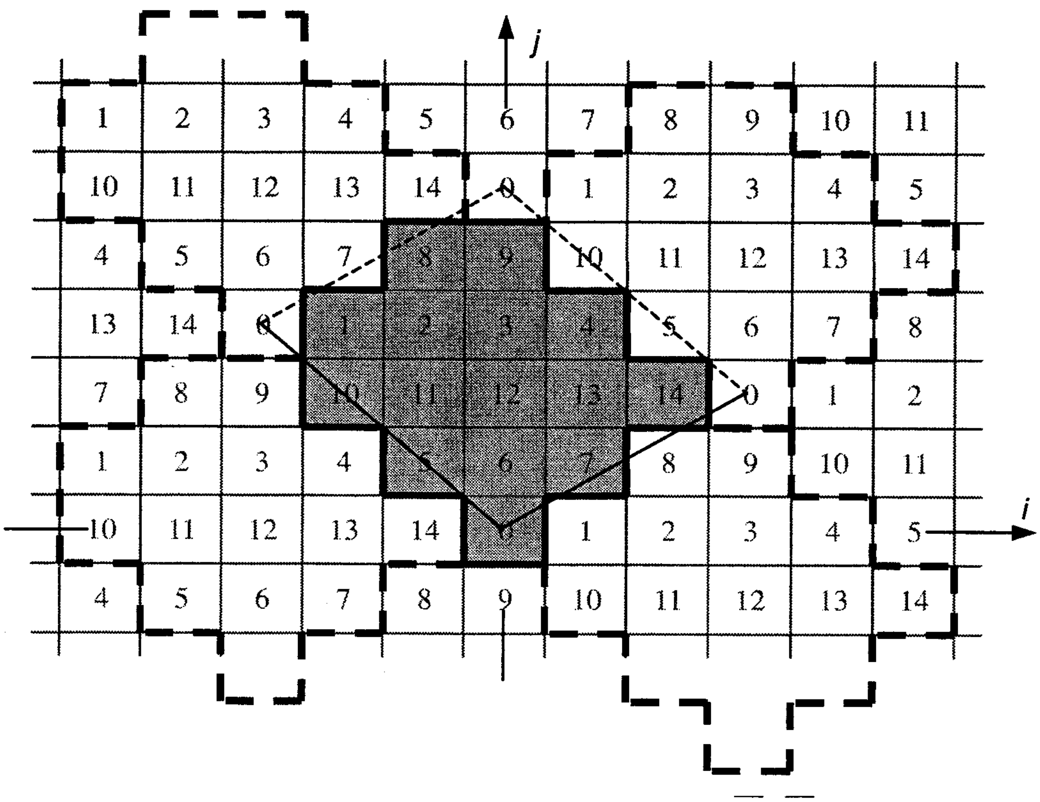

Fig. 2. Part of the infinite grid $G_{15,6}$ corresponding to the double-loop network $LL(15, 1, 6)$. We have shown a parallelogram and have shaded the lattice points within it.

Given $LL(n, 1, s)$, we construct an infinite grid $G_{n,s}$ in Z^2, labeling each lattice point (i, j) by $i + js \bmod n$. For example, part of the grid $G_{15,6}$ corresponding to $LL(15, 1, 6)$ is depicted in Fig. 2. Every label l $(0 \le l < n)$ is repeated in $G_{n,s}$ infinitely many times. Let $n = qs + r$, where r $(0 \le r < s)$ is the remainder of dividing n by s. Then, lattice points (r, q) and $(s, -1)$ correspond to 0. We refer to a lattice point with label i as an i-point. For any 0-point (u, v) on the lattice, there is an integer k such that $u + vs = kn$. Thus, we have:

$$(u, v) = k(r, q) + (kq - v)(s, -1) \tag{5}$$

This leads to the conclusion that every 0-point (u, v) can be represented as $t_1(r, q) + t_2(s, -1)$, where $t_1, t_2 \in Z$. We define

$$dist((x, y), (u, v)) = |x - u| + |y - v| \tag{6}$$

as the distance between lattice points (x, y) and (u, v).

It is clear that we have a path consisting of $x - u$ ring links and $y - v$ skip links from node $x + ys$ to node $u + vs$ in $LL(n, 1, s)$. Thus, the distance $d(x + ys, u + vs)$ between nodes $x + ys$ and $u + vs$ in $LL(n, 1, s)$ is no greater than $dist((x, y), (u, v))$ on the grid $G_{n,s}$. The following lemma corresponds to the result on L-shaped tiles in [11]; we supply its proof, both for completeness and to introduce our proof methods and constructions (Fig. 2).

Lemma 1: Suppose that 0-points B and D have coordinates (u, v) and $(-x, y)$, respectively, with $u, x \geq 0$ and $v, y > 0$. Consider the parallelogram ABCD, where points A and C have coordinates (0, 0) and $(u - x, v + y)$, as shown in Fig. 3. If the area of the region Σ covered by the parallelogram ABCD, excluding the two edges BC and CD (and by implication, the lattice points B, C, and D) is n, then Σ contains exactly n lattice points whose labels are $0, 1, \ldots, n - 1$.

Proof. It is easily verified that the area of Σ is $|uy + vx|$. The fact that Σ with area n contains n points is a direct consequence of the parallelogram ABCD tessellating the plane and there being one grid point in each unit of area on average. To complete the proof, we need to show that there is only one i-point within Σ for $0 \leq i < n$. The proof is by contradiction. Suppose there are multiple i-points for some i. The corresponding points in all other parallelograms tessellating the plane will also be i-points. This means that j-points do not exist in the lattice Z^2 for at least one j $(0 \leq j < n)$; a clear contradiction. □

Lemmas 2 and 3 that follow establish the distance between points at or near the center of a parallelogram and the closest 0-point on the grid. Because any double-loop network is node-symmetric, its diameter can be found by obtaining the worst-case distance to a 0-point from another point. Lemma 2 covers the case where the midpoint of the parallelogram of Fig. 3 is in the first quadrant $(i \geq 0, j > 0)$, while Lemma 3 deals with the case of the midpoint being in the second quadrant $(i \leq 0, j > 0)$.

Lemma 2: Suppose that region Σ and the four lattice points A, B, C, and D are as in Lemma 1 and that $x \leq y$, $v \leq u$, $x < u$, and $v < y$ (see Fig. 3). Consider the points P and Q with coordinates $(\lfloor (u - x)/2 \rfloor, \lceil (v + y)/2 \rceil)$ and $(\lceil (u - x)/2 \rceil, \lceil (v + y)/2 \rceil)$, respectively. If Σ includes n lattice points, then no 0-point is closer to P than the nearest of the points A, B, C, D. Thus, the shortest distance from node 0 to node $\lfloor (u - x)/2 \rfloor + \lceil (v + y)/2 \rceil s$ in $LL(n, 1,$

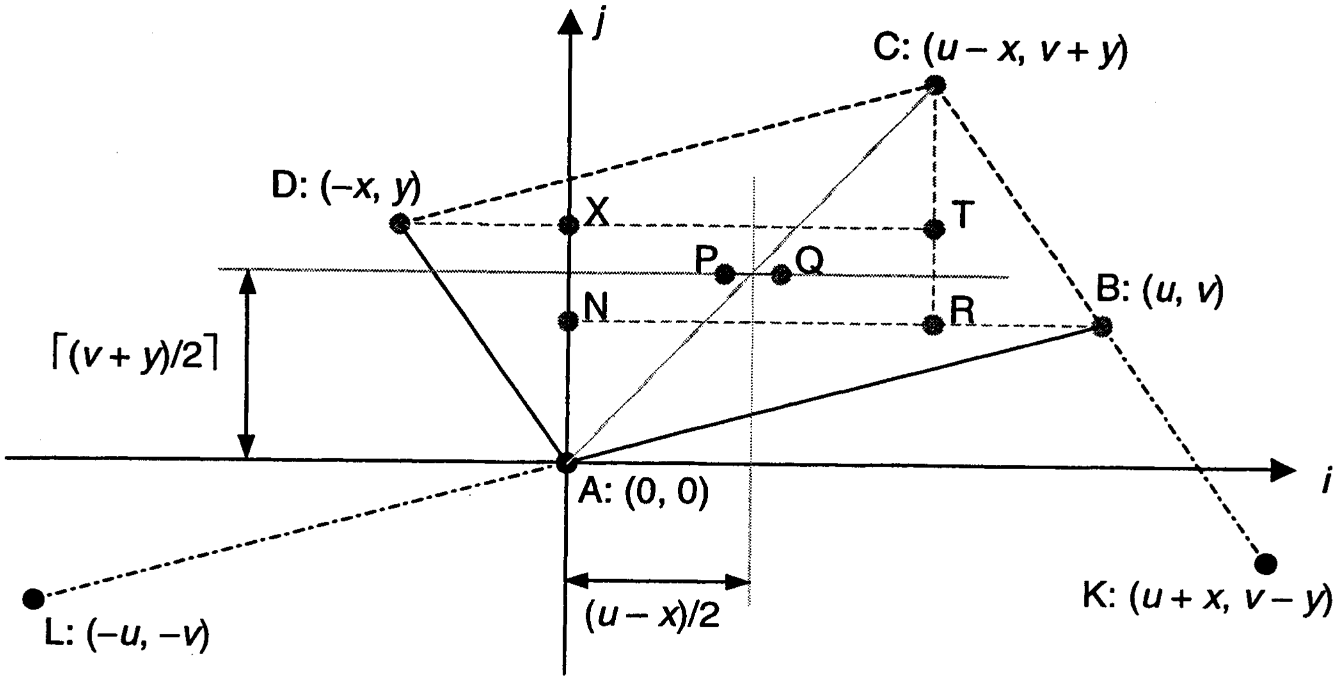

Fig. 3. Six 0-points (A, B, C, D, K, and L) on the lattice, along with other points used in proofs (some later on in the paper). Points P and Q coincide when $u - x$ is even.

s) is *min*(*dist*(A, P), *dist*(B, P), *dist*(C, P), *dist*(D, P)). The shortest distance from node 0 to node $\lceil (u-x)/2 \rceil + \lceil (v+y)/2 \rceil s$ is similarly related to point Q.

Proof. Let $d_P = min\{$dist(A, P), dist(B, P), dist(C, P), dist(D, P)$\}$. Clearly, $d(0, \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil s) \le d_P$. We begin by deriving the distances between P and the four 0-points A, B, C, D:

$$dist(\mathrm{A}, \mathrm{P}) = \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil \tag{7a}$$
$$dist(\mathrm{B}, \mathrm{P}) = u - \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil - v \tag{7b}$$
$$dist(\mathrm{C}, \mathrm{P}) = u - x - \lfloor (u-x)/2 \rfloor + v + y - \lceil (v+y)/2 \rceil \tag{7c}$$
$$dist(\mathrm{D}, \mathrm{P}) = x + \lfloor (u-x)/2 \rfloor + y - \lceil (v+y)/2 \rceil \tag{7d}$$

Suppose that the shortest path from node $\lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil s$ to node 0 contains m_l ring links and m_s skip links, where $m_l, m_s \in Z$. We have $m_l + \lfloor (u-x)/2 \rfloor + (m_s + \lceil (v+y)/2 \rceil) s \equiv 0 \bmod n$. We know that $(i, j) = (m_l + \lfloor (u-x)/2 \rfloor, m_s + \lceil (v+y)/2 \rceil)$ is a 0-point by definition. Because Σ contains n lattice points and the area of Σ is $uy + vx$, we have $uy + vx = n$. The 0-points (i, j), (u, v), and $(-x, y)$ imply, by definition, $i + js \equiv 0 \bmod n$, $u + vs \equiv 0 \bmod n$, and $-x + ys \equiv 0 \bmod n$. So, we have $iy + jx \equiv 0 \bmod n$ and $-iv + ju \equiv 0 \bmod n$. Thus, there exist two integers k_1 and k_2 such that $iy + jx = k_1 n$ and $-iv + ju = k_2 n$ by our definition of 0-points. Because $uy + vx = n$, we have:

$$(i, j) = (m_1 + \lfloor (u-x)/2 \rfloor, m_s + \lceil (v+y)/2 \rceil) = k_1(u, v) + k_2(-x, y) \tag{8}$$

Hence, $m_1 = k_1 u - k_2 x - \lfloor (u-x)/2 \rfloor$ and $m_s = k_1 v + k_2 y - \lceil (v+y)/2 \rceil$. There are four cases to be considered: (1) $k_1, k_2 \geq 1$, (2) $k_1, k_2 \leq 0$, (3) $k_1 \leq 0, k_2 > 0$, and (4) $k_1 > 0, k_2 \leq 0$.

Case 1: When $k_1, k_2 \geq 1$, $|m_1| + |m_s| \geq k_1(u+v) + k_2(y-x) - \lfloor (u-x)/2 \rfloor - \lceil (v+y)/2 \rceil \geq u + v + y - x - \lfloor (u-x)/2 \rfloor - \lceil (v+y)/2 \rceil = dist(\mathrm{C}, \mathrm{P})$.

Case 2: When $k_1, k_2 \leq 0$, $|m_1| + |m_s| \geq -k_1(u+v) - k_2(y-x) + \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil \geq \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil = dist(\mathrm{A}, \mathrm{P})$.

Case 3: When $k_1 \leq 0$ and $k_2 > 0$, $|m_1| + |m_s| \geq -k_1(u-v) + k_2(y+x) + \lfloor (u-x)/2 \rfloor - \lceil (v+y)/2 \rceil \geq x + y + \lfloor (u-x)/2 \rfloor - \lceil (v+y)/2 \rceil = dist(\mathrm{D}, \mathrm{P})$.

Case 4: When $k_1 > 0$ and $k_2 \leq 0$, $|m_1| + |m_s| \geq k_1(u-v) - k_2(y+x) - \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil \geq u - v - \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil = dist(\mathrm{B}, \mathrm{P})$.

Noting that $|m_1| + |m_s| \geq d_\mathrm{P}$ completes the proof of $d(0, \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil s) = d_\mathrm{P}$. □

Because the following Lemma can be proven in the same way as Lemma 2, we omit its proof.

Lemma 3: Suppose that region Σ and lattice points A, B, C, and D are defined as in Lemma 1 and that $x \geq y$, $v \geq u$, $x > u$, and $v > y$. Consider the points P′ and Q′ with coordinates $(-\lfloor (x-u)/2 \rfloor, \lceil (v+y)/2 \rceil)$ and $(-\lceil (x-u)/2 \rceil, \lceil (v+y)/2 \rceil)$, respectively. If Σ includes n lattice points, then no 0-point is closer to P′ than the nearest of the points A, B, C, D. Thus, the shortest distance from node 0 to node $-\lfloor (x-u)/2 \rfloor + \lceil (v+y)/2 \rceil s$ is $min(dist(\mathrm{A}, \mathrm{P}'), dist(\mathrm{B}, \mathrm{P}'), dist(\mathrm{C}, \mathrm{P}'), dist(\mathrm{D}, \mathrm{P}'))$. The shortest distance from node 0 to node $\lceil (u-x)/2 \rceil + \lceil (v+y)/2 \rceil s$ is similarly related to Q′. □

Lemma 4 below provides the worst-case distance from a point M inside the parallelogram ABCD of Fig. 3 to the nearest corner. It does so by exhaustively considering the various subregions in the parallelogram where M might lie, deriving the worst-case distance in each case, and using the results of Lemma 2 in the process. Lemma 5 does the same for the case where the center of the parallelogram is in the second quadrant, using Lemma 3 which covers this latter case.

Lemma 4: Suppose that region Σ and lattice points A, B, C, and D are defined as in Lemma 1 and that $x \leq y$, $v \leq u$, $x < u$, and $v < y$ (see Fig. 3). Let $r_0 = \lfloor (u+v)/2 \rfloor$, $r_1 = \lfloor (u - x + v + y)/2 \rfloor$, $r_2 = \lfloor (u + x + y - v)/2 \rfloor$, and $r_3 = \lfloor (x+y)/2 \rfloor$. Further, let $d_1 = min\{max\{r_0, r_1, r_3\}, max\{r_0, r_2, r_3\}\}$. If Σ includes n lattice points, then $D(n, 1, s)$ equals $r_1 - 1$ if $r_1 = r_2$

and $(u-x)(v+y) \equiv 1 \bmod 2$; otherwise, it equals d_1.

Proof. As $x \le y$ and $v \le u$, we have $r_0 \le r_1$ and $r_3 \le r_2$. Let $d_M = max\{min\{dist(A, M), dist(B, M), dist(C, M), dist(D, M)\} \mid M \text{ is a lattice point in } \Sigma\}$. From $r_0 \le r_1$ and $r_3 \le r_2$, we get $d_1 = min\{max\{r_1, r_3\}, max\{r_0, r_2\}\}$. Referring to Fig. 3, the following inequalities are seen to hold:

$$max\{min\{dist(A, M), dist(B, M)\} \mid M \text{ is a lattice point in the triangle ANB}\} \le r_0 \tag{9a}$$

$$max\{min\{dist(C, M), dist(D, M)\} \mid M \text{ is a lattice point in the triangle DTC}\} \le r_0 \tag{9b}$$

$$max\{min\{dist(A, M), dist(D, M)\} \mid M \text{ is a lattice point in the triangle AXD}\} \le r_3 \tag{9c}$$

$$max\{min\{dist(B, M), dist(C, M)\} \mid M \text{ is a lattice point in the triangle BRC}\} \le r_3 \tag{9d}$$

$$max\{min\{dist(B, M), dist(D, M)\} \mid M \text{ is a lattice point in the rectangle XTRN}\} \le r_2 \tag{9e}$$

Thus $d_M \le max\{r_0, r_2, r_3\} = max\{r_0, r_2\}$. Additionally, given that

$$max\{min\{dist(A, M), dist(C, M)\} \mid M \text{ is a lattice point in the rectangle XTRN}\} \le r_1 \tag{10}$$

we have $d_M \le max\{r_0, r_1, r_3\} = max\{r_1, r_3\}$. This leads to:

$$D(n, 1, s) \le d_M \le min\{max\{r_0, r_2\}, max\{r_1, r_3\}\} = d_1 \tag{11}$$

In the rest of our proof, we deal with three possible cases: (1) $r_1 < r_2$, (2) $r_1 > r_2$, and (3) $r_1 = r_2$. Each of the three cases has two subcases.

Case 1: If $r_1 < r_2$, then $d_1 = min\{max\{r_0, r_2\}, max\{r_1, r_3\}\} = max\{r_1, r_3\}$.

Subcase 1.1: If $r_3 \le r_1$, then $d_1 = r_1$. Let P be the point with coordinates $(\lfloor (u-x)/2 \rfloor, \lceil (v+y)/2 \rceil)$. We have $v < x$, given that $r_1 < r_2$. From the distance formulas (7) in the proof of Lemma 2, we have $d(0, \lfloor (u-x)/2 \rfloor + \lceil (v+y)/2 \rceil s) = min\{\text{dist}(A, P), \text{dist}(B, P), \text{dist}(C, P), \text{dist}(D, P)\} = r_1$. Thus, $D(n, 1, s) \ge r_1 = d_1$. So, $D(n, 1, s) = d_1$ in this subcase.

Subcase 1.2: If $r_3 > r_1$, then $d_1 = r_3$. Consider lattice points A, C, D in Fig. 3, plus the point L with coordinates $(-u, -v)$ and let the region Ω correspond to the parallelogram ACDL, excluding the edges CD and DL. As the area of the parallelogram Ω is $(u-x)(-v) - (v+y)(-u) = uy + vx$, it contains n lattice points labeled $0, 1, \ldots, n-1$. Let point R have coordinates $(-\lfloor x/2\rfloor, \lceil y/2\rceil)$. We have $dist(\text{A}, \text{R}) = \lfloor x/2\rfloor + \lceil y/2\rceil$, $dist(\text{C}, \text{R}) = u - x + \lfloor x/2\rfloor + v + y - \lceil y/2\rceil$, $dist(\text{D}, \text{R}) = x - \lfloor x/2\rfloor + y - \lceil y/2\rceil$, and $dist(\text{L}, \text{R}) = u + \lfloor x/2\rfloor + v + \lceil y/2\rceil$. As in the proof of Lemma 2, we can show that $d(0, -\lfloor x/2\rfloor + \lceil y/2\rceil s) = min\{\text{dist(A, R), dist(C, R), dist (D, R), dist(L, R)}\} = r_3$. Thus, $D(n, 1, s) \geq r_3 = d_1$, completing the proof that $D(n, 1, s) = d_1$ in this subcase.

Case 2: If $r_1 > r_2$, then $d_1 = min\{max\{r_0, r_2\}, max\{r_1, r_3\}\} = max\{r_0, r_2\}$.

Subcase 2.1: If $r_0 \leq r_2$, then $d_1 = r_2$. Let Q be the point with coordinates $(\lceil (u-x)/2\rceil, \lceil (v+y)/2\rceil)$. From Lemma 2, we have $d(0, \lceil (u-x)/2\rceil + \lceil (v+y)/2\rceil s) = min\{\text{dist(A, Q), dist(B, Q), dist(C, Q), dist(D, Q)}\} = r_2$. Thus, $D(n, 1, s) \geq r_2 = d_1$. So, $D(n, 1, s) = d_1$ in this subcase.

Subcase 2.2: If $r_0 > r_2$, then $d_1 = r_0$. Consider lattice points A, B, D in Fig. 5, plus the point K with coordinates $(u + x, v - y)$ and let the region Θ correspond to the parallelogram AKBD, excluding the edges BD and BK. As the area of Θ is $uy + vx$, it contains n lattice points labeled $0, 1, \ldots, n-1$. Let point S have coordinates $(\lfloor u/2\rfloor, \lceil v/2\rceil)$. As in the proof of Lemma 2, we can show that $d(0, \lfloor u/2\rfloor + \lceil v/2\rceil s) = min\{\text{dist(A, S), dist(K, S), dist (B, S), dist(D, S)}\} = r_0$. Thus, $D(n, 1, s) \geq r_0 = d_1$, completing the proof that $D(n, 1, s) = d_1$ in this subcase.

Case 3: If $r_1 = r_2$, then from $r_0 \leq r_1$ and $r_3 \leq r_2$, we get $d_1 = min\{max\{r_0, r_2\}, max\{r_1, r_3\}\} = r_1$.

Subcase 3.1: If $(u-x)(v+y) \equiv 0 \bmod 2$, let P be the point with coordinates $(\lfloor (u-x)/2\rfloor, \lceil (v+y)/2\rceil)$. We know that at least one of the two values $u - x$ and $v + y$ is even. Thus, all four of the distances $dist$(A, P), $dist$(B, P), $dist$(C, P), and $dist$(D, P) equal or exceed r_1. From Lemma 2, we have $d(0, \lfloor (u-x)/2\rfloor + \lceil (v+y)/2\rceil s) = min\{dist(\text{A, P}), dist(\text{B, P}), dist(\text{C, P}), dist(\text{D, P})\} = r_1$. Thus, $D(n, 1, s) \geq r_1 = d_1$. So, $D(n, 1, s) = d_1$ in this subcase.

Subcase 3.2: If $(u-x)(v+y) \equiv 1 \bmod 2$, then both values $u - x$ and $v + y$ are odd. Because $r_1 = r_2$, we have $x = v$, $r_0 < r_1$, and $r_3 < r_1$. Let P be the point with coordinates $(\lfloor (u-x)/2\rfloor, \lceil (v+y)/2\rceil)$. From Lemma 2, we have $D(n, 1, s) \geq d(0, \lfloor (u-x)/2\rfloor + \lceil (v+y)/2\rceil s) = min\{dist(\text{A, P}), dist(\text{B, P}), dist(\text{C, P}), dist(\text{D, P})\} = dist(\text{D, P}) = r_1 - 1$. Also, $D(n, 1, s) \leq max\{r_1, r_3\} = r_1$. To complete the proof in this subcase, we need to prove that $D(n, 1, s) < r_1$. If $max\{min\{dist(\text{A, P}), dist(\text{B, P}), dist(\text{C, P}), dist(\text{D, P})\} \mid \text{P is a lattice point in } \Sigma\} = r_1$, there must be a lattice point, say point J with coordinates (t, w), for which $min\{dist(\text{A, J}), dist(\text{B, J}), dist(\text{C, J}), dist(\text{D, J})\} = r_1$. From $r_0 < r_1$ and $r_3 < r_1$, it can be verified by the inequalities (9) in the proof of Lemma 4 that J must be in the rectangle NRTX (see Fig. 3). Given that $dist(\text{A, J}) + dist(\text{C, J}) = w + t + u - x - t + v + y - w = 2r_1$, $dist(\text{A, J}) \geq r_1$, and *dist*(C, J) $\geq r_1$, we have *dist*(A, J) = *dist*(C, J) $= t + w = r_1$. Thus, *dist*(B, J) $= u - t + r_1 - t - v = u - x + r_1 - 2t = 3r_1 - (v + y) - 2t = r_1 + [2r_1 - (v + y) - 2t]$ and *dist*(D, J) $= t + x + y - r_1 + t = v + y - r_1 + 2t = r_1 - [2r_1 - (v + y) - 2t]$. As $2r_1 - (v + y) - 2t$ is an odd integer, we know that *min*{*dist*(B, J), *dist*(D, J)} $< r_1$, leading to the desired result. □

Because the following Lemma can be proven in the same way as Lemma 4, we omit its proof.

Lemma 5: Suppose that region Σ and lattice points A, B, C, and D are defined as in Lemma 1 and that $x \geq y$, $v \geq u$, $x > u$, and $v > y$. Let $r_0 = \lfloor (u + v)/2 \rfloor$, $r_1' = \lfloor (x - u + v + y)/2 \rfloor$, $r_2' = \lfloor (u + x + v - y)/2 \rfloor$, and $r_3 = \lfloor (x + y)/2 \rfloor$. Further, let $d_1' = min\{max\{r_0, r_1', r_3\}, max\{r_0, r_2', r_3\}\}$. If Σ includes n lattice points, then $D(n, 1, s)$ equals $r_1' - 1$ if $r_1' = r_2'$ and $(u - x)(v + y) \equiv 1 \bmod 2$; otherwise, it equals d_1'. □

The stage is now set for the derivation of our diameter formulas.

3. Deriving the Diameter Formulas

In this section, we derive two diameter formulas for the undirected double loop networks corresponding to the two cases (1) $q > r$ and (2) $q \leq r$ with $b \leq aq + 1$. These are stated in Theorems 1 and 2, respectively.

Theorem 1: Consider the double-loop network $LL(n, 1, s)$ with chord lengths 1 and s ($2 \leq s < n/2$) and let $n = qs + r$, where r ($0 \leq r < s$) is the remainder of dividing n by s.

(1) If $n = qs$, i.e., $q > r = 0$, then $D(n, 1, s) = \lfloor (s + q - 1)/2 \rfloor$.

(2) If $q > r > 1$, then $D(n, 1, s) = max\{\lfloor (s + q - r + 1)/2 \rfloor, \lfloor (q + r)/2 \rfloor\}$.

(3) If $q > r = 1$, then $D(n, 1, s) = (s + q)/2 - 1$ if s and q are both even, and $\lfloor (s + q)/2 \rfloor$ otherwise.

Proof. (1) Let lattice points A, B′, C′, D′ have coordinates (0, 0), $(0, q)$, $(-s, q + 1)$, and $(-s, 1)$, respectively, and region Γ be the parallelogram AB′C′D′ with edges B′C′ and C′D′ not included. We know that the area of Γ is n. Let $w_0 = \lfloor q/2 \rfloor$, $w_1 = \lfloor (s + q + 1)/2 \rfloor$, $w_2 = \lfloor (s + q - 1)/2 \rfloor$, and $w_3 = \lfloor (s + 1)/2 \rfloor$. As $w_1 > w_2$, by Lemma 5 we have $D(n, 1, s) = \lfloor (s + q - 1)/2 \rfloor$.

(2) Let lattice points A, B′, C′, D′ have coordinates (0, 0), (r, q), $(-s + r, q + 1)$, and $(-s, 1)$, respectively, and region Γ be the parallelogram AB′C′D′ with edges B′C′ and C′D′ not included. We know that the area of Γ is n. Let $w_0 = \lfloor (r + q)/2 \rfloor$, $w_1 = \lfloor (s - r + q + 1)/2 \rfloor$, w_2

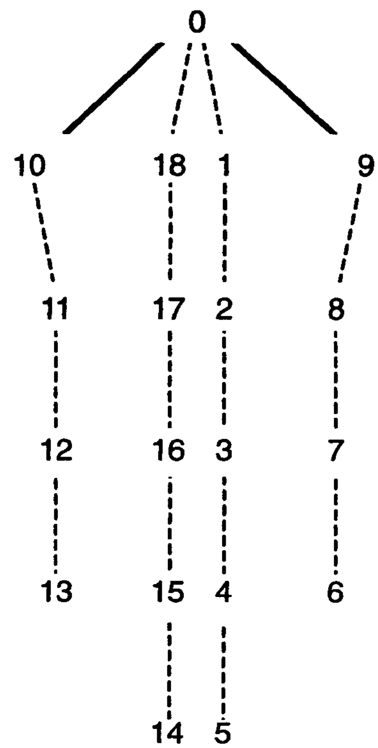

Fig. 4. Shortest paths from node 0 to all other nodes in $LL(19, 1, 9)$. Solid and dotted lines represent skip and ring links, respectively.

$= \lfloor (s + r + q - 1)/2 \rfloor$, and $w_3 = \lfloor (s + 1)/2 \rfloor$. As $r > 1$, we have $w_2 > w_1$. This combined with $w_0 \le w_2$, $w_3 < w_1$, and Lemma 5 yields $D(n, 1, s) = max\{\lfloor (s - r + q + 1)/2 \rfloor, \lfloor (q + r)/2 \rfloor\}$.

(3) Let lattice points A, B′, C′, D′ have coordinates (0, 0), $(1, q)$, $(-s + 1, q + 1)$, and $(-s, 1)$, respectively, and region Γ be the parallelogram AB′C′D′ with edges B′C′ and C′D′ not included. We know that the area of Γ is n. Let $w_0 = \lfloor (1 + q)/2 \rfloor$, $w_1 = w_2 = \lfloor (s + q)/2 \rfloor$, and $w_3 = \lfloor (s + 1)/2 \rfloor$. As $w_1 = w_2$, Lemma 5 yields the desired diameter formula. □

Example 1. Computing the diameter of $LL(19, 1, 9)$, a 19-node double-loop network with skip distance 9: Here, we have $n = 19$, $s = 9$, $q = 2$, $r = 1$. Case (3) of Theorem 1 yields $D(19, 1, 9) = \lfloor (9 + 2)/2 \rfloor = 5$. This is confirmed by Fig. 4, where shortest paths from node 0 to all other nodes are shown. □

Theorem 2: Consider the double-loop network $LL(n, 1, s)$ with chord lengths 1 and s ($2 \le s < n/2$) and let $n = qs + r$, where r ($0 \le r < s$) is the remainder of dividing n by s. Furthermore, assume that $s = ar + b$, where b ($0 \le b < r$) is the remainder of dividing s by

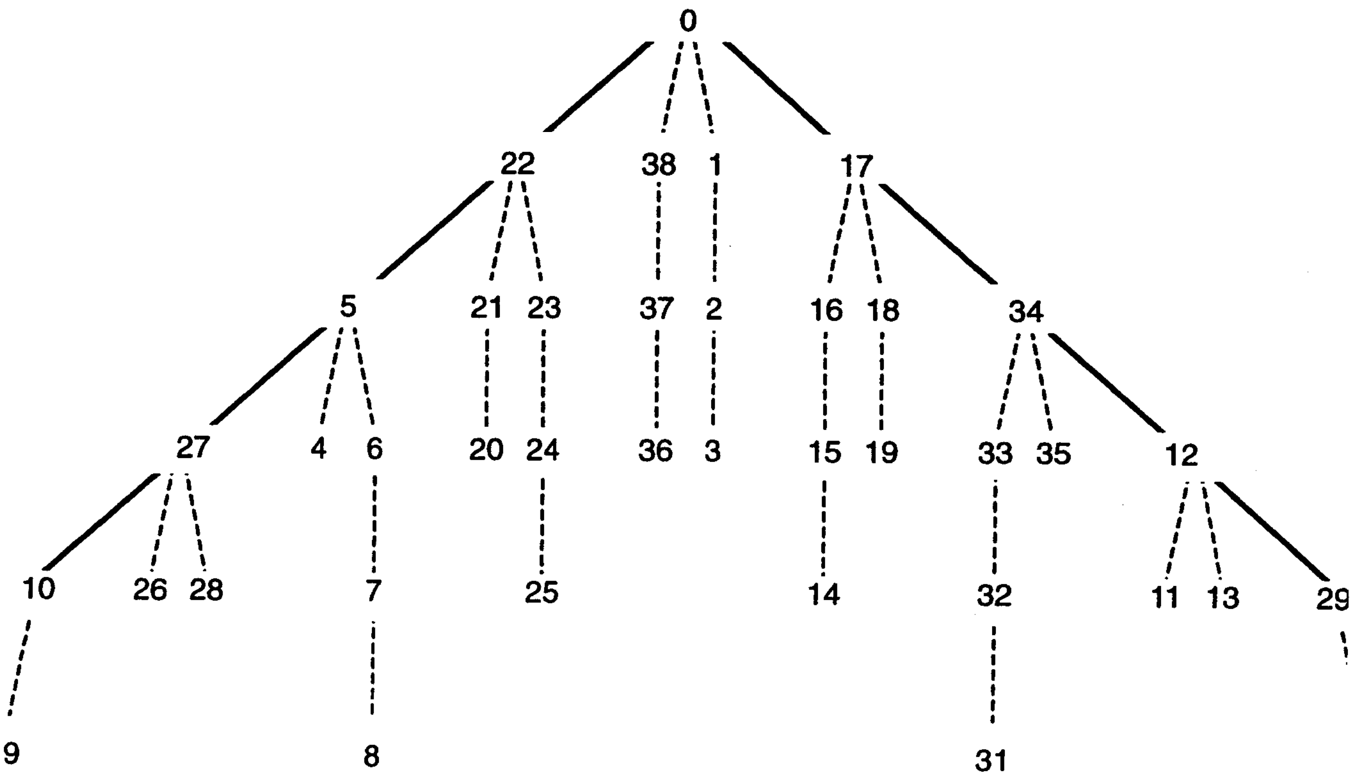

Fig. 5. Shortest paths from node 0 to all other nodes in *LL*(39, 1, 17). Solid and dotted lines represent skip and ring links, respectively.

r. Define $p_0 = \lfloor (q + r)/2 \rfloor$, $p_1 = \lfloor (r - b + (a + 1)q + 1)/2 \rfloor$, $p_2 = \lfloor (r + b + (a - 1)q + 1)/2 \rfloor$, and $p_3 = \lfloor (b + aq + 1)/2 \rfloor$. Let $e_1 = min\{max\{p_1, p_3\}, max\{p_0, p_2\}\}$. If $q \le r$ and $b \le aq + 1$, then $D(n, 1, s)$ is $p_1 - 1$ when $p_1 = p_2$ and $(r + b)(aq - q + 1) \equiv 1 \bmod 2$; otherwise, it is e_1.

Proof. Let lattice points A, B″, C″, D″ have coordinates (0, 0), (r, q), $(r - b, (a + 1)q + 1)$, and $(-b, aq + 1)$, respectively, and region Ψ be the parallelogram AB″C″D″ which does not include the edges B″C″ and C″D″. We know that the area of Ψ is n. Given that $q \le r$ and $b \le aq + 1$, we have $p_0 \le p_1$ and $p_3 \le p_2$. Thus, $min\{max\{p_0, p_1, p_3\}, max\{p_0, p_2, p_3\}\} = min\{max\{p_1, p_3\}, max\{p_0, p_2\}\} = e_1$. Because $q \le r$ and $b \le aq + 1$, we have $b < r$ and $q < aq + 1$. Thus, Lemma 4 applies and completes the proof. □

Example 2. Computing the diameter of *LL*(39, 1, 17), a 39-node double-loop network with skip distance 17: We have $n = 39$, $s = 17$, $q = 2$, $r = 5$, $a = 3$, $b = 2$, $p_0 = 3$, $p_1 = 6$, $p_2 = 6$, $p_3 = 4$. Because $p_1 = p_2 = 6$ and $(r + b)(aq - q + 1) = 35 \equiv 1 \bmod 2$, Theorem 2 yields $D(39, 1, 17) = p_1 - 1 = 5$. This is confirmed by Fig. 5, where shortest paths from node 0 to all other nodes are shown. □

4. Conclusion

Determination of the exact diameter of double-loop networks in the form of closed-form formulas, like many other combinatorial problems, is difficult. In this paper, we have provided closed-form formulas for the diameter of a double-loop network for the case $q > r$ (Theorem 1) and for a subcase of the case $q \leq r$ when $b \leq aq + 1$ (Theorem 2). In the complementary subcase of $q \leq r$, when $b > aq + 1$, the $O(\log n)$-time algorithm of Zerovnik and Pisanski [13] can be used to derive the diameter, but we do not have a closed-form formula for the diameter in this subcase. Deriving such a formula constitutes a possible direction for further research, as is the exploration of other topological and algorithmic properties of double-loop networks.

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