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Further mathematical properties of Cayley digraphs applied to hexagonal and honeycomb meshes

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Abstract

In this paper, we extend known relationships between Cayley digraphs and their subgraphs and coset graphs with respect to subgroups to obtain a number of general results on homomorphism between them. Intuitively, our results correspond to synthesizing alternative, more economical, interconnection networks by reducing the number of dimensions and/or link density of existing networks via mapping and pruning. We discuss applications of these results to well-known and useful interconnection networks such as hexagonal and honeycomb meshes, including the derivation of provably correct shortest-path routing algorithms for such networks.

Keywords: Cayley digraphs; Cellular networks; Coset graphs; Diameter; Distributed systems; Homomorphism; Interconnection networks; Internode distance; Parallel processing; Routing

1. Introduction

The fact that Cayley digraphs and coset graphs are excellent models for interconnection networks, studied in connection with parallel processing and distributed computation, is widely acknowledged [1,2,4]. Many well-known interconnection networks are Cayley digraphs or coset graphs. For example, hypercube, butterfly, and cube-connected cycle networks are Cayley graphs, while de Bruijn and shuffle-exchange networks are coset graphs [4,11].

Much work on interconnection networks can be categorized as ad hoc design and evaluation. Typically, a new interconnection scheme is suggested and shown to be superior to some previously studied network(s) with respect to one or more performance or complexity attributes. Whereas Cayley digraphs have been used to explain and unify interconnection networks with some success, much work remains to be done. As suggested by Heydemann [4], general theorems are lacking for Cayley digraphs and more group theory has to be exploited to find properties of Cayley digraphs.

In this paper, we explore the relationships between Cayley digraphs and their subgraphs and coset graphs with respect to subgroups and obtain general results on homomorphism between them. We provide several applications of

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Nomenclature[2]

$\bullet \leqslant \bullet$	subgroup relationship
$\bullet \lhd \bullet$	normal subgroup relationship
$^{\bullet}/_{\bullet}$	set of (right) cosets
$\bullet \times \bullet$	graph or set cross-product
$\bullet^{(i)}$	the symbol "•" repeated i times
$(\bullet, \bullet)$	edge
$\langle \bullet \rangle$	cyclic group
$\rightarrow$	mapping
$\leftrightarrow$	bijection
$\cong$	isomorphic to
$\equiv$	congruent to
Γ, Δ, Σ	graphs or digraphs
ϕ	homomorphism
1	identity element of a group
$Aut()$	automorphism group
C_k	cycle (ring network) of size k
$Cay()$	Cayley graph
CCC_q	cube-connected cycles of order q
$Cos()$	coset graph
$dis()$	distance function
$E()$	edge set of a graph
G, H	groups
K, N	subgroups
sign(e)	-1 if $e < 0$; 0 if $e = 0$; $+1$ if $e > 0$
S, T	generator sets, subsets of G
$V()$	vertex set of a graph
Z_q	cyclic group of order q
Z_d^q	elementary abelian d-group of order d^q

these results to well-known and practically useful interconnection networks such as hexagonal and honeycomb meshes. Our Cayley-graph-based addressing scheme for these networks allows us to use results from group theory to derive distance formulas as well as shortest-path routing algorithms that are simple and elegant. These results lead us to the conclusion that our addressing schemes are superior to those proposed in [6,10], and to earlier schemes reviewed and improved upon therein.

The hexagonal mesh network (Fig. 1) has been studied extensively. The infinite version of this network can be used to derive it structural properties, internode distances, and routing schemes. Finite versions can be defined in different ways by restricting the nodes to within a polygonal area (rectangle, hexagon, and so on). Hexagonal torus networks are derived from the latter finite versions by postulating various wraparound connectivity patterns at the edges. The honeycomb mesh network (Fig. 2) is similarly important in both parallel processing and data communication. It too can be studied as an infinite structure or in finite forms, and it can be turned into toroidal versions by the provision of wraparound links. For both of these network classes, the node indexing (addressing) scheme has been shown to be important to the simplicity and efficiency of message routing. Figs. 1 and 2 also depict the node indexing schemes proposed in [6,10], respectively, to facilitate the study of their structural properties and routing algorithms.

Before proceeding further, we introduce some definitions and notations related to digraphs, Cayley digraphs in particular, and interconnection networks. For more definitions and basic results on graphs and groups we refer the

[2] Unless explicitly specified, all graphs in this paper are undirected graphs.

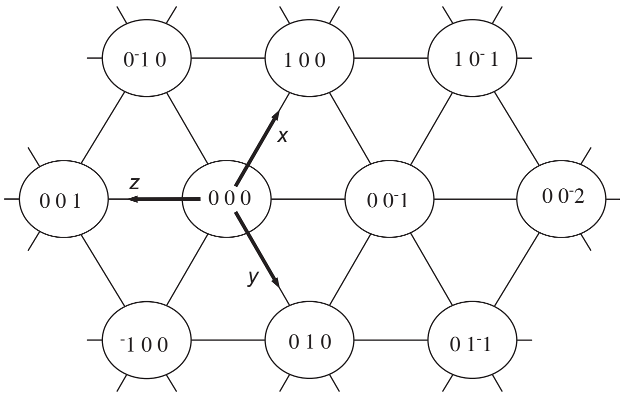

Fig. 1. Hexagonal mesh network, as usually drawn, and the *xyz* addressing scheme of [6].

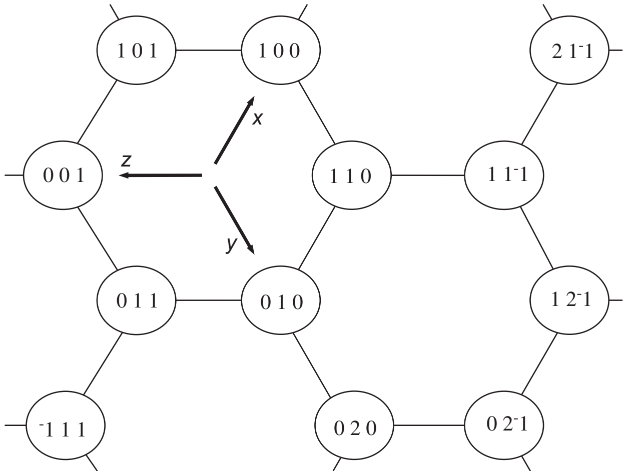

Fig. 2. Honeycomb mesh network, as usually drawn, and the *xyz* addressing scheme of [10].

reader to [3], for instance, and on interconnection networks to [5,7]. Unless noted otherwise, all graphs in this paper are undirected graphs.

A digraph $\Gamma = (V, E)$ is defined by a set V of vertices and a set E of arcs or directed edges. The set E is a subset of elements (u, v) of $V \times V$. If the subset E is symmetric, that is, $(u, v) \in E$ implies $(v, u) \in E$, we identify two opposite arcs (u, v) and (v, u) by the undirected edge (u, v). Because we deal primarily with undirected graphs in this paper, no problem arises from using the same notation (u, v) for a directed arc from u to v or an undirected edge between u and v.

Let G be a (possibly infinite) group and S a subset of G. The subset S is said to be a generating set for G, and the elements of S are called generators of G, if every element of G can be expressed as a finite product of their powers. We also say that G is generated by S. The Cayley digraph of the group G and the subset S, denoted by $Cay(G, S)$, has vertices that are elements of G and arcs that are ordered pairs (g, gs) for $g \in G$, $s \in S$. If S is a generating set of G then we say that $Cay(G, S)$ is the Cayley digraph of G generated by S. If $1 \notin S$ (1 is the identity element of G) and $S = S^{-1}$, then $Cay(G, S)$ is a simple graph.

Assume that Γ and Σ are two digraphs. The mapping ϕ of $V(\Gamma)$ to $V(\Sigma)$ is a homomorphism from Γ to Σ if for any $(u, v) \in E(\Gamma)$ we have $(\phi(u), \phi(v)) \in E(\Sigma)$. In particular, if ϕ is a bijection such that both ϕ and the inverse of ϕ are homomorphisms then ϕ is called an isomorphism of Γ to Σ. Let G be a (possibly infinite) group and S a subset of G. Assume that K is a subgroup of G (denoted as $K \leqslant G$). Let ${}^{G}/_{K}$ denote the set of the right cosets of K in G. The (right) coset graph of G with respect to subgroup K and subset S, denoted by $Cos(G, K, S)$, is the digraph with vertex set ${}^{G}/_{K}$ such that there exists an arc (Kg, Kg') if and only if there exist $s \in S$ and $Kgs = Kg'$.

The following basic theorem, which can be easily proven, is helpful in establishing some of our subsequent results.

Theorem 1. *For $g \in G$, $S \subseteq G$, and $K \leqslant G$, the mapping $\phi: g \to Kg$ is a homomorphism from $Cay(G, S)$ to $Cos(G, K, S)$.*

2. Hexagonal mesh networks

Let $G = Z \times Z$, where Z is the infinite cyclic group of integers, and consider $\Gamma = Cay(G, S)$ with $S = \{(\pm 1, 0), (0, \pm 1), (1, 1), (-1, -1)\}$. It is evident that Γ is isomorphic to the hexagonal (hex) mesh network [10]. Fig. 3 shows a small part of an infinite hex mesh in which the six neighbors of the "center" node (0, 0) are depicted. A finite hex mesh is obtained by simply using the same connectivity rules for a finite subset of the nodes located within a regular boundary (often a rectangle or hexagon). In the latter case, wraparound links are sometimes provided to keep the node degree uniformly equal to 6, leading to a hexagonal torus network. Here, we do not concern ourselves with these variations and deal only with the infinite hex mesh.

Let $N = \{(d, d, d) | d \in Z\}$. Then, N is a normal subgroup of $Z \times Z \times Z$ (denoted as $N \lhd Z \times Z \times Z$). Let $H = {}^{Z\times Z\times Z}/_{N}$ and $\Sigma = \mathrm{Cos}(Z \times Z \times Z, N, S')$, where $S' = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. Then, it is clear that Σ is isomorphic to the Cayley graph $Cay(H, NS')$ by Theorem 1, where $NS' = \{Ns' | s' \in S'\}$ is a subset of the group H. Now we are prepared to show the following result.

Proposition 1. *The network Σ, defined above, is isomorphic to the hex mesh network.*

Proof. Given that Γ, introduced earlier in this section, is isomorphic to hex mesh, we only need to show that Σ is isomorphic to Γ. Let $\phi: (a, b, c) + N \to (a - c, b - c)$ be the correspondence of Σ to Γ. It is evident that ϕ is a bijection of Σ to Γ. Moreover, $((a, b, c + N, (a, b, c) + s' + N)$ is an edge of Σ for $s' \in S'$ if and only if $((a - c, b - c), (a - c, b - c) + s)$ is an edge of Γ for $s \in S$, where $\phi: s' + N \to s$. Hence ϕ is an isomorphism of Σ to Γ. $\square$

Proposition 1 has interesting applications to parallel and distributed systems, including in certain problems pertaining to cellular communication networks [6].

Using the Cayley-graph formulation of hex mesh networks, we can easily derive the distance $dis((a, b), (c, d))$ between the vertices (a, b) and (c, d) in such networks.

Proposition 2. *In the hex mesh Γ, $dis((0, 0), (a, b))$ equals $\max(|a|, |b|)$ if a and b have the same sign and $|a| + |b|$ otherwise.*

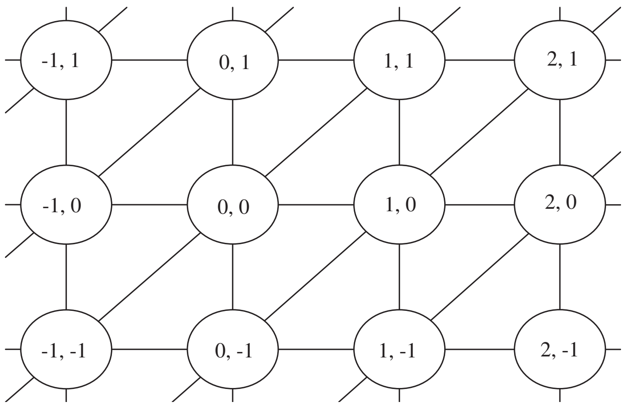

Fig. 3. Connectivity pattern for hexagonal mesh network, where node (i, j) is connected to nodes $(i \pm 1, j)$, $(i, j \pm 1)$, $(i + 1, j + 1)$, and $(i - 1, j - 1)$.

Proof. We only prove the proposition in the two cases $a \geqslant b \geqslant 0$ and $a \geqslant 0 \geqslant b$; other cases can be demonstrated similarly. When $a \geqslant b \geqslant 0$, we have the following shortest path of length a from $(0,0)$ to (a,b): $(0,0) \to (1,0) \to \cdots \to (a-b,0) \to (a-b+1,1) \to \cdots \to (a-1,b-1) \to (a,b)$. Clearly, $dis((0,0),(a,b)) = \max(|a|,|b|)$ in this case. When $a \geqslant 0 \geqslant b$, we have the following shortest path of length $a+|b|$: $(0,0) \to (1,0) \to \cdots \to (a,0) \to (a,-1) \to \cdots \to (a,b)$. Therefore, $dis((0,0),(a,b)) = |a|+|b|$ in this case. □

By symmetry of Cayley graphs, we can easily obtain the distance between any two vertices in the graph Γ from Proposition 2, using $dis((a,b),(c,d)) = dis((0,0),(c-a,d-b))$. This observation and the preceding discussion lead to a simple distributed routing algorithm for Γ. Note that Algorithm 1 gives precedence to "diagonal" links over horizontal or vertical links and to horizontal links over vertical links, and that at most two of the three link types are viable options in any given step of shortest-path routing. It is fairly easy to modify the algorithm to distribute the traffic more evenly. The function sign(e) returns a value in $\{-1,0,+1\}$: -1 if $e<0$; 0 if $e=0$; $+1$ if $e>0$.

Algorithm 1. The first node p on a shortest path from (a,b) to (c,d) in hex mesh
$\gamma = \text{sign}(a-c)$; $\delta = \text{sign}(b-d)$;
if $\gamma\delta \geqslant 0$ then $p = (a-\gamma, b-\delta)$ else $p = (a-\gamma, b)$.

We now consider the automorphism group $Aut(\Gamma)$ of the graph Γ. We know that $Aut(\Gamma)$ contains the (left) regular automorphism group of Γ which is isomorphic to the group $Z \times Z$; we still denote it as $Z \times Z$. Furthermore, we know that $Aut(\Gamma) = (Z \times Z)(Aut(\Gamma))_{(u,v)}$, where $(Aut(\Gamma))_{(u,v)}$ is the stabilizer (subgroup) of $Aut(\Gamma)$ which fixes the vertex (u,v).

Let σ be the mapping $(x,y) \to (x, x-y)$. Then it easily verified that $\sigma \in (Aut(\Gamma))_{(0,0)}$ and $\sigma^2 = 1$. Let λ be the mapping $(x,y) \to (x-y, x)$. Then, $\lambda \in (Aut(\Gamma))_{(0,0)}$ and $\lambda^6 = 1$. We now establish that $(Aut(\Gamma))_{(0,0)} = \langle \lambda, \sigma \rangle$, where $\sigma\lambda\sigma = \lambda^{-1}$.

It is easily verified that $\sigma\lambda\sigma = \lambda^{-1}$. In order to prove $(Aut(\Gamma))_{(0,0)} = \langle \lambda, \sigma \rangle$, let $\varphi \in (Aut(\Gamma))_{(0,0)}$. Clearly $\varphi(S) = S$. Let Δ be the graph whose vertex set is S and edge set is the hexagonal cycle formed by the vertices of S. Given that φ fixes $(0,0)$, we have $\varphi \in Aut(\Delta)$. But $Aut(\Delta) = D_6$ and $\langle \lambda, \sigma \rangle \cong D_6$, where D_6 is the dihedral group of order 12. Thus we obtain that $(Aut(\Gamma))_{(0,0)} = \langle \lambda, \sigma \rangle$, proving the following.

Proposition 3. *Let* $\sigma: (x,y) \to (x, x-y)$ *and* $\lambda: (x,y) \to (x-y, x)$ *be mappings from* $Z \times Z$ *to* $Z \times Z$. *Then,* $Aut(\Gamma) = (Z \times Z)\langle \lambda, \sigma \rangle$, *where* $\sigma^2 = \lambda^6 = 1$ *and* $\sigma\lambda\sigma = \lambda^{-1}$.

3. Honeycomb and other networks

Let G be a (possibly infinite) group and S a subset of G and consider the problem of constructing a group G'' and its generating set S'' such that $G'' = G$ as sets and $S'' \subseteq S$, and a homomorphism $\phi: \Gamma'' \to \Gamma$, where $\Gamma = Cay(G,S)$ and $\Gamma'' = Cay(G'', S'')$. It is easily shown that a number of pruning schemes, including the one studied in [8], are equivalent to the construction above. Pruning of interconnection networks constitutes a way of obtaining variants with lower implementation cost, and greater scalability and packageability [9]. If pruning is done with care, and in a systematic fashion, many of the desirable properties of the original (unpruned) network, including (node, edge) symmetry and regularity, can be maintained while reducing both the node degree and wiring density which influence the network cost.

Example 1. Let $G = Z_2^q \times Z_q$, $s_i = (0^{(i-1)}, 1, 0^{(q-i+1)})$, $i = 1, \ldots, q$, and $s = (0^{(q)}, 1)$. Then $s_i, s \in G$. Let $S = \{s_1, \ldots, s_q, s\}$ and $\Gamma = Cay(G,S)$. Define $N = Z_2^q$, $K = \langle s \rangle$, and $G'' = N \otimes K$ as the semidirect product of N by K, where $s^{-1} \otimes s_1 \otimes s = s_2, \ldots, s^{-1} \otimes s_{q-1} \otimes s = s_q, s^{-1} \otimes s_q \otimes s = s_1$. Let $\Gamma'' = Cay(G'', S'')$, where $S'' = \{s_1, s\}$. Then, $\Gamma'' = CCC_q$ (cube-connected cycle of order q), which is a pruned network from $\Gamma = Cay(G,S)$.

Example 2. In [8], the authors studied the honeycomb torus network as a pruned 2D torus. They also proved that the honeycomb torus network is a Cayley graph, without explicating its associated group. We fill this gap in the following, while also showing (in the proof of Proposition 4 below) why the parameter k in [8] must be even. Let $G = (\langle c \rangle \langle b \rangle)\langle a \rangle$

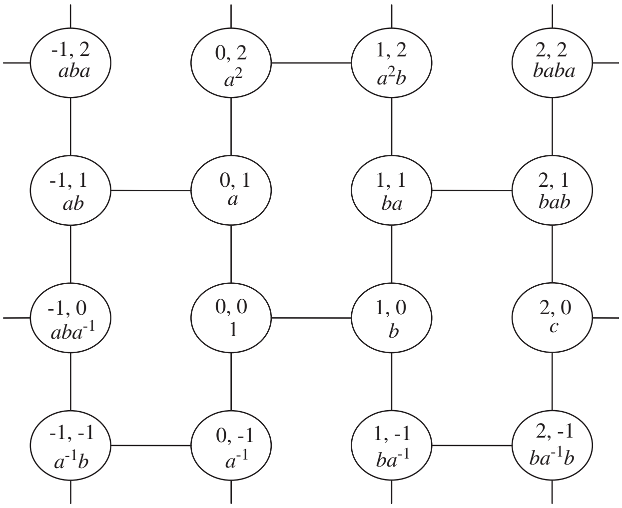

Fig. 4. Connectivity pattern for the honeycomb mesh network. Each node is labeled in two ways corresponding to its integer coordinates on the grid (upper label) and the notation in Remark 1 (lower label), with the associations being $(0, 1) = a$, $(1, 0) = b$, $(2, 0) = c$.

be the group generated by the elements a, b, c, satisfying the relations $a^k = b^2 = c^{l/2} = 1$, $bcb = c^{-1}$, $aba^{-1} = c^{-1}b$, $aca^{-1} = c^{-1}$. Here, k and l are even integers. Thus the group $\langle c\rangle\langle b\rangle = \langle c, b\rangle$ is a semidirect product of $\langle c\rangle$ by $\langle b\rangle$, and G is a semidirect product of $\langle c, b\rangle$ by $\langle a\rangle$. Let $S = \{a, a^{-1}, b\}$ and $\Delta = Cay(G, S)$. We prove that Δ is isomorphic to the honeycomb torus network in [8] (denoted as Σ).

Proposition 4. *The Cayley digraph Δ, defined in Example 2, is isomorphic to the honeycomb torus network Σ [8]; that is, $\Delta \cong \Sigma$.*

Proof. Let $[0, 1]^T = a'$ and $[1, 0]^T = b'$ in the proof of Theorem 1 in [8], and let $c' = b' \otimes a'^{-1} \otimes b' \otimes a'$. Note that if either k or l is an odd integer, the group operator $\otimes$ will not be well defined. For example, for $k = 5$ and $l = 4$ we have $[1, -1]^T = [1, 4]^T$. However, because $[1, -1]^T \otimes [-1, 1]^T = [0, 0]^T \neq [2, 0]^T = [1, 4]^T \otimes [-1, 1]$, the group operator is ill defined. With the assumptions above, we easily verify that $a'^k = b'^2 = c'^{l/2} = 1$, $b' \otimes c' \otimes b' = c'^{-1}$, $a' \otimes b' \otimes a'^{-1} = c'^{-1} \otimes b'$, and $a' \otimes c' \otimes a^{-1} = c'^{-1}$. Let ϕ be the mapping $a \to a'$, $b \to b'$. Then, ϕ may be extended into an isomorphism of the group G to the group $G' = (\langle c'\rangle\langle b'\rangle)\langle a'\rangle$, because their defining relations are the same. Hence ϕ is also an isomorphism of the graph Δ to the graph Σ. □

Remark 1. We may consider the infinite honeycomb mesh network as a Cayley graph of a different group. Let $G = (\langle c\rangle\langle b\rangle)\langle a\rangle$, where $\langle c\rangle$ and $\langle a\rangle$ are infinite cyclic groups, and c, b, a satisfy the relations $b^2 = 1$, $bcb = c^{-1}$, $aba^{-1} = c^{-1}b$, $aca^{-1} = c^{-1}$. Let $S = \{a, a^{-1}, b\}$ and $\Delta_\infty = Cay(G, S)$. Then Δ_∞ is isomorphic to the infinite honeycomb mesh network (see Fig. 4).

Now let $N = \langle a^k\rangle\langle c^{l/2}\rangle$, where k and l are even integers. We can easily verify that $N \lhd G$. Construct the quotient group $G' = {}^G/_N$ and let $S' = \{Na, Na^{-1}, Nb\}$; the graph $Cay(G', S')$ is isomorphic to the honeycomb torus network.

Remark 2. An important case in the construction above arises for $G = Z_{k_1} \times \cdots \times Z_{k_n} (n \geqslant 2)$, where $k_1, \ldots, k_n$ are positive integers. In general, $G'' = N \otimes K$ is a semidirect product of groups N and K. If ϕ is the identity mapping of G to G'', then for $s'' \in S''$ we have $(x_1, \ldots, x_n) \otimes s'' = (x_1, \ldots, x_n) + s$ for some $s \in S$. In particular, if $(x_1, \ldots, x_n)$ is the identity element of G, we obtain that $s'' = s$ for some $s \in S$. Hence $S'' \subseteq S$. For instance, for the honeycomb torus network, we have $N = \langle c\rangle\langle b\rangle$, $K = \langle a\rangle$, $S = \{a, a^{-1}, b, b^{-1}\}$, $S'' = \{a, a^{-1}, b\}$.

As an application of the method above, we now consider the problem of finding the distance between two vertices in the honeycomb mesh network Δ_∞. We know that the infinite honeycomb mesh network $\Delta_\infty = Cay(G, S)$, where

$G=(\langle c\rangle\langle b\rangle)\langle a\rangle$, $S=\{a, a^{-1}, b\}$, $\langle c\rangle$ and $\langle a\rangle$ are infinite cyclic groups and c, b, a satisfy the relations $b^2=1$, $bcb=c^{-1}$, $aba^{-1}=c^{-1}b$, $aca^{-1}=c^{-1}$. Thus, any element of G can be expressed as the product $c^jb^la^i$, where l is 0 or 1 and j and i are integers. We first formulate the distance between vertex 1 (the identity of G) and vertex $c^jb^la^i$ in the following theorem.

Theorem 2. *For* $|i|\leqslant|2j+l|$, *we have* $dis(1, c^jb^la^i)=|4j+l+1/2[(-1)^{i+l}-(-1)^l]|$; *otherwise*, $dis(1, c^jb^la^i)=|i|+|2j+l|$.

Proof. We first consider the case of $l=0$, proving that if $|i|\leqslant|2j|$, we have $dis(1, c^ja^i)=|4j|$ for $i\equiv 0 \bmod 2$ and $dis(1, c^ja^i)=|i|+2|j|$ otherwise. We limit our proof to two cases: (1) $0\leqslant i/2\leqslant j$, and (2) $0\leqslant j\leqslant i/2$, where i is even. The other cases may be verified similarly.

(1) Because $c=ba^{-1}ba$ and $a^2b=ba^2$, we have $c^ja^i=(ba^{-1}ba)^ja^i=(ba^{-1}ba)^{j-i/2}(ba^{-1}ba)^{i/2}a^i=(ba^{-1}ba)^{j-i/2}(baba)^{i/2}$. This gives $dis(1, c^ja^i)\leqslant 4(j-i/2)+4(i/2)=4j$. It is clear that $dis(1, c^j)=4j$ and $dis(1, c^ja^i)\geqslant dis(1, c^j)$ when $0\leqslant i/2\leqslant j$ (i is even). Hence we obtain $dis(1, c^ja^i)=|4j|$.

(2) We have $c^ja^i=(ba^{-1}ba)^ja^i=(baba)^ja^{i-2j}$, leading to $dis(1, c^ja^i)\leqslant 4j+i-2j=i+2j$. Using induction on j, we show that $dis(1, c^ja^i)=i+2j$ when $0\leqslant j\leqslant i/2$ and i is even. We begin with $dis(1, a^i)=i$. Suppose that $dis(1, c^ja^i)=i+2j$ when $0\leqslant j\leqslant i/2$. Let $0\leqslant j+1\leqslant i/2$. Then by our induction assumption ($j\geqslant 0$): $dis(1, c^{j+1}a^i)\geqslant dis(1, c^ja^i)+2\geqslant i+2j+2$. This, combined with $dis(1, c^ja^i)\leqslant i+2j$, derived earlier, leads to $dis(1, c^{j+1}a^i)=i+2(j+1)$, completing the induction.

We next consider the remaining case of $l=1$, proving that for $|i|\leqslant|2j+1|$, $dis(1, c^jba^i)=|4j+1+(i \bmod 2)|$, while for $|i|\geqslant|2j+1|$, $dis(1, c^jba^i)=|i|+|2j+1|$. As in the case of $l=0$, we only consider two cases: (1) $0\leqslant i/2\leqslant j$, and (2) $0\leqslant j\leqslant i/2$, where i is even.

(1) We have $c^jba^i=(ba^{-1}ba)^{j-i/2}(ba^{-1}ba)^{i/2}a^ib=(ba^{-1}ba)^{j-i/2}(baba)^{i/2}b$. This yields $dis(1, c^jba^i)\leqslant 4(j-i/2)+4(i/2)+1=4j+1$. As in the case of $l=0$, we can show that $dis(1, c^jba^i)=|4j+1|$ when $0\leqslant i/2\leqslant j$ and i is even.

(2) We have $c^jba^i=(ba^{-1}ba)^ja^{2j}a^{i-2j}b=(baba)^ja^{i-2j}b$. This leads to $dis(1, c^jba^i)\leqslant 4j+i-2j+1=i+2j+1$. As in the case of $l=0$, we can easily verify that $dis(1, c^jba^i)=i+2j+1$ when $0\leqslant j\leqslant i/2$ and i is even. This completes the proof. □

Applying the pruning scheme to the infinite mesh, we obtain the infinite honeycomb mesh. By Remark 1, it is isomorphic to the Cayley graph Δ_∞. Thus by Theorem 2 we have the following.

Corollary 1. *In the infinite honeycomb mesh, the distance between nodes* (x, y) *and* (u, v) *is obtained as follows: if* $|v-y|\leqslant|u-x|$, *then* $dis((x, y), (u, v))$ *equals* $|2(u-x)+1/2[(-1)^{u+v}-1]|$ *when* $x+y\equiv 0 \bmod 2$, *and* $|2(x-u)+1/2[(-1)^{u+v+1}-1]|$ *otherwise. In the case of* $|v-y|\geqslant|u-x|$, *we have* $dis((x, y), (u, v))=|u-x|+|v-y|$.

Proof. By Theorem 2, $dis(1, (k, i))$ equals $|2k+1/2[(-1)^{i+k}-1]|$ if $|i|\leqslant|k|$, and $|i|+|k|$ otherwise. By symmetry of Cayley graphs, $dis((x, y), (u, v))$ equals $dis(1, (u-x, v-y))$ if $x+y\equiv 0 \bmod 2$, and $dis(1, (x-u, v-y))$ otherwise. This leads to the desired result. □

The discussion above leads to a simple way of determining p, the first node after (x, y) on a shortest path from node (x, y) to node $(0, 0)$. To do this, we assume $xy\neq 0$ and distinguish four cases depending on the signs of x and y:

(1) $x\geqslant 0, y\geqslant 0$: if $x=0$ or $x+y\equiv 0 \bmod 2$, then $p=(x, y-1)$ else $p=(x-1, y)$;
(2) $x\geqslant 0, y<0$: if $x=0$ or $x+y\equiv 0 \bmod 2$, then $p=(x, y+1)$ else $p=(x-1, y)$;
(3) $x<0, y\leqslant 0$: if $x+y\equiv 0 \bmod 2$, then $p=(x+1, y)$ else $p=(x, y+1)$;
(4) $x<0, y>0$: if $x+y\equiv 0 \bmod 2$, then $p=(x+1, y)$ else $p=(x, y-1)$.

The preceding discussion allows us to formulate a simple distributed routing algorithm. Note that the algorithm is used only if $x\neq u$ ($\gamma\neq 0$) or $y\neq v$ ($\delta\neq 0$), although it does yield the consistent result $p=(x, y)$ if used with $\gamma=\delta=0$.

Algorithm 2. The first node p on a shortest path from (x, y) to (u, v) in honeycomb mesh

```
γ = sign(x − u); δ = sign(y − v);
β = (x + y) mod 2; β′ = 1 − β;
if γ = 0
then p = (x, y − δ)
else
        if γ = +1
        then if δ = −1 then p = (x − β, y + β′) else p = (x − β, y − β′) endif
        else if δ = +1 then p = (x + β′, y − β) else p = (x + β′, y + β) endif
        endif
endif
```

Finally, we embark on determining the automorphism group of the infinite honeycomb mesh network Δ_∞. Let σ be the mapping of the set G of Δ_∞ to itself such that $1 \leftrightarrow 1$, $a \leftrightarrow a^{-1}$, $b \leftrightarrow b$, and $\sigma^2 = 1$. This is the reflection to the straight line through two vertices 1 and b. Similarly, let λ be the mapping of the set G to itself such that $1 \leftrightarrow 1$, $a \leftrightarrow b$, $a^{-1} \leftrightarrow a^{-1}$, and $\lambda^2 = 1$. The latter is the reflection to the straight line through two vertices 1 and a^{-1}. Then we have $(\sigma\lambda)^3 = 1$, which leads to $\langle \sigma, \lambda \rangle \cong S_3$, where S_3 is the symmetric group of degree 3. As in the proof of Proposition 3, we have $(Aut(\Delta_\infty))_1 = \langle \sigma, \lambda \rangle$. Thus we have proved the following.

Proposition 5. *Let the mapping σ be defined as $1 \leftrightarrow 1$, $a \leftrightarrow a^{-1}$, $b \leftrightarrow b$, $\sigma^2 = 1$, and the mapping λ as $1 \leftrightarrow 1$, $a \leftrightarrow b$, $a^{-1} \leftrightarrow a^{-1}$, $\lambda^2 = 1$. Then, $Aut(\Delta_\infty) = G\langle \sigma, \lambda \rangle$.*

4. Conclusion

In this paper, we have provided a number of general results on homomorphism between Cayley digraphs and their subgraphs and coset graphs. We have also demonstrated the applications of these results to some well-known interconnection networks, including hexagonal and honeycomb meshes and related networks. Such networks represent alternative, more economical, interconnection networks that are obtained by reducing the number of dimensions and/or link density of existing networks via mapping and pruning.

In particular, we showed that our results lead to very simple, elegant, and provably correct shortest-path routing algorithms for hexagonal mesh and honeycomb mesh networks. A key to the latter achievement is the Cayley-graph-based node addressing schemes that we introduce, thus allowing results from group theory to be applied to the determination of internode distance and, thereby, to routing. The generality of our results lead us to the expectation that they will find many more applications.

We are currently investigating the applications of our method to the problems related to routing and average internode distance in certain subgraphs of the infinite honeycomb mesh network. These results along with potential applications in the following areas will be reported in future:

- load balancing and congestion control;
- scheduling and resource allocation;
- fault tolerance and graceful degradation.

These constitute important practical problems in the design, evaluation, and efficient operation of parallel and distributed computer systems.

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