Impact of Heterogeneous Link Qualities and Network Connectivity on Binary Consensus

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***Abstract*— In this paper we consider a network that is trying to reach consensus over the occurrence of an event while communicating over Additive White Gaussian Noise (AWGN) channels. We characterize the impact of different link qualities and network connectivity on consensus performance by analyzing both the asymptotic and transient behaviors. More specifically, we derive a tight approximation for the second largest eigenvalue of the probability transition matrix. We furthermore characterize the dynamics of each individual node.**

keywords– binary Consensus, link qualities, network connectivity

I. Introduction

Cooperative decision-making and control has received considerable attention in recent years. Such problems arise in many different areas such as environmental monitoring, surveillance and security, smart homes and factories, target tracking and military systems. Consider a scenario where a network of agents wants to perform a task jointly. Each agent has limited sensing capabilities and has to rely on the group for improving its estimation/detection quality. Consensus problems arise when the agents need to reach an agreement on the value of a parameter. These problems have been categorized into two groups: *Estimation Consensus* and *Detection Consensus* [1]. *Estimation consensus* refers to the problems where the parameter of interest can take values over an infinite set or an unknown finite set. These problems received considerable attention over the past few years. Convergence and equilibrium state of continuous-time and discrete-time consensus protocols have been studied for both time-invariant and time-varying topologies [2]-[5]. Furthermore, consensus protocols have been applied to formation problems [6]-[9] as well as distributed filtering [10]. The uncertainty in the exchanged information was considered and accounted for in [11] where conditions for achieving consensus were derived. [12], [13] provide a comprehensive survey of the literature on such consensus problems.

Detection Consensus, on the other hand, refers to the problems in which the parameter of interest takes values from a finite known set. *Binary consensus* [1] then refers to a subset of detection consensus problems where the network is trying to reach an agreement over a parameter that can only have two values. For instance, networked detection of fire falls into this category. While estimation consensus problems have received considerable attention, detection consensus problems have mainly remained unexplored. [14] considered and characterized phase transition of a binary consensus problem in the presence of a uniformly distributed communication noise. Since the probability density function of this noise is bounded, there exists a transition point beyond which consensus will be reached in this case [14]. In most consensus applications, the agents will communicate their status wirelessly. Therefore, the received data will be corrupted by the receiver noise, which is best modeled as an additive Gaussian noise [15]. Therefore, it becomes important to analyze the performance of consensus problems in the presence of Gaussian communication noise. To address this, binary consensus with Gaussian communication noise was considered in [1]. Since the noise is not bounded in this case, there is no transition point beyond which consensus is guaranteed. Instead, a probabilistic approach was utilized to characterize the behavior of the network. It was shown that the steady-state behavior of such systems is undesirable, independent of the amount of communication noise variance, as the network loses the memory of the initial state. The network, however, may still be in consensus for a long period of time. To characterize this, an expression for the second largest eigenvalue of the underlying dynamical system was then derived. In [16], [17], results of [1] are extended to the case where knowledge of link qualities is available in the receiver as well as to fading environments.

The analysis of [1] was carried out under the assumption that the graph is fully connected, with all the link noises having the same variances. In this paper, we extend the analysis of [1] to embrace the impact of heterogeneous link qualities and graph connectivity. More specifically, we derive an expression for the second largest eigenvalue of the probability transition matrix when links have different noise variances. We then explore the impact of graph connectivity on the performance. We make the assumption that all the nodes have the same degree (regular graphs) in order to derive an expression for the second largest eigenvalue. We finally characterize the behavior of each individual node. We show that the asymptotic voting pattern of each node becomes purely random. The paper is organized as follows.

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Section II introduces the problem and describes our system model. Section III explores the impact of different link qualities on the performance. Section IV characterizes the impact of network connectivity on consensus. This is followed by the characterization of the behavior of an individual node in Section V and conclusions in Section VI.

II. System Model

Consider M agents that want to reach consensus on the occurrence of an event. Each agent makes a decision on the occurrence of the event based on its one-time local sensor measurement. Let $b_i(0) \in \{0,1\}$ represent the initial decision of the i^{th} agent, at time step $k = 0$, based on its local measurement, where $b_i = 1$ indicates that the i^{th} agent votes that the event occurred whereas $b_i = 0$ denotes otherwise. Each agent sends its vote to the rest of the group, *using only one bit of information*, and revises its vote based on the received information. This process will go on for a while. Adopting the same language of [1], we say that *accurate* consensus is achieved if each agent reaches the majority of the initial votes. For instance, if 70% of the agents start by voting 1, it is desirable that all the agents vote one after communicating a number of times.

Each transmission gets corrupted by the receiver noise, which is best modeled as an Additive White Gaussian Noise (AWGN channel) [15]. When a node receives the decisions of other nodes, the receptions can happen in different frequencies or time slots [18]. Then each reception will experience a different (uncorrelated) sample of the receiver thermal noise. Let $n_{j,i}(k)$ represent the noise at k^{th} time step in the transmission of the information from the j^{th} node to the i^{th} one. $n_{j,i}(k)$ is a zero-mean Gaussian random variable with the variance of $\sigma^2_{j,i}(k)$. Let $\hat{b}_{j,i}(k)$ represent the reception of the i^{th} agent from the transmission of the j^{th} one at k^{th} time step. We will have $\hat{b}_{j,i}(k) = b_j(k) + n_{j,i}(k)$ for $j \in \Psi_i(k)$, where $\Psi_i(k)$ represents the set of those agents that can communicate to the i^{th} one (including itself) at time step k. Let $N_i(k)$ represent the size of $\Psi_i(k)$. Each agent will then update its vote based on the received information as follows:

$$b_i(k+1) = \text{Dec}\left(\frac{1}{N_i(k)} \sum_{j\in\Psi_i(k)} \hat{b}_{j,i}(k)\right)$$
$$= \text{Dec}\left(\frac{1}{N_i(k)} \sum_{j\in\Psi_i(k)} b_j(k) + \frac{1}{N_i(k)} \sum_{j\in\Psi_i(k), j\neq i} n_{j,i}(k)\right), \quad (1)$$

where Dec(.) represents a decision function for binary 0-1 detection: $\text{Dec}(x) = \begin{cases} 1 & x \geq .5 \\ 0 & x < .5 \end{cases}$.

III. Impact of Heterogeneous link qualities

In this section we explore the impact of different link qualities on binary networked consensus. In order to focus on the impact of link qualities, we consider a fully connected time-invariant graph in this section, and relax these assumptions in the subsequent parts. We will then have $N_i(k) = M$ and

$$b_i(k+1) = \text{Dec}\left(\frac{1}{M} \sum_{j=1}^{M} \hat{b}_{j,i}(k)\right)$$
$$= \text{Dec}\left(\frac{S(k)}{M} + \frac{1}{M} \sum_{j=1, j\neq i}^{M} n_{j,i}(k)\right), \quad (2)$$

where $S(k) = \sum_{i=1}^{M} b_i(k)$ is the sufficient information to represent the state of the network in this case. Without loss of generality and for the purpose of mathematical derivations, we take M to be even. Let $\kappa_{j,i}$ represent the probability that the j^{th} agent votes one given that the current state is i ($S(k) = i$). We will have,

$$\begin{aligned}\kappa_{j,i} &= \text{Prob}(b_j(k+1) = 1 | S(k) = i) \\ &= \text{Prob}(\frac{i}{M} + n_j(k) \geq \frac{1}{2}) \\ &= Q(\frac{\frac{1}{2} - \frac{i}{M}}{\sigma_j}),\end{aligned} \quad (3)$$

where $n_j = \frac{\sum_{z=1, z\neq j}^{M} n_{z,j}}{M}$, $\sigma_j^2 = \frac{\sum_{z=1, z\neq j}^{M} \sigma^2_{z,j}}{M^2}$ and $Q(\eta) = \frac{1}{\sqrt{2\pi}} \int_\eta^\infty e^{-u^2/2} du$. Let $\Pi(k) = \begin{bmatrix} \text{Prob}(S(k) = 0) \\ \vdots \\ \text{Prob}(S(k) = M) \end{bmatrix}$. We will have

$$\Pi(k+1) = P^T \Pi(k), \quad (4)$$

where $P_{i,j} = \text{Prob}(S(k+1) = j | S(k) = i)$. Matrix P is row stochastic and positive (assuming $\sigma_j \neq 0, \forall j$). Let λ_0, $\lambda_1, \ldots, \lambda_M$ represent the eigenvalues of P, where $|\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_M|$. Then $\lambda_0 = 1$, $|\lambda_i| < 1$ for $1 \leq i \leq M$ and $\lim_{k\to\infty} (P^T)^k \to xy^T$ where $x = P^T x$, $y = Py$, and $x^T y = 1$ [1], [19]. Furthermore, it can be easily confirmed that $\kappa_{j,M-i} = 1 - \kappa_{j,i}$ for $0 \leq i \leq M$ and $1 \leq j \leq M$.

In [1], binary consensus over communication links with the same noise variances was considered. It was shown that the asymptotic behavior of the network was undesirable as it would lose its memory of the initial state. To understand the transient behavior, an expression for the second largest eigenvalue was derived. In this part, we will extend that analysis to derive a tight approximation for the second largest eigenvalue when links have different noise variances.

Lemma 1: We have $P_{M-i,M-j} = P_{i,j}$.

Proof: We will have the following by noting that different noise samples are independent:

$$P_{M-i,M-j} = \underbrace{\sum_{i_1=0}^{1} \cdots \sum_{i_M=0}^{1}}_{\sum_{u=1}^{M} i_u = M-j} \text{Prob}(b_1(k+1) = i_1, \cdots, b_M(k+1) = i_M | S(k) = M-i)$$
$$= \underbrace{\sum_{i_1=0}^{1} \cdots \sum_{i_M=0}^{1}}_{\sum_{u=1}^{M} i_u = M-j} \prod_{u=1}^{M} \text{Prob}(b_u(k+1) = i_u | S(k) = M-i). \quad (5)$$

Since for $i_u \in \{0,1\}$

$$\text{Prob}(b_u(k+1)=i_u|S(k)=M-i)$$
$$=\text{Prob}(b_u(k+1)=1-i_u|S(k)=i), \tag{6}$$

we will have $P_{M-i,M-j}=P_{i,j}$. ■

Lemma 2: $\sum_{j=0}^{M} j\times P_{i,j}=\sum_{j=1}^{M}\kappa_{j,i}$ for $0\le i\le M$.

Proof:

$$\begin{aligned}\sum_{j=0}^{M} j\times P_{i,j} &= E\big(S(k+1)|S(k)=i\big)\\ &= \sum_{j=1}^{M} E\big(b_j(k+1)|S(k)=i\big)\\ &= \sum_{j=1}^{M}\kappa_{j,i}.\end{aligned} \tag{7}$$

■

Lemma 3: We have $\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)(P_{i,j}-P_{M-i,j})=\frac{M}{2}-\sum_{j=1}^{M}\kappa_{j,i}$.

Proof: The following can be shown using Lemma 1 and 2:

$$\begin{gathered}\sum_{j=0}^{\frac{M}{2}-1}(\tfrac{M}{2}-j)(P_{i,j}-P_{M-i,j})=\\ \sum_{j=0}^{\frac{M}{2}-1}(\tfrac{M}{2}-j)P_{i,j}+\sum_{j'=\frac{M}{2}+1}^{M}(\tfrac{M}{2}-j')P_{M-i,M-j'}=\\ \sum_{j=0}^{\frac{M}{2}-1}(\tfrac{M}{2}-j)P_{i,j}+\sum_{j'=\frac{M}{2}+1}^{M}(\tfrac{M}{2}-j')P_{i,j'}=\\ \sum_{j=0}^{M}(\tfrac{M}{2}-j)P_{i,j}=\tfrac{M}{2}-\sum_{j=1}^{M}\kappa_{j,i}.\end{gathered} \tag{8}$$

■

We next find a tight approximation for λ_1.

Assumption 1: For large enough σ_js, we will have the following approximation based on the linearization of the Q function: $\kappa_{j,i,\text{approx}}=\frac{i}{M}+(1-\frac{2i}{M})\kappa_{j,0}$ for $0\le i\le M$. See [1] for more details.

Theorem 1: Let P_{approx} and $\lambda_{1,\text{approx}}$ represent the approximation of matrix P and its second largest eigenvalue under Assumption 1 respectively. We will have $\lambda_{1,\text{approx}}=1-\frac{2}{M}\sum_{j=1}^{M}Q(\frac{1}{2\sigma_j})$.

Proof: Let P^T be partitioned as follows: $P^T=\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$, where P_1 is the matrix with the first $\frac{M}{2}$ rows of P^T, P_2 is the $\frac{M}{2}$th row of P^T and P_3 is the matrix with the last $\frac{M}{2}$ rows of P^T (note that P is a square matrix of dimension $M+1$). Let λ represent an eigenvalue of P with $\beta=\begin{bmatrix}\beta_0 & \beta_1 & \ldots & \beta_M\end{bmatrix}^T$ representing the corresponding eigenvector: $\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}\beta=\lambda\beta$. By utilizing the special structure of matrix P denoted in Lemma 1, we have $P_1\beta-DP_3\beta=\lambda\begin{bmatrix}\beta_0-\beta_M\\ \beta_1-\beta_{M-1}\\ \vdots\\ \beta_{\frac{M}{2}-1}-\beta_{\frac{M}{2}+1}\end{bmatrix}$, where D represents the backward identity matrix. This results in $\Sigma\zeta=\lambda\zeta$, where $\zeta=\begin{bmatrix}\zeta_0 & \zeta_1 & \ldots & \zeta_{\frac{M}{2}-1}\end{bmatrix}^T$ with $\zeta_i=\beta_i-\beta_{M-i}$ for $0\le i\le\frac{M}{2}-1$ and $\Sigma_{j,i}=P_{i,j}-P_{M-i,j}$ for $0\le i,j\le\frac{M}{2}-1$. Let $\Sigma_{j,i,\text{approx}}$ represent the approximation of $\Sigma_{j,i}$. Let $\chi=\Sigma_{\text{approx}}^T\begin{bmatrix}\frac{M}{2}\\ \frac{M}{2}-1\\ \vdots\\ 1\end{bmatrix}$, then the i^{th} element of χ will be as follows, using Lemma 3:

$$\chi(i)=\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)(P_{i,j,\text{approx}}-P_{M-i,j,\text{approx}})$$
$$=\frac{M}{2}-\sum_{j}\kappa_{j,i,\text{approx}}=(\frac{M}{2}-i)(1-\frac{2}{M}\sum_{j}Q(\frac{1}{2\sigma_j})). \tag{9}$$

Then $1-\frac{2}{M}\sum_j Q(\frac{1}{2\sigma_j})$ is an eigenvalue of Σ_{approx}. Since the eigenvalues of Σ_{approx} are also eigenvalues of P_{approx}, $1-\frac{2}{M}\sum_j Q(\frac{1}{2\sigma_j})$ is an eigenvalue of P_{approx}. Furthermore, as $\sigma_j\to 0$, this eigenvalue goes to one. Therefore it is the second largest eigenvalue (see [1] for details). ■

The second largest eigenvalue plays a key role in determining how fast the network is approaching its steady-state. The closer the second eigenvalue is to the unit circle, the network will be in consensus for a longer period of time. This can be seen from Theorem 1. The higher the noise variances are, the smaller the second largest eigenvalue will be.

So far in this section we assumed that link variances are time-invariant. For stochastic link variances, Eq. 4 can be easily modified to reflect the impact of non-stationary link qualities by changing P to $P(k)$ and σ_j of Eq. 3 to $\sigma_j(k)$. For a general distribution of $\sigma_{j,i}(k)$, $\Pi(k)$ may not converge. To see this, note that for $\Pi(k)$ to converge, there should exist a vector c such that $c^TP(k)=c^T$, for any $P(k)$. However, by checking the eigenvectors of $P(k)$ for $\sigma_{j,i}(k_1)=0$ and $\sigma_{j,i}(k_2)\to\infty$, for $j\neq i$, it can be easily confirmed that no common c exists. For such stochastic cases, the average dynamics should instead be considered, i.e. the average of $P(k)$, averaged over the distribution of link variances should be considered (see [17] for related discussions).

IV. Impact of Graph Connectivity

In this section we explore the impact of graphs that are not fully connected on consensus behavior. More specifically, we derive an expression for the second largest eigenvalue assuming a time-invariant connected graph. Since different nodes have different neighbor sets in this case, let $S_i(k)$ represent the sum of the votes of the neighbors of the i^{th} node (including itself) at k^{th} time step: $S_i(k)=\sum_{j\in\Psi_i}b_j(k)$. We have the following for $0\le r\le N_i$,

$$\begin{aligned}\kappa_{i,r,N_i} &= \text{Prob}[b_i(k+1)=1|S_i(k)=r]\\ &= Q(\frac{\frac{1}{2}-\frac{r}{N_i}}{\sigma_i}),\end{aligned} \tag{10}$$

where $\sigma_i^2 = \frac{\sum_{j\in\Psi_i, j\neq i}\sigma_{j,i}^2}{N_i^2}$ and N_i is the size of the neighbor set of node i (including itself).

In this case, $S(k)$ is no longer sufficient information to represent the state of the network. Instead, we define $D(k) = [b_1(k)\ b_2(k)\ \cdots\ b_M(k)]$ as the state of the network at k^{th} time step. Given $D(k)$, $b_i(k+1)$s become independent:

$$\text{Prob}\big[D(k+1) = [z_1\ \cdots z_M]|D(k)\big] = \prod_{i=1}^{M}\text{Prob}[b_i(k+1) = z_i|D(k)], \tag{11}$$

where $z_i \in \{0,1\}$ for $1 \leq i \leq M$. Let $\Xi(k)$ represent a $2^M \times 1$ vector that contains the probabilities of being in different states:

$$\Xi(k) = \begin{bmatrix} \text{Prob}\big[D(k) = [00\cdots0]\big] \rightarrow S(k) = 0 \\ \left.\begin{matrix}\text{Prob}\big[D(k) = [00\cdots1]\big] \\ \vdots \\ \text{Prob}\big[D(k) = [01\cdots0]\big] \\ \text{Prob}\big[D(k) = [10\cdots0]\big]\end{matrix}\right\} \rightarrow S(k) = 1 \\ \vdots \\ \text{Prob}\,[D(k) = [11\cdots1]] \rightarrow S(k) = M \end{bmatrix}, \tag{12}$$

where $S(k)$ is the sum of all the votes as defined in Section III. In Eq. 12, without loss of generality, possible states are ordered such that $S(k)$ increases. Within each group where $S(k)$ is constant, the states are ordered increasingly. Then, $\Xi_i(k) = \text{Prob}[D(k) = \phi_i]$ for $0 \leq i \leq 2^M - 1$, where ϕ_i is the i^{th} state chosen from the ordered list. We will have,

$$\Xi(k+1) = T^T\Xi(k), \tag{13}$$

where $T = [T_{i,j}]$ represents a $2^M \times 2^M$ state transition matrix. Next we derive an expression for the second largest eigenvalue of the state transition matrix. For this derivation, we assume that all the nodes have the same number of neighbors, i.e. the graph is regular. We furthermore, assume that all the links have the same noise variances. We are currently working on relaxing these assumptions as discussed in Section VI on further extensions.

Extended Assumption 1 (extended to not fully connected graphs): For large enough σ_i, κ_{i,r,N_i} of Eq. 10 can be tightly approximated as follows:

$$\kappa_{i,r,N_i} \approx \frac{r}{N_i} + (1 - 2\frac{r}{N_i})\kappa_{i,0,N_i}. \tag{14}$$

Lemma 4: Let T_{approx} represent the transition probability matrix generated under extended Assumption 1. Let $N_i = N$ for $1 \leq i \leq M$ and $\sigma_{i,j} = \sigma$ for $i \neq j$ and $1 \leq i,j \leq M$. Then, $\lambda_{1,\text{approx}} = 1 - 2Q(\frac{1}{2\sigma_N})$ is the second largest eigenvalue of T_{approx}, where $\sigma_N^2 = \frac{(N-1)\sigma^2}{N^2}$.

Proof: For any $0 \leq i \leq 2^M - 1$, we will have the following:

$$\begin{aligned} &\sum_{j=0}^{M} j\text{Prob}[S(k+1) = j|D(k) = \phi_i] \\ &= E[S(k+1)|D(k) = \phi_i] \\ &= \sum_{m=1}^{M} E[b_m(k+1)|D(k) = \phi_i] \\ &= \sum_{m=1}^{M} \kappa_{m,S_m(k),N}, \end{aligned} \tag{15}$$

where $S_m(k)$ is a function of ϕ_i. By applying the approximation of Eq. 14 to $\kappa_{m,S_m(k),N}$, we will have

$$\begin{aligned} &\sum_{j=0}^{M} j\text{Prob}[S(k+1) = j|D(k) = \phi_i] \\ &= \sum_{m=1}^{M}\frac{S_m(k)}{N} + \big(M - 2\sum_{m=1}^{M}\frac{S_m(k)}{N}\big)\kappa_{m,0,N} \\ &= S(k) + \big(M - 2S(k)\big)\kappa_{m,0,N}, \end{aligned} \tag{16}$$

where $\kappa_{m,0,N} = Q(\frac{1}{2\sigma_N})$, for $1 \leq m \leq M$, and $S(k) = \text{sum}(\phi_i)$ with sum(.) representing the sum of the vector. The last equality is written using $\sum_m S_m(k) = NS(k)$.

Similarly, we have

$$\sum_{j=0}^{M}\frac{M}{2}\text{Prob}[S(k+1) = j|D(k) = \phi_i] = \frac{M}{2}. \tag{17}$$

Therefore,

$$\begin{aligned} &\sum_{j=0}^{M}(\frac{M}{2} - j)\text{Prob}[S(k+1) = j|D(k) = \phi_i] \\ &= \big(\frac{M}{2} - S(k)\big)\big(1 - 2Q(\frac{1}{2\sigma_N})\big). \end{aligned} \tag{18}$$

Let ζ represent a $2^M \times 1$ vector, where $\zeta_d = \frac{M}{2} - \text{sum}(\phi_d)$ and $\chi = T_{\text{approx}}\zeta$. Then we will have the following using Eq. 18,

$$\begin{aligned} \chi(d) &= \sum_{j=0}^{M}(\frac{M}{2} - j)\text{Prob}[S(k+1) = j|D(k) = \phi_d] \\ &= \big(\frac{M}{2} - \text{sum}(\phi_d)\big)(1 - 2Q(\frac{1}{2\sigma_N})) \\ &= \zeta_d(1 - 2Q(\frac{1}{2\sigma_N})). \end{aligned} \tag{19}$$

Therefore, $1 - 2Q(\frac{1}{2\sigma_N})$ is one of the eigenvalues of T_{approx}. As σ_N goes to 0, $\lambda_{1,\text{approx}}$ goes to one. Therefore, similar to Section III, it can be easily confirmed that this is the second largest eigenvalue. ■

As discussed earlier, the second largest eigenvalue plays a key role in the overall consensus behavior. The closer it is to one, the better the performance will be as the network will stay in consensus for a longer period of time with higher probability. It can be seen from Lemma 4 that as N increases, λ_1 gets closer to one. This means that the denser the graph is, the better the performance will be, as expected. Fig. 1 shows the impact of graph density on the performance. The network has 6 nodes with 5 out of 6 initially voting 0. All the links have $\sigma = 0.6$. The figure shows the probability of accurate consensus as a function of time. It can be seen that as N decreases, the network stays in accurate consensus with lower probability and for a shorter period of time.

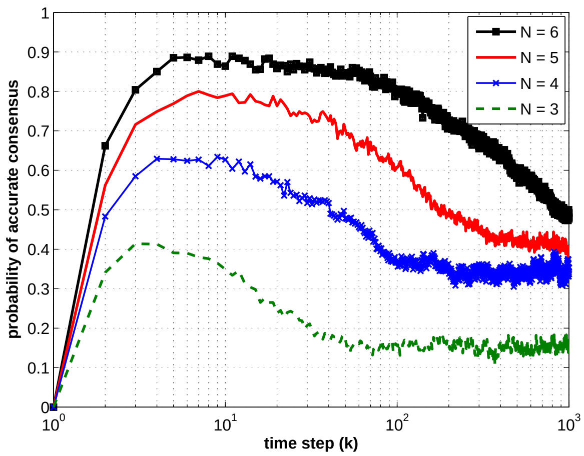

Fig. 1. Impact of network connectivity on binary consensus over regular graphs with noisy links – $M = 6$.

V. Behavior of an individual node

In this section we explore the voting dynamics of an individual node. We will have the following for a general connected graph with different and stochastic link qualities:

$$b_i(k+1) = \text{Dec}\left(\frac{S_i(k)}{N_i(k)} + n_i(k)\right), \qquad (20)$$

where $1 < N_i(k) \le M$ is as defined in Section II and $n_i(k) = \frac{1}{N_i(k)}\sum_{j\in\Psi_i(k), j\neq i} n_{j,i}(k)$ has a variance of $\sigma_i^2(k) = \frac{1}{N_i^2(k)}\sum_{j\in\Psi_i(k), j\neq i}\sigma_{j,i}^2(k)$. We will have the following for the i^{th} node under extended Assumption 1:

$$\begin{aligned}
&\text{Prob}[b_i(k+1) = 1] \\
&= \sum_{r=0}^{N_i(k)} \text{Prob}[b_i(k+1) = 1 | S_i(k) = r]\text{Prob}[S_i(k) = r] \\
&= \sum_{r=0}^{N_i(k)} \kappa_{i,r,N_i(k)}\text{Prob}[S_i(k) = r] \\
&= \sum_{r=0}^{N_i(k)} \left(\frac{r}{N_i(k)} + (1 - 2\frac{r}{N_i(k)})\kappa_{i,0,N_i(k)}\right) \times \\
&\quad \text{Prob}[S_i(k) = r] \\
&= \kappa_{i,0,N_i(k)} + \frac{1 - 2\kappa_{i,0,N_i(k)}}{N_i(k)}\overline{S_i(k)} \\
&= \kappa_{i,0,N_i(k)} + \frac{1 - 2\kappa_{i,0,N_i(k)}}{N_i(k)}\sum_{j\in\Psi_i(k)}\text{Prob}[b_j(k) = 1].
\end{aligned} \qquad (21)$$

Then we will have

$$\Lambda(k+1) = B(k)\Lambda(k) + A(k), \qquad (22)$$

where $\Lambda(k) = \begin{bmatrix} \text{Prob}(b_1(k) = 1) \\ \text{Prob}(b_2(k) = 1) \\ \vdots \\ \text{Prob}(b_M(k) = 1) \end{bmatrix}$, $A_i(k) = \kappa_{i,0,N_i(k)}$, $B_i'(k) = \frac{1-2\kappa_{i,0,N_i(k)}}{N_i(k)}$, $B(k)$ is an $M \times M$ matrix with $B_{i,j}(k) = \begin{cases} B_i'(k) & j \in \Psi_i(k) \\ 0 & \text{else} \end{cases}$ and $A(k)$ is a column vector whose i^{th} element is $A_i(k)$.

Therefore:

$$\Lambda(k) = \prod_{i=0}^{k-1} B(i)\Lambda(0) + \sum_{j=0}^{k-1}\left[\left(\prod_{t=j+1}^{k-1} B(t)\right)A(j)\right]. \qquad (23)$$

For the next derivation, we assume that the network has different link qualities that are stochastic, but it is fully connected. Then we will have, $N_i(k) = M, \forall i$ and $B(k) = \begin{bmatrix} B_1'(k) \\ B_2'(k) \\ \vdots \\ B_M'(k) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} = B'(k)\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$. Let

$$\begin{aligned}
\alpha(j) &= \sum_{r=1}^{M} A_r(j) = \sum_{r=1}^{M} Q\left(\frac{1}{2\sigma_r(j)}\right), \\
\beta(j) &= \sum_{r=1}^{M} B_r'(j) = 1 - \frac{2}{M}\sum_{r=1}^{M} Q\left(\frac{1}{2\sigma_r(j)}\right) = 1 - \frac{2}{M}\alpha(j),
\end{aligned} \qquad (24)$$

and $\hat{\Lambda}(0) = \text{sum}\,(\Lambda(0))$. We will have,

$$\begin{aligned}
\Lambda(k) &= \prod_{i=0}^{k-1} B(i)\Lambda(0) + \sum_{j=0}^{k-1}\left(\prod_{t=j+1}^{k-1} B(t)\right)A(j) \\
&= A(k-1) + B'(k-1) \\
&\quad \times \left[\prod_{i=0}^{k-2}\beta(i)\hat{\Lambda}(0) + \sum_{j=0}^{k-2}\alpha(j)\prod_{t=j+1}^{k-2}\beta(t)\right]
\end{aligned} \qquad (25)$$

By using Eq. 24, we will have

$$\begin{aligned}
\Lambda(k) = A(k-1) + B'(k-1)\Bigg[&\prod_{i=0}^{k-2}\beta(i)\hat{\Lambda}(0) \\
&+ \frac{M}{2}\left(\sum_{j=0}^{k-2}\prod_{t=j+1}^{k-2}\beta(t) - \sum_{j=0}^{k-2}\prod_{t=j}^{k-2}\beta(t)\right)\Bigg],
\end{aligned} \qquad (26)$$

which results in

$$\Lambda(k) =$$

$$A(k-1) + B'(k-1)\left[\prod_{i=0}^{k-2}\beta(i)\hat{\Lambda}(0) + \frac{M}{2}(1-\prod_{j=0}^{k-2}\beta(j))\right]$$

$$= A(k-1) + \frac{M}{2}B'(k-1) + (\hat{\Lambda}(0) - \frac{M}{2})B'(k-1)\prod_{i=0}^{k-2}\beta(i)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T + (\hat{\Lambda}(0) - \frac{M}{2})B'(k-1)\prod_{i=0}^{k-2}\beta(i). \quad (27)$$

It is easy to see that $\beta(i) = \sum_{r=1}^{M} B'_r(i) < 1$ (assuming that at least one link has a non-zero variance). Therefore,

$$\lim_{k\to\infty} \Lambda(k) = \frac{1}{2}\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T. \quad (28)$$

This suggests that at steady state, each node's voting pattern becomes purely random, which is consistent with the fact that the network becomes memoryless. Fig. 2 shows the convergence of $\Lambda(k)$ to 0.5 for a network of 4 nodes, where 3 out of 4 start by voting 1. The simulation confirms the derivations.

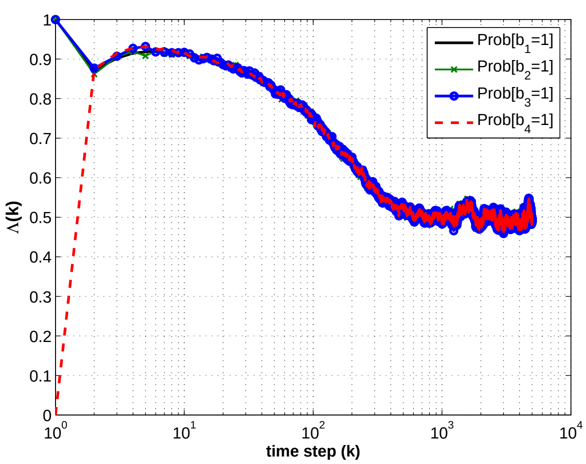

Fig. 2. Dynamics of each individual node

VI. Conclusion and Further Extensions

In this paper we considered the dynamics of binary consensus over Gaussian communication links. We explored the impact of link qualities with different variances on the performance by deriving an expression for the second largest eigenvalue. Furthermore, we characterized the impact of network connectivity on binary consensus by deriving an expression for the second largest eigenvalue for regular connected graphs. Finally, we characterized the behavior of an individual node and showed that each node will asymptotically vote purely random. We made a number of assumptions in our derivations. For instance, we assumed regular graphs in order to characterize the impact of graph connectivity. We are currently working on relaxing these assumptions. For the stochastic case, we are working on characterizing the dynamics of the average of the system with pdfs that are relevant in fading environments.

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