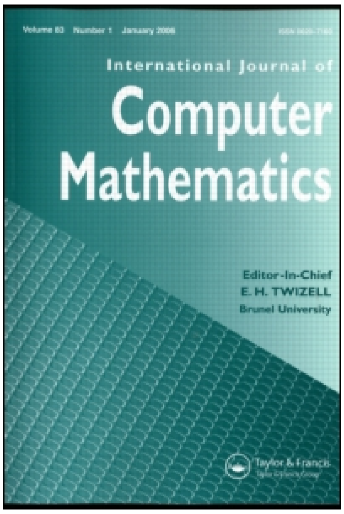

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713455451

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Wenjun Xiao [a]; Behrooz Parhami [b]
[a] Department of Computer Science, South China University of Technology, Guangzhou, People's Republic of China [b] Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA, USA

First Published:January2009

To cite this Article Xiao, Wenjun and Parhami, Behrooz(2009)'On routing and diameter of metacyclic graphs',International Journal of Computer Mathematics,86:1,21 — 30

To link to this Article: DOI: 10.1080/00207160801965222

URL: http://dx.doi.org/10.1080/00207160801965222

International Journal of Computer Mathematics
Vol. 86, No. 1, January 2009, 21–30

On routing and diameter of metacyclic graphs

Wenjun Xiao[a] and Behrooz Parhami[b]*

[a] *Department of Computer Science, South China University of Technology, Guangzhou, People's Republic of China;* [b] *Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA, USA*

(*Received 20 February 2007; revised version received 05 September 2007; second revision received 21 November 2007; third revision received 16 December 2007; accepted 20 December 2007*)

Metacyclic graphs, which include supertoroids as a subclass, have been shown to possess interesting properties and potential applications in implementing moderate- to large-size parallel processors with fairly small node degrees. Wu, Lakshmivarahan, and Dhall (J. Parallel Distrib. Comput. 60 (2000), pp. 539–565) have described a deterministic, distributed routing scheme for certain subclasses of metacyclic graphs. However, they offer no proof that the scheme is a shortest-path routing algorithm and do not indicate whether or how their scheme may be extended to the entire class of metacyclic graphs. In this paper, we provide a near-shortest-path, deterministic routing algorithm that is applicable to any metacyclic graph and derive a bound for the diameter of such graphs.

Keywords: Cayley graph; internode distance; metacyclic graph; network diameter; routing algorithm

2000 AMS Subject Classification: 68R10; 05C12; 94C15

1. Introduction

It is now well known that Cayley graphs [1,2] and Cayley coset graphs [2,7] provide an extremely useful framework that can facilitate the design and analysis of interconnection structures for parallel processing. Computer architecture researchers continually seek new interconnection networks that offer a variety of desirable topological features and cost/performance attributes [8]. This quest has led to the exploration of Cayley graphs of finite groups obtained from two or more groups through well-defined combining operations. These operations, which are quite varied, include direct product, wreath product, and semi-direct product [2,5,9]. For example, the two-dimensional toroid (torus) is an example of a Cayley graph formed by the direct product of two cyclic groups [4].

Draper and Faber [5] initiated the study of a class of interconnection networks whose topologies are based on supertoroidal graphs, which are Cayley graphs of semi-direct product of two cyclic groups of order $m = ck$ and $n = c^2 l$, where $c \geq 2$, $k \geq 1$, and $l \geq 1$ are integers. A supertoroid or

*Corresponding author. Email: parhami@ece.ucsb.edu

ISSN 0020-7160 print/ISSN 1029-0265 online

DOI: 10.1080/00207160801965222
http://www.informaworld.com

supertoroidal graph, with parameters c, k, and l is denoted as $\Gamma(c, k, l)$. Wu *et al.* [9] described a distributed, deterministic routing scheme for supertoroids and other special classes of metacyclic graphs. Based on an experimental verification for $4 \leq c \leq 20$ and various values of the parameters k and l, they conjectured that their scheme was indeed a correct, shortest-path routing algorithm, but were unable to construct a proof of optimality.

In this paper, we provide a near-shortest-path, deterministic routing algorithm for metacyclic graphs, which are Cayley graphs formed by the semi-direct product $Z/m \otimes Z/n$ of two cyclic groups Z/m (of order m) and Z/n (of order n), for integers m and n [4]. We also derive some bounds on the diameter of metacyclic graphs. Our results, which can be applied to all metacyclic graphs, can be viewed as generalizations of those of Wu *et al.* [9], where determining the diameter and deriving a shortest-path routing algorithm for metacyclic graphs are identified as open problems. Thus, our results constitute some progress towards solving these problems.

The rest of this paper is organized as follows. In Section 2, we cover the needed definitions and notational conventions and also introduce metacyclic graphs, which cover toroids and supertoroids as special cases, as a subclass of Cayley graphs of semi-direct products. In Section 3, we derive a deterministic routing algorithm, which is applicable to the entire class of metacyclic graphs, and discuss its properties and examples of use. In Section 4, we present some general bounds on the diameter of metacyclic graphs and also discuss some special cases of interest. Section 5 contains our conclusions and directions for further research.

2. Definitions and notations

We continue our discussion by defining a number of key concepts and notations [4]. Let G be a finite group with I as its identity element. Let $\Omega \subset G$ be a generating set of G such that $I \notin \Omega$ and $g^{-1} \in \Omega$ if $g \in \Omega$. Define the Cayley graph $C(G, \Omega) = (V, E)$, where $V = G$ and $E = \{(x, y) | y = xg$ for some $g \in \Omega\}$. Then, $C(G, \Omega)$ is a regular, vertex-transitive graph of degree $|\Omega|$. Given any integer n, let h and m be integers such that $\gcd(n, h) = 1$ and $h^m \equiv 1 \bmod n$. The rest of our presentation is in terms of a particular fixed value of h thus chosen. Let $G = Z/m$ and $H = Z/n$ be two cyclic groups, where $Z/n = \{0, 1, \ldots, n-1\}$ and all operations are modulo n. The semi-direct product $Z/m \otimes Z/n$ is referred to as a metacyclic group. In other words, $G \otimes H = Z/m \otimes Z/n$, with the product operator defined by $(a, b) \otimes (u, v) = (a, b)(u, v) = (a + u, h^{-u}b + v)$, is a semi-direct product of the two cyclic groups, where the first component is computed mod m and the second one mod n [9]. Consider the generating set $\Omega = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. For the Cayley graph $C(Z/m \otimes Z/n, \Omega)$, which is known as a metacyclic graph, taking $\varepsilon \in \{-1, 1\}$, we have,

$$(a, b)(\varepsilon, 0) = (a + \varepsilon, h^{-\varepsilon} b), \tag{1}$$

$$(a, b)(0, \varepsilon) = (a, b + \varepsilon). \tag{2}$$

Clearly, for $h = 1$, we have $G \otimes H = G \times H$ and the semi-direct product becomes a direct product of G and H [4]. Hence, the toroid $C(G \times H, \Omega)$ is a metacyclic graph, where $G = Z/m$, $H = Z/n$, and $\Omega = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Consider $G' = Z/ck$ and $H' = Z/c^2l$, with c, k, and l integers. Then, $\gcd(c^2l, 1 + cl) = 1$. Taking $h = 1 + cl$, $m = ck$, $n = c^2l$, we can easily find $h^m \equiv (1 + cl)^{\mathrm{ck}} \equiv 1 + c^2kl \equiv 1 \bmod n$ by the binomial theorem [4]. Hence, $G' \otimes H'$ is a semi-direct product. For $(a, b), (u, v) \in G' \otimes H'$, we can readily establish the following identity, which is the same as Equation (2.3) of Wu *et al.* [9]:

$$(a, b)(u, v) = (a + u, (1 - ucl)b + v). \tag{3}$$

The Cayley graph $C(G' \otimes H', \Omega) = \Gamma(c, k, l)$, known as the supertoroidal graph, with $G' = Z/ck$, $H' = Z/c^2l$, and $\Omega = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$, is a metacyclic graph. Thus, recalling that $\varepsilon \in \{-1, 1\}$, we have,

$$(a, b)(\varepsilon, 0) = (a + \varepsilon, (1 - \varepsilon cl)b), \tag{4}$$

$$(a, b)(0, \varepsilon) = (a, b + \varepsilon). \tag{5}$$

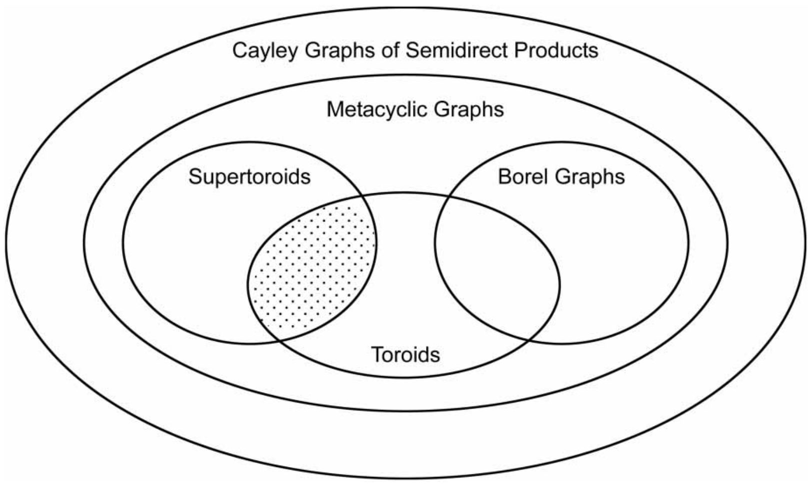

Figure 1. Cayley graphs of semi-direct products and their various subclasses [9].

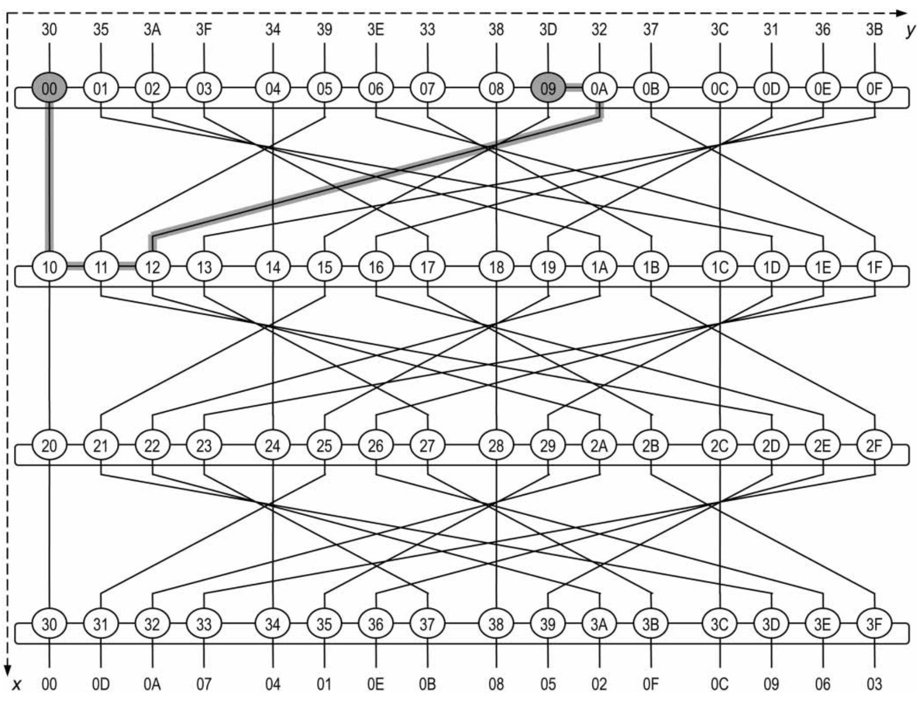

Figure 2. The 64-node supertoroidal network $\Gamma(4, 1, 1)$ of Example 1 with nodes of degree 4. The b component in the index of node (a, b) is shown as a hexadecimal digit (0–F). To avoid clutter, vertical wraparound links have not been drawn fully; instead, they are labelled using the node number on the other side.

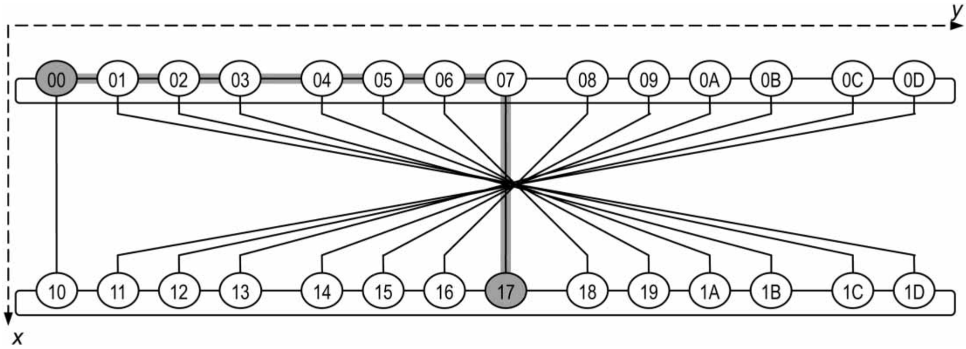

Figure 3. The 28-node instance of the metacyclic graph of Example 2 with nodes of degree 3. The *b* component in the index of node (a, b) is shown as a hexadecimal digit (0–D).

Figure 1, adapted from Wu *et al.* [9], shows the relationships between the class of Cayley graphs of semi-direct products (the most general) and its various important subclasses: metacyclic graphs, supertoroids, Borel graphs, and toroids (tori). Borel graphs, which will not be discussed in this paper, constitute a family of Cayley graphs that are based on a special class of matrix groups [3]. As shown by the shaded area in Figure 1, some, but not all, toroids are supertoroids. An example of a supertoroid (Figure 2), as well as a metacyclic graph that is not a supertoroid (Figure 3), will be discussed in Section 3 (Examples 1 and 2) in conjunction with the application of our routing algorithm (Algorithm 1).

3. Routing algorithm

Assume that $G = Z/m$, $H = Z/n$, and $\Omega = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Because the metacyclic graph $\Gamma = C(G \otimes H, \Omega)$ is vertex-transitive, we only need to consider routing from any node (p, q) to the node $(0, 0)$, where $0 \le p < m$ and $0 \le q < n$. If a link connects node (p, q) to node $(p, q)\ (1, 0)$, then this link is called a $[1, 0]$ link; $[-1, 0]$, $[0, 1]$, and $[0, -1]$ links are defined in an analogous manner. A path from node (p, q) to node $(0, 0)$ contains zero or more of each of the $[1, 0]$, $[-1, 0]$, $[0, 1]$, and $[0, -1]$ links. We denote the number of $[1, 0]$, $[-1, 0]$, $[0, 1]$, and $[0, -1]$ links of a path by w, x, y, and z, respectively. Such a path from node (p, q) to node $(0, 0)$ is of length $w + x + y + z$ and may be denoted as $w[1, 0] + x[-1, 0] + y[0, 1] + z[0, -1]$. Because w and x are modulo m, and y and z are modulo n, it is the case that $0 \le w,\ x < m$ and $0 \le y,\ z < n$. Without loss of generality, we may assume by Equations (1) and (2) that a route from node (p, q) to node $(0, 0)$ satisfies the equations that follow, where $\varepsilon_i \in \{1, -1\}$ and $1 \le i \le w + x$.

$$h^{\varepsilon_{w+x}}(\cdots h^{\varepsilon_2}(h^{\varepsilon_1}(q + v_1) + v_2) + \cdots + v_{w+x}) + v_{w+x+1} \equiv 0 \bmod n \tag{6}$$

$$x - w \equiv p \bmod m, \tag{7}$$

$$v_1 + v_2 + \cdots + v_{w+x+1} = y - z, \tag{8}$$

$$|v_1| + |v_2| + \cdots + |v_{w+x+1}| = y + z, \tag{9}$$

where the number of $h^{\varepsilon i}$ elements in Equation (6) is $w + x$ and $v_1, v_2, \ldots, v_{w+x+1}$ are integers produced by successive applications of Equation (2). Noting that Equation (6) can be rewritten as Equation (6′) and given the equalities $h^m \equiv 1 \bmod n$ and $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{w+x} = x - w \equiv p$

mod m, we have,

$$h^{\varepsilon_1+\varepsilon_2+\cdots\varepsilon_{w+x}}(q+v_1)+h^{\varepsilon_2+\cdots+\varepsilon_{w+x}}v_2+\cdots+v_{w+x+1} \quad \equiv 0 \bmod n \tag{6$'$}$$

$$h^{x-w}(q+v_1+h^{-\varepsilon_1}v_2+\cdots+h^{-p}v_{w+x+1}) \equiv 0 \bmod n. \tag{10}$$

Because gcd(n, h) = 1, Equation (10) yields

$$q+v_1+h^{-\varepsilon_1}v_2+\cdots+h^{-p}v_{w+x+1} \quad \equiv 0 \bmod n \tag{11}$$

The preceding discussion leads to the following result.

Observation 1 *The path* $w[1,0]+x[-1,0]+y[0,1]+z[0,-1]$ *from node* (p,q) *to node* $(0,0)$ *is a shortest path iff there exist integers* $w, x, y, z, v_1, v_2, \ldots, v_{w+x+1}$ *satisfying Equations* (7)*,* (8)*,* (9)*, and* (11) *such that* $w+x+y+z$ *is minimal, where* $0 \le w, x < m$ *and* $0 \le y, z < n$*.*

Because the exponent ε_i of h in Equation (11) may be 1 or -1, which is not fixed, we consider the following more general congruent equation, with fixed exponents for h, which generalizes Equation (11)

$$q+v_1h^{-x}+v_2h^{-x+1}+\cdots+v_xh^{-1}+v_{x+1}+v_{x+2}h+\cdots+v_{x+w+1}h^{w} \equiv 0 \bmod n. \tag{12}$$

Corollary 1 *If there exist integers* $w, x, y, z, v_1, v_2, \ldots, v_{w+x+1}$ *satisfying Equations* (7)*,* (8)*,* (9)*, and* (12)*, such that* $w+x+y+z$ $(0 \le w, x < m, 0 \le y, z < n)$ *is minimal, then we have the routing equation*

$$h^x\{\underbrace{h^{-1}\{\cdots h^{-1}}_{w-1}\{h^{-x-1}[\underbrace{h(\cdots h(h}_{x}(q+v_{x+1})+v_x)+\cdots+v_2)+v_1]+v_{x+2}\}$$
$$+\cdots+v_{x+w}\}+v_{x+w+1}\} \equiv 0 \bmod n, \tag{13}$$

where $d(p,q)$*, the distance between node* (p,q) *and node* $(0,0)$*, satisfies the inequalities* $w+x+y+z \le d(p,q) \le w+3x+y+z$*.*

Proof It is clear that if the integers $w', x', y', z', v'_1, v'_2, \ldots, v'_{w+x+1}$ satisfy Equations (7), (8), (9), and (11) such that $w'+x'+y'+z'$ is minimal, with $0 \le w', x' < m$ and $0 \le y', z' < n$, then they satisfy Equations (7), (8), (9), and (12). Hence, we have $w+x+y+z \le d(p,q) = w'+x'+y'+z'$ from Observation 1. Repeatedly multiplying (x times in all) Equation (12) by h, we get,

$$h(\cdots h(h(q+v_{x+1})+v_x)+\cdots+v_2)+v_1+v_{x+2}h^{x+1}$$
$$+\cdots+v_{x+w}h^{x+w+1}+v_{x+w+1}h^{x+w} \equiv 0 \bmod n. \tag{14}$$

Multiplying Equation (14) by h^{-x-1}, we get,

$$h^{-x-1}[h(\cdots h(h(q+v_{x+1})+v_x)+\cdots+v_2)+v_1]+v_{x+2}+\cdots+v_{x+w}h^{w-2}$$
$$+v_{x+w+1}h^{w-1} \equiv 0 \bmod n \tag{15}$$

Repeatedly multiplying ($w-1$ times in all) Equation (15) by h^{-1}, we obtain,

$$\underbrace{h^{-1}\{\cdots h^{-1}}_{w-1}\{h^{-x-1}[\underbrace{h(\cdots h(h}_{x}(q+v_{x+1})+v_x)+\cdots+v_2)+v_1]$$
$$+v_{x+2}\}+\cdots+v_{x+w}\}+v_{x+w+1} \equiv 0 \bmod n \tag{16}$$

Multiplying Equation (16) by h^x, we arrive at Equation (13). It is apparent that in the sequence of operations described above, Equation (1) has been applied $x+(w-1)+(x+1)+x = w+3x$

times and Equation (2) has been applied $|v_1| + |v_2| + \cdots + |v_{w+x+1}| = y + z$ times. Hence, we have $d(p, q) \leq w + 3x + y + z$. ■

We can obtain a deterministic routing algorithm for the metacyclic graph $\Gamma = C(G \otimes H, \Omega)$ based on Corollary 1, which also establishes the correctness of our algorithm.

ALGORITHM 1 *Routing from node* (p, q) *to node* $(0, 0)$ *in a metacyclic graph*
For any given (w, x) *satisfying* $x - w \equiv p \bmod m$ *and* $0 \leq w, x < m$:

Step 1. Find integers y, z, $v_1, v_2, \ldots, v_{w+x+1}$ *satisfying Equations* (8), (9), *and* (12) *such that* $w + x + y + z$ *is minimal, where* $0 \leq y, z < n$.
Step 2. Construct the routing Equation (13), *as noted in the preceding discussion.*

Let $l(p, q)$ be the length of the routing path from node (p, q) to node $(0, 0)$. Then, $l(p, q) \leq w + 3x + y + z$. Hence, we have $l(p, q) - d(p, q) \leq 2x$. In particular, since $0 \leq x < m$, Algorithm 1 is a near-shortest-path routing algorithm when m is small. Furthermore, Algorithm 1 is a shortest-path routing algorithm if $x = 0$. Our algorithm is provably correct and is applicable to all metacyclic graphs, as opposed to the scheme of Wu *et al.* [9], which has not been formally proven correct and, additionally, only pertains to specific classes of metacyclic graphs. It is noteworthy, however, that the application of our algorithm requires that for any (w, x), with $x - w = p \bmod m$ $(0 \leq w, x < m, 0 \leq y, z < n)$, we find the integers y, z, $v_1, v_2, \ldots, v_{w+x+1}$ satisfying Equations (8), (9), and (12) such that $w + x + y + z$ is minimal. This is not a trivial task, and we discuss it further in the following.

For any given (w, x) with $x - w \equiv p \bmod m$ and $0 \leq w, x < m$, based on Equation (12), we first sort $|h^{-x}|, |h^{-x+1}|, \ldots, |h^{-1}|, |h|, |h^2|, \ldots, |h^w|$. The time required for this phase is O(m log m). Next, we use repeated divisions with remainders to find $v_1, v_2, \ldots, v_{w+x+1}$, beginning from the greatest of $|h^{-x}|, |h^{-x+1}|, \ldots, |h^{-1}|, |h|, |h^2|, \ldots, |h^w|$, where q is the dividend for the first step. This phase too requires O(m log m) time. The total time complexity of computing the routing equation in Algorithm 1 is thus O(m log m) for any given (w, x), with $x - w = p \bmod m$ and $0 \leq w, x < m$. Consequently, the total time complexity for computing the routing equation in Algorithm 1, for all (w, x), with $x - w = p \bmod m$ and $0 \leq w, x < m$, is O(m^2 log m).

Example 1: Consider $\Gamma(c, k, l) = C(Z/ck \otimes Z/c^2l, \Omega)$, where $\Omega = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. This supertoroidal graph is depicted in Figure 2. For comparison purposes, we have shown a 64-vertex three-dimensional toroid in Figure 4. Let $c = 4$ and $k = l = 1$. Then, $h = 1 + cl = 5$, $h^{-1} = 1 - cl = -3$. Now consider routing from node (0, 9) to node (0, 0), that is, let $p = 0$, $q = 9$. Setting $w = x = 1$, by Equation (12) we have $9 + (-3)v_1 + v_2 + 5v_3 \equiv 0 \bmod 16$. Because $|h| > |h^{-1}|$, we first perform division with the dividend $q = 9$ and the divisor $h = 5$, obtaining the remainder -1 (i.e., $9 = 5 \times 2 - 1$). Next, the dividend is taken to be -1 and the divisor $h^{-1} = -3$, which yields the remainder of -1. Thus, we obtain $9 + (-3) \times 0 + 1 + 5 \times (-2) \equiv 0 \bmod 16$. Hence, we have $v_1 = 0$, $v_2 = 1$, $v_3 = -2$. By Equation (13), we have $5 \times [(-3)^2 \times 5 \times (9 + 1) - 2] \equiv 0 \bmod 16$. Clearly, $-3 \times 5 \equiv 1 \bmod 16$. Hence, we get $5 \times [(-3) \times (9 + 1) - 2] \equiv 0 \bmod 16$. The preceding steps lead to the routing path $(0, 9) \to (0, 10) \to (1, 2) \to (1, 1) \to (1, 0) \to (0, 0)$. This path from node (0, 9) to node (0, 0), traced by a heavy line in Figure 2, is of length 5. Generally, there exist other paths. For example, here we have the alternate path: $(0, 9) \to (1, 5) \to (2, 1) \to (2, 0) \to (3, 0) \to (0, 0)$.

Example 2: This is an example metacyclic graph that is not a supertoroidal graph. Let $m = 2$ and assume that n is even but not a multiple of 4. Assume that $h = n - 1$ and $h^{-1} = -1$. By

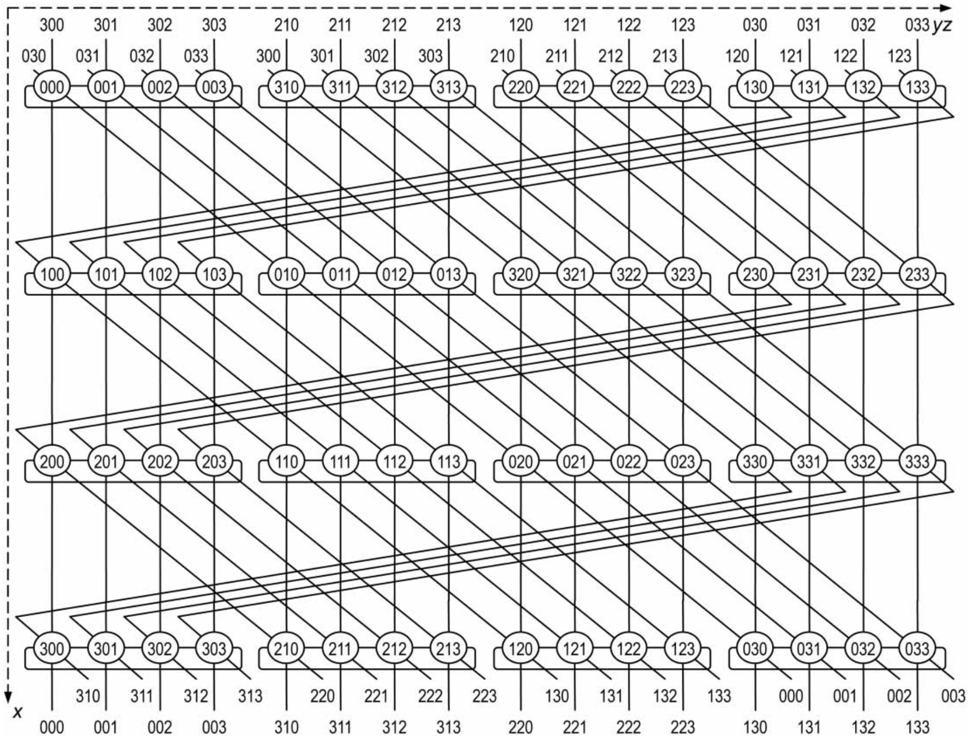

Figure 4. Three-dimensional $4 \times 4 \times 4$ toroid (of node degree 6) drawn so as to facilitate comparison with the super-toroidal network of Figure 2. To avoid clutter, vertical wraparound links have not been drawn fully; instead, they are labelled using the node number on the other side. Despite its larger node degree, and thus higher implementation cost, this network has the same diameter as the one in Figure 2.

Equation (12), we have $q + v_1 h^{-1} + v_2 \equiv 0 \bmod n$. For $p = 1$ and $q = n/2$, we may set $v_1 = n/2$ and $v_2 = 0$. By Equation (13), we have the routing equation $hq + v_1 \equiv (-1)n/2 + n/2 \equiv 0 \bmod n$. Algorithm 1 yields the following routing path from node $(1, n/2)$ to node $(0, 0)$ in this network: $(1, n/2) \to (0, n/2) \to (0, n/2 - 1) \to \cdots \to (0, 1) \to (0, 0)$. The length of this path is $n/2 + 1$. Figure 3 shows an instance of this network for $n = 14$, with the aforementioned routing path from node (1, 7) to node (0, 0) traced by a heavy line.

Remark 1 As is evident from Examples 1 and 2, some h^{-1} and h elements in Equation (13) cancel each other out, which frequently leads to a shortest path from node (p, q) to node (0, 0).

Remark 2 Subtracting Equation (8) from Equation (12), we obtain:

$$q + y - z + v_1(h^{-x} - 1) + v_2(h^{-x+1} - 1) + \cdots + v_x(h^{-1} - 1) + v_{x+2}(h - 1) + \cdots + v_{w+x+1}(h^w - 1) \equiv 0 \bmod n \tag{17}$$

Because $h^m \equiv 1 \bmod n$, we have $h^{-1} \equiv h^{m-1} \bmod n$, and thus $h^{-l} - 1 \equiv h^{(m-1)l} - 1 \bmod n$, where l is an integer. Given that $h^l - 1$ is divisible by $h - 1$, and based on Equation (17), we obtain,

$$q \equiv z - y + f|h - 1| \bmod n, \text{ for some integer } f. \tag{18}$$

Similarly, given that $h^l - 1$ is divisible by $h^{-1} - 1$, we can deduce the following.

$$q \equiv z - y + g|h^{-1} - 1| \bmod n, \text{ for some integer } g \tag{19}$$

We will use Equation (19) to establish a diameter lower bound in Section 4. Note that in the special case of $\Gamma(c, k, l)$, Equation (19) reduces to $q \equiv z - y \bmod cl$.

4. Diameter bounds

Assume that $\Gamma = C(G \otimes H, \Omega)$ is a metacyclic graph, where $\Omega = \{(\pm 1, 0), (0, \pm 1)\}$. Let $D(X)$ denote the diameter of graph X and $d(p, q)$ be the distance from the node (p, q) to the node (0, 0). Then we have the following result.

THEOREM 1 *Assume that $|h^{-1} - 1|$ divides n, $p = \lfloor m/2 \rfloor$, and $q = \lfloor |h^{-1} - 1|/2 \rfloor$. Then, the diameter of Γ satisfies $D(\Gamma) \geq d(p, q) \geq \lfloor m/2 \rfloor + \lfloor |h^{-1} - 1|/2 \rfloor$.*

Proof Because $|h^{-1} - 1|$ divides n, we have by Equation (19)

$$q \equiv z - y \bmod |h^{-1} - 1|.$$

Hence, we have $q + 2y = y + z + g|h^{-1} - 1|$, where $g \geq 0$. For $g = 0$, we have $y + z = q + 2y \geq q = \lfloor |h^{-1} - 1|/2 \rfloor$. For $g \geq 1$, we have $y - z = g|h^{-1} - 1| - q$, which leads to $y + z = g|h^{-1} - 1| - q + 2z \geq g|h^{-1} - 1| - \lfloor |h^{-1} - 1|/2 \rfloor \geq \lfloor |h^{-1} - 1|/2 \rfloor$. On the other hand, we have by Equation (7)

$$x - w \equiv \lfloor m/2 \rfloor \bmod m.$$

Thus, $x + w \equiv 2w + \lfloor m/2 \rfloor \bmod m$. For $x \leq w$, we have $w = m - \lfloor m/2 \rfloor + x \geq \lfloor m/2 \rfloor$. For $x \geq w$, we have $x \geq w + \lfloor m/2 \rfloor \geq \lfloor m/2 \rfloor$. Hence, $w + x \geq \lfloor m/2 \rfloor$ in either case. Based on the lower bounds derived for $w + x$ and $y + z$, we can write $w + x + y + z \geq \lfloor m/2 \rfloor + \lfloor |h^{-1} - 1|/2 \rfloor$ and, so, $D(\Gamma) \geq d(p, q) \geq \min(w + x + y + z) \geq \lfloor m/2 \rfloor + \lfloor |h^{-1} - 1|/2 \rfloor$. ■

The following corollary of Theorem 1 is a known result from the study of Draper and Faber [5].

COROLLARY 2 *For $\Gamma = \Gamma(c, k, l)$, we have $D(\Gamma) \geq \lfloor ck/2 \rfloor + \lfloor cl/2 \rfloor$.*

Draper and Faber [5] have shown that $D(\Gamma(c, k, l)) = \lfloor ck/2 \rfloor + \lfloor cl/2 \rfloor$ when $c \geq 8$. We have been unable to obtain a similar closed-form representation for the diameter of metacyclic graphs as a function of the parameters m, n, and h. In the absence of such an exact diameter formula, the following diameter upper bound is useful.

Assume that $|h^{-1}| \neq 1$ and $q < n$ has the following polynomial expansion [4] in terms of h^{-1}

$$q = u_1 h^{-i} + u_2 h^{-i+1} + \cdots + u_i h^{-1} + u_{i+1},$$

where $|h^{-i}| \leq q < |h^{-i-1}|$ and $|u_j| \leq |h^{-1}|$, $j = 1, 2, \ldots, i + 1$. Then, because $|h^{-i}| \leq q < n$, we have $i \leq \log n / \log |h^{-1}|$, where the logarithms are in base 2. Hence, we have:

$$|u_1| + |u_2| + \cdots + |u_{i+1}| \leq (i + 1)|h^{-1}| \leq (\log n / \log |h^{-1}| + 1)|h^{-1}|.$$

As a result, $d(p, q) \leq O(m) + (\log n / \log |h^{-1}| + 1)|h^{-1}|$. We have thus shown the following.

THEOREM 2 *Suppose that $\Gamma = C(G \otimes H, \Omega)$ is a metacyclic graph. Then, the diameter of Γ satisfies $D(\Gamma) \leq O(m) + (\log n / \log |h^{-1}| + 1)|h^{-1}|$, when $|h^{-1}| \neq 1$.*

By Theorems 1 and 2, we have the following special results.

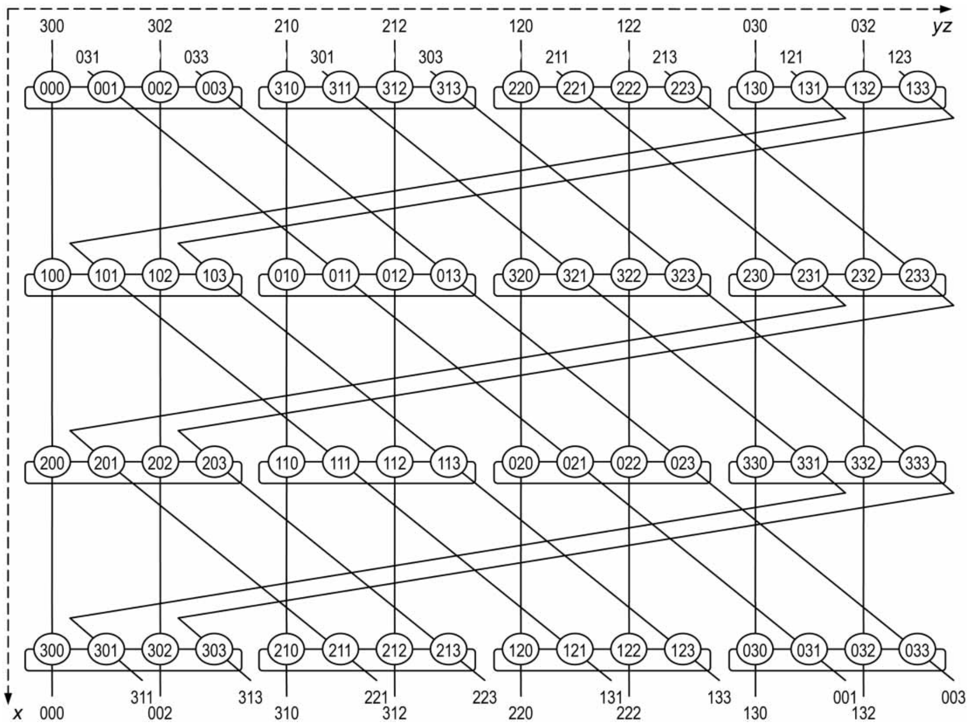

Figure 5. Three-dimensional $4 \times 4 \times 4$ pruned toroidal network (node degree of 4) for comparison with Figures 2 and 4. To avoid clutter, vertical wrap-around links have not been drawn fully; instead, they are labelled using the node number on the other side.

Corollary 3 *Consider* $\Gamma = \Gamma(c, k, l)$. *Then* $h^{-1} = 1 - cl$, *and so* $D(\Gamma) \leq O(m + cl) = O(ck + cl)$, *where* $cl \neq 2$. *Therefore, we have* $D(\Gamma) = \Theta(ck + cl)$.

Corollary 4 *Let* $m = 4$ *and* $n = h^2 + 1 > 2$. *Then* $h^{-1} = -\sqrt{n-1}$ *and we have* $D(\Gamma) \leq O(\sqrt{n})$.

We see that in this case, the network diameter is less than those of the corresponding torus and supertoroid networks, with the same parameters m and n, when the number of nodes is sufficiently large.

5. Conclusion

In this paper, we have extended the results of Wu *et al.* [9] on routing for certain subclasses of metacyclic graphs, by presenting a deterministic routing algorithm that is applicable to all metacyclic graphs. We have also derived bounds on diameters of such graphs. Our routing algorithm comes with a correctness proof, and even though it is not a shortest-path algorithm, its chosen routing paths are not much longer than the shortest paths when m is small. The derivation of an exact formula for the diameter of metacyclic graphs as a function of m, n, and h, is still an open problem, but our bounds constitute some progress towards solving this problem. Because metacyclic groups (semi-direct products of two cyclic groups) are not commutative, the difficulty in constructing a general shortest-path routing algorithm is not unexpected. In the absence of an optimal routing algorithm, our near-shortest-path routing algorithm is a useful tool.

Further research can proceed in several directions. Other than tackling the open problems cited in the preceding paragraph, one may endeavour to compare metacyclic graphs in general, and the subclass of supertoroids in particular, to other networks with similar cost/performance attributes. For example, pruned torus networks, which also are Cayley graphs [10] and offer reduced node degree compared with ordinary or unpruned tori, may be viewed in some respects as being intermediate between ordinary tori and supertoroids. An example of pruned three-dimensional torus, with node degree 4 and diameter equal to the corresponding unpruned torus of Figure 4, is depicted in Figure 5. It would be interesting to compare such networks of equal node degree with regard to their topological parameters and robustness attributes. Comparing the performance of such networks under realistic implementation and packaging considerations, as studied by Gupta and Dally [6], constitutes another potentially fruitful research direction.

Acknowledgements

Detailed comments, offered in two rounds, by an anonymous reviewer have led to clear presentation and removal of some redundancies and errors. We are grateful for this reviewer's contributions. The research of W. Xiao has been supported by the Natural Science Foundation of Guangdong Province.

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