Optimal Placement of Spare Modules in a Cascaded Chain

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Key Words – Redundant cascade, Standby redundancy, Series system, Optimal redundancy, Reconfiguration

Reader Aids –
Purpose: Report an analysis
Special math needed: Probability
Results useful to: Reliability theoreticians

Summary & Conclusions – A class of redundant cascaded chains with i.i.d. modules is considered in which recovery from a failure takes place by replacing the faulty module by a spare module. The complexity of the reconfiguration process depends upon the location of spare modules in the cascade. This paper deals with the question of optimally placing the spare modules in order to minimize the *s*-expected recovery time (down time) of the system. Exact analysis is carried out for cascades with one and two spare modules and an approximate analysis is given for three or more spares. Even though exact analysis does not seem to be practical in the general case, the symmetry of spare module positions in the special cases discussed here and linearity of the system suggest that one might expect the optimal positions to be symmetric in general. Because of this symmetry, one can reduce the number of variables to be considered in the general case, however, some inaccuracies might be introduced.

INTRODUCTION

Consider a cascaded chain of i.i.d. modules, with the numbering of modules representing either a physical ordering, based on module locations, or a logical ordering, based on a 1-way linear intercommunication between modules. Examples of such systems include shift-register, rotating, and other equivalent types of memories, array and associative processors, pipeline processors with identical programmable stages, and byte-sliced processors and computers. Clearly, if provisions are not made for replacing faulty modules in a cascaded chain (series system), a single failed module results in system failure.

A technique for replacing a failed module, which has been studied in connection with the design of fault-tolerant associative processors [1], is the use of shorting networks [2] to bypass the failed module and unbypass one of the spares. In this scheme, the interconnection between modules is established through a number of 2-state cells as in Fig 1. Each cell is easily implemented using a single flip-flop and several gates [2] and its two states are designated as bent (pass) and crossed (interchange). Fig 1 also shows the reconfiguration process in a cascade with four modules after the failure of the second one from the left.

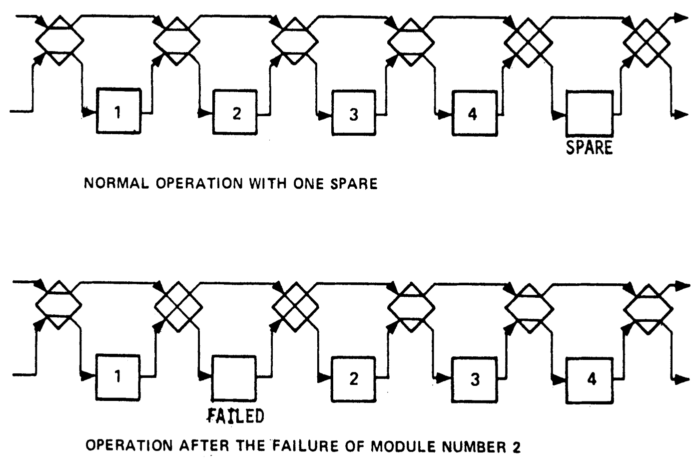

Fig. 1. Reconfiguration with a Shorting Network.

In addition to changing the cell states, the recovery process, as shown in Fig 1, consists of assigning new designations to three of the modules. This can involve loading new programs into the modules and/or transferring data between neighboring modules. In the case of Fig 1, a maximum of 4 and an average of 2.5 such steps are needed upon failure. Had the spare been placed in the middle position, then a maximum of 2 and an average of 1.5 new designations would have been needed. This suggests that the position of spare modules in the cascade affects the complexity of, and hence the time needed for, the recovery procedure. In the subsequent discussion, optimal placement of spares refers to an arrangement which minimizes the *s*-expected number of new designations needed for recovery from failures.

ASSUMPTIONS AND NOTATION

m	number of active (operating) modules.
s	number of spare modules.
$j_1, j_2, \ldots, j_s$	indices of active modules which initially have spare modules immediately to their right $(j_i \leqslant j_{i+1})$.
k_i	index of highest numbered module replaced by spare i $(i = 1, 2, \ldots, s-1)$ upon the first module failure $(j_i \leqslant k_i \leqslant j_{i+1})$; for convenience, define $k_0 = 0$ and $k_s = m$.
$D(m, j_1, j_2, \ldots, j_s)$	*s*-expected number of new designations after s failures.

X	index of the first failed module; random variable with $Pr\{X = i\} = 1/m$ for $i = 1, 2, \ldots, m$.
$j_i^{opt}(m, s); k_i^{opt}(m, s)$	optimal values for j_i and k_i.
$D_p(m, j_1, j_2)$	s-expected number of new designations, given that the second failure occurs with probability p.
$j_i^{opt}(m, s, p); k_i^{opt}(m, s, p)$	optimal values for j_i and $k_i (i = 1$ or 2), given that the second failure occurs with probability p.
p	probability of a second module failure, given that a first failure has occurred.
r	reliability of each operating module (exponential function of time); assume spares cannot fail.
c	probability of recovery from each failure (coverage factor).
p_i	probability that spare module i is used.
$R(m, s, r)$	reliability of cascade with m operating and s spare modules.

CASCADES WITH ONE OR TWO SPARES

The result of the following analysis for $s = 1$ is intuitively obvious and is given only to familiarize readers with the notation and method of analysis for the subsequent discussion. The quantity to be minimized is:

$$D(m, j_1) = \sum_{i=1}^{j_1} \Pr\{X = i\} \times (j_1 - i + 1) + \sum_{i=j_1+1}^{m} \Pr\{X = i\} \times (i - j_1) \quad (1)$$

$$= \frac{1}{m}\left[\sum_{i=1}^{j_1} (j_1 - i + 1) + \sum_{i=j_1+1}^{m} (i - j_1)\right]$$

$$= \frac{m+1}{2} - \frac{j_1(m - j_1)}{m}.$$

The minimum value of $D(m, j_1)$ is obtained for

$$j_1^{opt}(m, 1) = m/2; \quad (2)$$

if m is odd, $(m - 1)/2$ and $(m + 1)/2$ are equally optimal choices for j_1. The intuitive interpretation of (2) is that the single spare module must be placed at the middle point of the cascade.

A cascade with two spares is converted to one with a single spare after the first failure. Thus:

$$D(m, j_1, j_2) = \sum_{i=1}^{j_1} Pr\{X = i\} \times [(j_1 - i + 1) + D(m, j_2)] \quad (3)$$

$$+ \sum_{i=j_1+1}^{k_1} Pr\{X = i\} \times [(i - j_1) + D(m, j_2)]$$

$$+ \sum_{i=k_1+1}^{j_2} Pr\{X = i\} \times [(j_2 - i + 1) + D(m, j_1)]$$

$$+ \sum_{i=j_2+1}^{m} Pr\{X = i\} \times [(i - j_2) + D(m, j_1)]$$

$$= \frac{1}{2} + \frac{k_1}{m} D(m, j_2) + \frac{m - k_1}{m} D(m, j_1)$$

$$+ \frac{j_1^2 + (k_1 - j_1)^2 + (j_2 - k_1)^2 + (m - j_2)^2}{2m}.$$

Equating $\partial D/\partial j_1$, $\partial D/\partial j_2$, $\partial D/\partial k_1$ to zero, the resulting set of equations yields, after tedious computation, only one set of acceptable solutions as follows:

$$j_1^{opt}(m, 2) = m/3; j_2^{opt}(m, 2) = 2m/3; k_1^{opt}(m, 2) = m/2. \quad (4)$$

Since the optimal values given by (4) might not be integers, a question arises as to the actual values to be selected. intuitively, one feels that the rounded values of $m/3$, $2m/3$, and $m/2$ are the best choices for j_1, j_2, and k_1, respectively. Even though no mathematical proof has been found for this conjecture, many examples have shown this to be true.

THE GENERAL CASE: APPROXIMATE ANALYSIS

When more than two spares are used, exact equations such as (4) become extremely difficult to obtain because of the larger number of variables involved. On the other hand, in the analysis of the previous section, the s-expected number of total new designations after the first and second failures was minimized. One might argue that in highly reliable systems, the probability of actually reaching the i-th reconfiguration is a sharply decreasing function of i. Thus, a more reasonable approach for such systems would be to minimize the sum of terms obtained by multiplying the s-expected number of new designations after failure i by the probability of that failure actually occurring during the system's useful life.

The following approximate equation is obtained if the probability of having two or more failures is negligible compared to that of the first failure:

$$D(m, j_1, j_2, \ldots, j_s) \approx \sum_{i=1}^{j_1} Pr\{X = i\} \times (j_1 - i + 1) \quad (5)$$

$$+ \sum_{i=j_1+1}^{k_1} Pr\{X=i\} \times (i-j_1) + \sum_{i=k_1+1}^{j_2} Pr\{X=i\}$$

$$\times (j_2 - i + 1)$$

$$+ \ldots\ldots + \sum_{i=j_s+1}^{m} Pr\{X=i\} \times (i-j_s)$$

$$= \frac{1}{m} \sum_{l=1}^{s} \left[\sum_{i=k_{l-1}+1}^{j_l} (j_l - i + 1) + \sum_{i=j_l+1}^{k_l} (i - j_l) \right]$$

$$= \frac{1}{2m} \sum_{l=1}^{s} \left[(j_l - k_{l-1})^2 + (k_l - j_l)^2 + k_l - k_{l-1} \right].$$

Equating $\partial D/\partial j_i$ and $\partial D/\partial k_i$ to zero, the resulting set of $2s$-1 equations yields, after tedious computation:

$$j_i^{opt}(m, s) \approx m(2i-1)/2s; k_i^{opt}(m, s) \approx mi/s. \qquad (6)$$

As in the case with two spares, the optimal values given by (6) might not be integers. To determine the position of spare modules, one can either use the rounded values or consider both the higher and lower integers for each j_i and k_i and select the best set of positions by comparing the values obtained from (5). In any case, since the analysis is itself approximate, such variations are not very important.

PROBABILISTIC ANALYSIS FOR TWO SPARES

As mentioned in the previous section, in a highly reliable system with two spares, the probability of actually using the second spare module is considerably less than that of the first spare. This is taken into account by multiplying the terms $D(m, j_1)$ and $D(m, j_2)$ in (3) by the conditional probability p of a second failure given that a first has occurred. Thus:

$$D_p(m, j_1, j_2) = 1/2 + (pk_1/m)D(m, j_2)$$
$$+ [p(m-k_1)/m]D(m, j_1) \qquad (7)$$
$$+ [j_1^2 + (k_1 - j_1)^2 + (j_2 - k_1)^2 + (m - j_2)^2]/2m$$

Equating $\partial D_p/\partial j_1$, $\partial D_p/\partial j_2$, $\partial D_p/\partial k_1$ to zero, the resulting set of equations yields after tedious computation, only one set of acceptable solutions as follows:

$$j_1^{opt}(m, 2, p) = \frac{1+p}{2+p} \cdot \frac{m}{2}; j_2^{opt}(m, 2, p)$$
$$= \frac{3+p}{2+p} \cdot \frac{m}{2}; k_1^{opt}(m, 2, p) = \frac{m}{2}. \qquad (8)$$

These results reduce to (4) if $p = 1$, and to the special case of (6) with $s = 2$ if $p = 0$.

The following reliability model of Bouricius et al. [3] can be used to estimate the value of p:

$$R(m, s, r) = r^m \times \sum_{i=0}^{s} (-c\, m \ln r)^i / i! \qquad (9)$$

Spare module i is used only if the system with $i-1$ spare modules fails and a successful recovery is carried out. Thus:

$$p_i = c[1 - R(m, i-1, r)]. \qquad (10)$$

The value of p in (8) can now be obtained from p_1 and p_2 as:

$$p = p_2/p_1 = 1 + c \times m \times r^m \times \ln r/(1 - r^m). \qquad (11)$$

The value of p, given by (11), is a decreasing function of r, going from 1 (for $r = 0$) to $1 - c$ (for $r = 1$).

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Manuscript received 1976 October 12; revised 1977 June 13.

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