Binary Consensus with Gaussian Communication Noise: A Probabilistic Approach

Yasamin Mostofi

Department of Electrical and Computer Engineering

University of New Mexico, Albuquerque, NM 87113, USA

Abstract— **In this paper we consider the impact of Gaussian communication noise on a network that is trying to reach consensus on the occurrence of an event. We take a probabilistic approach and formulate the consensus problem using Markov chains. We show that the steady state behavior in the presence of any amount of non-zero communication noise is unfavorable as the network loses the memory of the initial state. However, we show that the network can still reach and stay in accurate consensus for a long period of time. In order to characterize this, we derive a close approximation for the second largest eigenvalue of the network and show how it is related to the size of the network and communication noise variance.**

I. Introduction

Cooperative decision-making and control has gotten considerable interest in recent years. Such problems arise in many different areas such as environmental monitoring, surveillance and security, smart homes and factories, target tracking and military systems. Consider a scenario where a network of agents wants to perform a task jointly. Each agent has limited sensing capabilities and has to rely on the group for improving its estimation/detection quality. Consensus problems arise when the agents need to reach an agreement on a value of a parameter. We categorize these problems into two main groups: *Estimation Consensus* and *Detection Consensus*.

By *estimation consensus* we refer to the problems in which each agent has an estimate of the parameter of interest, where the parameter of interest (and therefore the states of the system) can take values over an infinite set or an unknown finite set. Then the agents want to reach a consensus on the value of the parameter. These problems received considerable attention over the past few years. Convergence and equilibrium state of continuous-time and discrete-time consensus protocols have been studied for both time-invariant and time-varying topologies [1], [2], [3], [4]. Furthermore, consensus protocols have been applied to formation problems [5], [6], [7], [8] as well as distributed filtering [9]. The uncertainty in the exchanged information was considered and accounted for in [11] where conditions for achieving consensus were derived. [10] provides a comprehensive survey of literature on such consensus problems.

By *detection Consensus*, on the other hand, we refer to problems in which the parameter of interest takes values from a finite known set. Then the update protocol that each agent will utilize becomes non-linear. We refer to a subset of detection consensus problems where the network is trying

to reach an agreement over a parameter that can only have two values as *binary consensus*. For instance, networked detection of fire falls into this category. [12] considered and characterized phase transition of a binary consensus problem in the presence of a uniformly distributed communication noise. Since the probability density function of this noise is bounded, there exists a transition point beyond which consensus will be reached in this case [12].

In most consensus applications, the agents will communicate their status wirelessly. On the bit level, receiver noise, which is best modeled as an additive Gaussian noise [13], will corrupt the transmitted bits. Therefore, it becomes important to analyze the performance of consensus problems in the presence of Gaussian communication noise. In this paper we consider a networked binary consensus problem with Gaussian communication noise. Since the noise is not bounded, there is no transition point beyond which consensus is guaranteed. Instead, we utilize a probabilistic approach to characterize and understand the behavior of the network. Introducing Gaussian communication noise, in itself, will result in complex behaviors. Therefore, in this paper we assume that the underlying graph of the network is fully connected in order to focus on the impact of noisy communication links. We characterize the network in terms of probability of reaching and staying in consensus as well as steady-state behavior, by utilizing the properties of binomial coefficients, $Q(.)$ function and the underlying Markov chain. We show that steady-state behavior of such systems is undesirable independent of the amount of communication noise variance as the network loses the memory of the initial state. However, we show that the network can still reach and stay in agreement for a long period of time before it heads towards its steady-state value. Our main contribution is characterizing this probabilistically by deriving a close approximation for the second largest eigenvalue of the underlying transition probability matrix. The paper is organized as follows. System model and the underlying Markov chain are presented in Section II and III respectively. Steady-state behaviors are characterized in Section IV. Transient behaviors such as probability of having a consensus at k^{th} time step is analyzed in Section V, where our main result is presented: deriving an expression for the second largest eigenvalue of the transition probability matrix. This is followed by conclusions and future directions in Section VI.

II. SYSTEM MODEL

Consider M agents that want to reach consensus on the occurrence of an event. Each agent makes a decision on the occurrence of the event based on its one-time local sensor measurement. Let $b_i(0) \in \{0,1\}$ represent the initial decision of the ith agent, at time step $k = 0$, based on its local measurement. $b_i = 1$ indicates that the ith agent votes that the event occurred whereas $b_i = 0$ denotes otherwise. Each agent sends its binary vote to the rest of the group and revises its vote based on the received information. This process will go on for a while. We say that *accurate consensus* is achieved if each agent reaches the majority of the votes. In this paper, we assume a fully connected underlying graph to devote our attention to the interesting and unexplored behaviors that result from voting over Gaussian noise channels. This means that every agent can exchange information with every other one. However, the received information will be corrupted by communication noise.

Each transmission gets corrupted by receiver noise, which is best modeled by Additive White Gaussian Noise (AWGN channel) [13]. When a node receives the decisions of other nodes, the receptions can happen in different frequencies or time slots [14]. Then each reception will experience a different (uncorrelated) sample of the receiver thermal noise.[1] Let $n_{j,i}(k)$ represent the noise at k^{th} time step in the transmission of the information from the jth node to the ith one. $n_{j,i}(k)$ is a zero-mean Gaussian random variable with variance of σ_n^2. We take the variance of the noise of all the transmissions to be the same in this paper. This is a fair assumption as variance of this noise is a function of the receiver bandwidth and thermal noise [14]. Let $\hat{b}_{j,i}(k)$ represent the reception of the ith agent from transmission of the jth one at k^{th} time step. We will have, $\hat{b}_{j,i}(k) = b_j(k) + n_{j,i}(k)$ for $1 \le i, j \le M$. Each agent will then update its vote based on the received information as follows:

$$
\begin{aligned}
b_i(k+1) &= Dec\left(\frac{1}{M}\sum_{j=1}^{M}\hat{b}_{j,i}(k)\right) \\
&= Dec\left(\frac{1}{M}\sum_{j=1}^{M}b_j(k) + \frac{1}{M}\sum_{j=1,j\neq i}^{M}n_{j,i}(k)\right) \\
&= Dec\left(\frac{S(k)}{M} + \frac{1}{M}\sum_{j=1,j\neq i}^{M}n_{j,i}(k)\right),
\end{aligned}
\tag{1}
$$

where $Dec(.)$ represents a decision function for binary 0-1 detection: $Dec(x) = \begin{cases} 1 & x \ge .5 \\ 0 & x < .5 \end{cases}$. Let $\Omega_M = \{0, 1, 2, \ldots, M\}$ represent a finite set. Then $S(k) =$

---

[1]If the transmissions are using the same frequency and time slots, they should be differentiated in codes. In this case, a node will experience different interference terms in the reception of the decisions of others.

$\sum_{j=1}^{M} b_j(k) \in \Omega_M$ represents the state of the network (sum of 1 votes) at k^{th} time step. $S(k)$ is a measure of the closeness to consensus at time step k.

Definition 1: We say that the network is in an *accurate* consensus state in the kth time step if and only if the following holds:

$$
\begin{aligned}
\text{if} \quad S(0) &\ge \lceil\frac{M}{2}\rceil \Rightarrow \forall i \quad b_i(k) = 1 \\
\text{if} \quad S(0) &< \lceil\frac{M}{2}\rceil \Rightarrow \forall i \quad b_i(k) = 0,
\end{aligned}
\tag{2}
$$

where $\lceil.\rceil$ represents the ceiling function. Note that, without loss of generality, we decided that an agent with half votes of zeros and half votes of ones, will vote one.

Definition 2: The network can be in a state of consensus that is not related to its initial state. We refer to this state as *memoryless consensus*. For instance, if 70% of the nodes start by voting 1 but the network ends up in a state where every node is voting 0, the network has reached memoryless consensus. This is undesirable since the final agreement is not related to the initial state of the system and is merely a function of the communication noise. In this paper, the term consensus will refer to the *accurate* consensus state unless otherwise is stated.

Remark 1: Due to the presence of Gaussian noises, there is no guarantee in reaching or staying in consensus. In other words, there is no transition point beyond which consensus is reached permanently. Rather than that, we are interested in evaluating the probability of reaching and staying in different states of the network, steady state probabilistic behavior of the system as well as relating these parameters to the noise variance and size of the network.

III. MARKOV CHAIN REPRESENTATION

Let $\Pi_i(k)$ represent the probability that $i \in \Omega_M$ of the agents are voting 1 at k^{th} time step: $\Pi_i(k) = \text{Prob}\left[S(k) = i\right], \quad i \in \Omega_M$. Let $P_{i,j}$ represent the probability of the network going from the ith state to the jth one in one time step,

$$
P_{i,j} = \text{Prob}\left[S(k+1) = j | S(k) = i\right], \quad i, j \in \Omega_M. \tag{3}
$$

Note that, given the current state, $P_{i,j}$ becomes independent of the earlier states: $S(k-1)$, $S(k-2)$, ..., $S(0)$. Furthermore, $P_{i,j}$ is independent of k, as indicated by Eq. 3. More specifically, $P_{i,j}$ will have a binomial distribution as follows,

$$
P_{i,j} = \binom{M}{j}\kappa_i^j(1-\kappa_i)^{M-j}, \tag{4}
$$

where κ_i represents the probability that any agent votes 1 in the next time step, given a current state of i ($S(k) = i$). This probability is the same for all the agents. Consider l^{th} agent. We will have:

$$
\kappa_i = \text{Prob}\left[\frac{i}{M} + w_l(k) > .5\right], \tag{5}
$$

where $w_l(k) = \frac{1}{M} \sum_{j=1, j \neq l}^{M} n_{j,l}(k)$ is a weighted sum of i.i.d Gaussian random variables, which will in turn have a Gaussian distribution: $w_l \sim N(0, \sigma_M)$, where $\sigma_M^2 = \frac{(M-1)\sigma_n^2}{M^2}$ and $N(\mu, \sigma)$ represents a normal distribution with average of μ and standard deviation of σ. Therefore,

$$\kappa_i = Q\left(\frac{.5 - \frac{i}{M}}{\sigma_M}\right), \tag{6}$$

where $Q(.)$ represents the Q function: $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-z^2}{2}} dz$.

The consensus of M agents that communicate over noisy links can, therefore, be modeled as a Markov Chain with the sequence of random variables, $S(0), S(1), S(2), \ldots$, where the possible states for $S(k)$ form a countable set Ω_M with $M + 1$ elements. The states represent all the possibilities for the sum of all the votes. $P_{i,j}$ will then represent the transition probability from state i to state j for $i, j \in \Omega_M$. We will have the following time update,

$$\Pi(k+1) = P^T \Pi(k), \tag{7}$$

where $\Pi(k) = \begin{bmatrix} \Pi_0(k) & \Pi_1(k) & \ldots & \Pi_M(k) \end{bmatrix}^T$ and $P = [P_{i,j}]$.

A. Properties of Matrix P and the Underlying Markov Chain

The Markov chain, represented by Eq. 7, is irreducible with aperiodic states, and therefore ergodic for $\sigma_n \neq 0$. Matrix P is a stochastic matrix, i.e. $\forall i, \quad \sum_{j=0}^{M} P_{i,j} = 1$. Therefore, one is an eigenvalue of P. Let $\lambda_0, \lambda_1, \ldots, \lambda_M$ represent the eigenvalues of matrix P in a decreasing order: $|\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_M|$.

Remark 2: Using Gersgorin disk theorem [15], the eigenvalues of P are located in the following area: $\bigcup_{i=0}^{M} \{z \in C : |z - P_{i,i}| \leq 1 - P_{i,i}\}$. This implies that $\forall i, |\lambda_i| \leq 1$.

Remark 3: Assume $\sigma_n \neq 0$. Then $P > 0$ (element-wise). From stochastic property of matrix P, we know that one is an eigenvalue. From Remark 2 and applying Perron's theorem [15], we will have,

a) $\lambda_0 = 1$ as a simple eigenvalue of P,
b) $|\lambda_i| < 1$ for $i \neq 0$ and
c) $[P^T]^m \to L$ as $m \to \infty$, where $L = xy^T$, $x = P^T x$, $y = Py$, and $x^T y = 1$.

Lemma 1: The $(M-i)^{th}$ row of matrix P is a reverse repeated version of the ith row for $i \in \Omega_M$, i.e. $P_{i,j} = P_{M-i, M-j}$ for $i, j \in \Omega_M$.

Proof: $\kappa_{M-i} = Q\left(\frac{.5 - \frac{M-i}{M}}{\sigma_M}\right) = Q\left(\frac{\frac{i}{M} - .5}{\sigma_M}\right) = 1 - \kappa_i$. Then from the expression of $P_{i,j}$ of Eq. 4, it can be easily seen that $P_{i,j} = P_{M-i, M-j}$. ∎

IV. STEADY-STATE BEHAVIOR

In this section, we are interested in steady state analysis of the aforementioned networked consensus problem. Using Eq. 7, we will have

$$\Pi(k) = [P^T]^k \Pi(0). \tag{8}$$

Note that $\Pi(0)$ has only one element of one and all the rest zero.

A. Case of $\sigma_n = 0$

In this part we look at a special ideal case to get more insight into the behavior of the networked consensus problem. If $\sigma_n = 0$, we will have the following by noting the definition of the utilized decision function of Section II, $\kappa_{i, \sigma_n = 0} = \begin{cases} 1 & i \geq \lceil \frac{M}{2} \rceil \\ 0 & i < \lceil \frac{M}{2} \rceil \end{cases}$ and $P_{i,j,\sigma_n=0} = \begin{cases} 1 & j = 0 \text{ and } i < \lceil \frac{M}{2} \rceil \\ 1 & j = M \text{ and } i \geq \lceil \frac{M}{2} \rceil \\ 0 & \text{else} \end{cases}$. Therefore, in this case, $\lambda_0 = \lambda_1 = 1$ and the rest of the eigenvalues are zero. This corresponds to two right eigenvector for P^T: $x_1 = \begin{bmatrix} 1 & 0 & 0 \ldots 0 \end{bmatrix}^T$ and $x_2 = \begin{bmatrix} 0 & 0 & 0 \ldots 1 \end{bmatrix}^T$. In this ideal case, the network will reach an accurate consensus that corresponds to the majority of the initial votes in one step and stays there, as expected. x_1 and x_2 correspond to the two possible consensus outcome in this case: all agents voting zero and all agents voting one. It can be observed that there is no unique initial-state-independent steady state. Based on the initial state, the network gets to the accurate consensus state, as is desirable.

B. Case of $\sigma_n \neq 0$

If $\sigma_n \neq 0$, From Remark 3 we know that the network reaches a steady state asymptotically. For $\sigma_n \neq 0$, we will have $\lim_{k \to \infty} \Pi(k) = xy^T \Pi(0)$ where x and y are as defined in Remark 3.

Lemma 2: The steady-state value of vector Π is independent of the initial state for $\sigma_n \neq 0$.

Proof: Since $\Pi(k)$ contains probabilities of being in different states, we have $\forall k, \sum_{i=0}^{M} \Pi_i(k) = 1$. Therefore, the scalar $y^T \Pi(0)$ should act as a normalization factor: $y^T \Pi(0) = \frac{1}{\text{sum}(x)}$, where sum$(x)$ refers to the sum of the elements of vector x. In other words, $\Pi(0)$ has exactly one element equal to one and the rest zero while y is a vector whose elements are all the same. Then, $y^T \Pi(0)$ loses the information of the initial state. Therefore, the steady-state value will be independent of the initial state. ∎

Remark 4: Lemma 2 shows that for any amount of non-zero communication noise, the system can never get to the accurate consensus asymptotically in an intelligent manner as it loses its memory of the initial state. It may only visit an accurate consensus state asymptotically in a random manner. This makes the asymptotic behavior of networked consensus with Gaussian noise considerably undesirable. However, the system can still show some intelligent behavior and can reach and stay in consensus for a long period of time (which could be enough for practical purposes). We will discuss this shortly. Before doing so, we will look at two special cases.

1) Special Case: $\sigma_n \to \infty$: In this part, we consider the behavior of networked consensus as communication noise variance approaches ∞.

Lemma 3: $\lim_{k\to\infty}\lim_{\sigma_n\to\infty}\Pi(k) = B$, where $B = .5^M \times \left[\binom{M}{0} \binom{M}{1} \binom{M}{2} \cdots \binom{M}{M} \right]^T$.

Proof: In this case, we will have, $\lim_{\sigma_n\to\infty}\kappa_i = 0.5$ and $\lim_{\sigma_n\to\infty}P_{i,j} = \binom{M}{j}0.5^M$. Therefore, we will have,

$$P^T_{\sigma_n\to\infty} = \lim_{\sigma_n\to\infty}P^T = \left[\begin{array}{cccc} B & B & \ldots & B \end{array} \right]. \quad (9)$$

Then P^T becomes a rank one matrix with $\lambda_0 = 1$ and $\lambda_i = 0$ for $1 \le i \le M$. By noting that $\text{sum}(B) = .5^M \sum_{i=0}^{M}\binom{M}{i} = 1$, it can be easily confirmed that $P^T_{\sigma_n\to\infty}B = B$. Therefore, B is the eigenvector of $P^T_{\sigma_n\to\infty}$ that corresponds to $\lambda_0 = 1$ and is the steady-state vector. ∎

2) Special Case: $M \to \infty$: This case has similar properties to the case of $\sigma_n = 0$. It can be easily shown that $\kappa_{i,M\to\infty} = \kappa_{i,\sigma_n=0}$ and $P_{i,j,M\to\infty} = P_{i,j,\sigma_n=0}$. This means that as $M \to \infty$, the network will reach consensus in one step and will stay there.

V. PROBABILITY OF BEING IN CONSENSUS AFTER k TIME STEPS: TRANSIENT BEHAVIORS

We observed in the previous section that for any amount of communication noise, accurate consensus can not be achieved in an intelligent way asymptotically. However, this does not mean that networked consensus in the presence of Gaussian noise will never work. Fig. 1 shows $\Pi(k)$ as a function of time step, k, for $M = 4$ agents. The initial condition in this example is $\Pi[0] = \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \end{array} \right]^T$, meaning that 3 out of 4 members are voting 1. Then the system is in accurate consensus at time step k if $\Pi[k] = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \end{array} \right]^T$, i.e. all members voting one. The solid line is showing the probability of having an accurate consensus ($\Pi_4(k)$). It can be seen that for a good amount of time (enough in practical situations), the system will be in accurate consensus with high probability. Another interesting feature is the steady-state behavior. In steady-state, for this case, the system will be in consensus with high probability (Π_0 and Π_4 close to .5). However, it will be a memoryless consensus. In general, the smaller the communication noise variance is, the higher the probability of reaching and maintaining an accurate consensus state will be. The second largest eigenvalue, (λ_1), plays a key role in determining how fast the network is approaching its steady-state. The closer the second eigenvalue is to the unit circle, the network will be in consensus for a longer period of time. We will then derive an expression for the second largest eigenvalue of matrix P next.

A. Second Largest Eigenvalue

In this part, we will derive an expression for the second largest eigenvalue of P as it becomes a crucial factor in determining the transient probability of the network being in consensus before it aims towards its steady-state undesirable
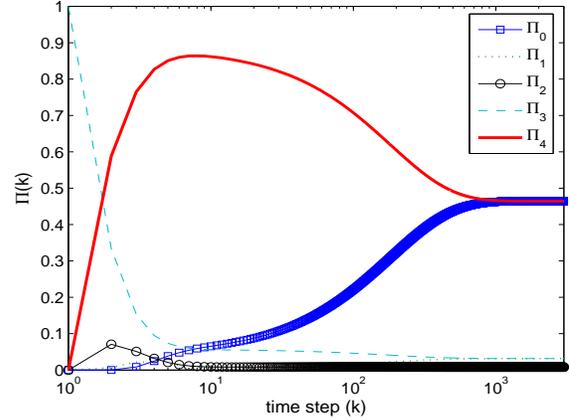

Fig. 1. Consensus behavior for M=4

behavior. Before deriving an expression for the general P matrix, we will consider a 3×3 case to gain some insight.

1) Case of $M = 2$: In this part, we will consider a case where two agents are trying to achieve consensus. We are interested in finding an analytical expression for the second largest eigenvalue of matrix P.

Lemma 4: Let $P_{3\times 3}$ represent P matrix for $M = 2$. Let $\lambda_{1,3\times 3}$ represent the second largest eigenvalue of $P_{3\times 3}$. Then, the second largest eigenvalue of $P_{3\times 3}$ will be as follows: $\lambda_{1,3\times 3} = 1 - 2Q\left(\frac{1}{2\sigma_M}\right)$.

Proof: For $M = 2$, we will have the following utilizing Lemma 1, $P^T_{3\times 3} = \left[\begin{array}{ccc} P_{0,0} & P_{1,0} & P_{2,0} \\ P_{0,1} & P_{1,1} & P_{2,1} \\ P_{2,0} & P_{1,0} & P_{0,0} \end{array} \right]$. Let λ represent a general eigenvalue of $P^T_{3\times 3}$. Then

$$P^T_{3\times 3}\beta = \lambda\beta, \quad (10)$$

with $\beta = \left[\begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \right]^T$ representing the corresponding eigenvector of $P^T_{3\times 3}$. By subtracting the first and third equations of Eq. 10, we will have, $\beta_1(P_{0,0} - P_{2,0}) - \beta_3(P_{0,0} - P_{2,0}) = \lambda(\beta_1 - \beta_3)$. Therefore, $\lambda = P_{0,0} - P_{2,0} = 1 - 2\kappa_0 = 1 - 2Q\left(\frac{1}{2\sigma_M}\right)$. Since $\lim_{\sigma_n\to 0}\lambda = 1$, λ represents the second largest eigenvalue of $P_{3\times 3}$. The smaller eigenvalues will go to zero as $\sigma_n \to 0$ (see Section IV-A). Therefore, $\lambda_{1,3\times 3} = 1 - 2Q\left(\frac{1}{2\sigma_M}\right)$. ∎

Note that $Q\left(\frac{1}{2\sigma_M}\right) \le .5$ since $\sigma_M \ge 0$.

2) General Case: In general, finding an expression for the second largest eigenvalue of the $M \times M$ case is extremely difficult without approximation. In this part, we will find an approximation for the second largest eigenvalue of matrix P, that matches well with its true value. Without loss of generality, we derive the results of this part for M even.

Lemma 5: Let P represent a transition probability matrix generated using κ_i, as indicated by Eq. 4. We will have,

$$\sum_{j=0}^{\frac{M}{2}-1}\left(\frac{M}{2} - j\right)(P_{i,j} - P_{M-i,j}) = \frac{M}{2}(1 - 2\kappa_i). \quad (11)$$

Proof: We have, $\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)(P_{i,j}-P_{M-i,j})=$
$\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)\binom{M}{j}\kappa_i^j(1-\kappa_i)^{M-j}$
$-\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)\binom{M}{j}(1-\kappa_i)^j\kappa_i^{M-j}.$

By noting that $\sum_{j=\frac{M}{2}+1}^{M}(\frac{M}{2}-j)\binom{M}{j}\kappa_i^j(1-\kappa_i)^{M-j}=$
$-\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)\binom{M}{j}(1-\kappa_i)^j\kappa_i^{M-j}$, we will have,
$\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)(P_{i,j}-P_{M-i,j})=$
$\sum_{j=0}^{M}(\frac{M}{2}-j)\binom{M}{j}\kappa_i^j(1-\kappa_i)^{M-j}=$
$\frac{M}{2}-\underbrace{\sum_{j=0}^{M}j\binom{M}{j}\kappa_i^j(1-\kappa_i)^{M-j}}_{\text{average of a binomially distributed variable}}=\frac{M}{2}(1-2\kappa_i).$

■

Assumption 1: For large enough σ_M, we can approximate κ_i by:

$$\begin{aligned}\kappa_{i,\text{approx}}&=\frac{i}{M}+(1-\frac{2i}{M})\kappa_0\quad\text{for}\quad 0\le i\le\frac{M}{2}-1,\\\kappa_{i,\text{approx}}&=1-\kappa_{M-i,\text{approx}}\quad\text{for}\quad\frac{M}{2}\le i\le M,\end{aligned}\quad(12)$$

where κ_0 is as defined previously. This approximation is based on the linearization of $Q(.)$ function around the origin as follows: $Q(\alpha)\approx Q(0)-\frac{\alpha}{\sqrt{2\pi}}$. Note that $\kappa_{i,\text{approx}}$ represent a probability value, i.e. $0\le\kappa_{i,\text{approx}}\le 1$. The larger the communication noise variance is, the more accurate this approximation will be. Cases with considerably small communication noise variances have the second largest eigenvalue very close to one. Therefore, we are more interested in finding the eignevalues of the cases with non-negligible communication noise variances, which makes this approximation more accurate.

Lemma 6: Let $P_{i,j,\text{approx}}$ represent the transition probability matrix generated using $\kappa_{i,\text{approx}}$ of Eq. 12 of Assumption 1. Let $\Omega_{\frac{M}{2}}=\{0,1,2,\ldots,\frac{M}{2}-1\}$. Let Γ represent an $\frac{M}{2}\times\frac{M}{2}$ matrix with the following coefficients: $\Gamma(j,i)=P_{i,j,\text{approx}}-P_{M-i,j,\text{approx}}$ for $i,j\in\Omega_{\frac{M}{2}}$. Then $\begin{bmatrix}\frac{M}{2}&\frac{M}{2}-1&\ldots&2&1\end{bmatrix}^T$ will be a right eigenvector of Γ^T with $1-2\kappa_0$ representing the corresponding eigenvalue.

Proof: Let $\chi=\Gamma^T\begin{bmatrix}\frac{M}{2}\\\frac{M}{2}-1\\\ldots\\2\\1\end{bmatrix}$. Then the ith element of vector χ will be as follows for $i\in\Omega_{\frac{M}{2}}$:

$$\chi(i)=\sum_{j=0}^{\frac{M}{2}-1}(\frac{M}{2}-j)(P_{i,j,\text{approx}}-P_{M-i,j,\text{approx}}),\quad(13)$$

which will be as follows using Lemma 5 (note that Lemma 5 is valid as long as κ_i represents a probability value):

$$\chi(i)=\frac{M}{2}(1-2\kappa_{i,\text{approx}}).\quad(14)$$

From Eq. 12, we will have, $\chi(i)=\frac{M}{2}(1-\frac{2i}{M})(1-2\kappa_0)=(\frac{M}{2}-i)(1-2\kappa_0),\forall i\in\Omega_{\frac{M}{2}}$. This means that $\begin{bmatrix}\frac{M}{2}&\frac{M}{2}-1&\ldots&2&1\end{bmatrix}^T$ will be a right eigenvector of Γ^T with $1-2\kappa_0$ representing the corresponding eigenvalue.

■

Remark 5: Let P_{approx} represent an approximation of matrix P under Assumption 1. It can be easily confirmed that P_{approx} is stochastic. Furthermore, by applying Perron's theorem, it can be seen that P_{approx} has a simple eigenvalue at 1 and all the rest within the unit circle for $\sigma_n\ne 0$.

Lemma 7: The first and second largest eigenvalues of $P_{\text{approx},\sigma_n=0}$ are one while the rest are within the unit circle.

Proof: We have $\lim_{\sigma_n\to 0}\kappa_{i,\text{approx}}=\frac{i}{M}$. From stochastic property of $P_{\text{approx},\sigma_n=0}$ we know that one is one of its eigenvalues. It can be easily confirmed that $P_{\text{approx},\sigma_n=0}^T\times Z_1=Z_1$ and $P_{\text{approx},\sigma_n=0}^T\times Z_2=Z_2$, where $Z_1=[1\quad 0\quad 0\ldots 0]^T$ and $Z_2=[0\quad\ldots 0\quad 1]^T$. This suggests that $P_{\text{approx},\sigma_n=0}$ has one as its eigenvalue with multiplicity of at least two. Let $\Phi_\lambda(A)=\det(A-\lambda I)$, where I is the unitary matrix. Then $\Phi_\lambda(P_{\text{approx},\sigma_n=0})=(1-\lambda)\Phi_\lambda(P_{\text{approx},\sigma_n=0}[1,1])$, where $A[1,1]$ represents a matrix that results from omitting the first row and column of A. Similarly, let $P'_{\text{approx},\sigma_n=0}$ represent the matrix that results from omitting the last row and column of $P_{\text{approx},\sigma_n=0}[1,1]$. Then, $\Phi_\lambda(P_{\text{approx},\sigma_n=0})=(1-\lambda)^2\Phi_\lambda(P'_{\text{approx},\sigma_n=0})$. By applying Gersgorin theorem, it can be easily confirmed that all the eigenvalues of $P'_{\text{approx},\sigma_n=0}$ are inside the unit circle. Therefore, the first and second largest eigenvalues of $P_{\text{approx},\sigma_n=0}$ are one while the rest are inside the unit circle.

■

Theorem 1: Let P_{approx} represent an approximation of matrix P under Assumption 1. Let $\lambda_{1,\text{approx}}$ represent the second largest eigenvalue of P_{approx}. Then we will have, $\lambda_{1,\text{approx}}=1-2\kappa_0=1-2Q\left(\frac{1}{2\sigma_M}\right)$.

Proof: Let P^T be partitioned as follows: $P^T=\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$, where P_1 is the matrix with the first $\frac{M}{2}$ rows of P^T, P_2 is the $\frac{M}{2}$th row of P^T and P_3 is the matrix with the last $\frac{M}{2}$ rows of P^T (note that we assumed an even M, without loss of generality, in this part). Let λ represent an eigenvalue of P with $\beta=\begin{bmatrix}\beta_0&\beta_1&\ldots&\beta_M\end{bmatrix}^T$ representing the corresponding eigenvector. We will have,

$$\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}\beta=\lambda\beta,\quad(15)$$

resulting in $P_1\beta=\lambda\begin{bmatrix}\beta_0&\beta_1&\ldots&\beta_{\frac{M}{2}-1}\end{bmatrix}^T$ and $P_3\beta=\lambda\begin{bmatrix}\beta_{\frac{M}{2}+1}&\beta_{\frac{M}{2}+2}&\ldots&\beta_M\end{bmatrix}^T$. Let D represent the backward identity matrix: $D(i,j)=1$ only if $j=M-i$ for $i,j\in\Omega_M$. Then $DP_3\beta=\lambda\begin{bmatrix}\beta_M&\beta_{M-1}&\ldots&\beta_{\frac{M}{2}+1}\end{bmatrix}^T$. By utilizing the special structure of matrix P denoted in Lemma 1, we have,

$$P_1\beta - DP_3\beta = \lambda \begin{bmatrix} \beta_0 - \beta_M \\ \beta_1 - \beta_{M-1} \\ \vdots \\ \beta_{\frac{M}{2}-1} - \beta_{\frac{M}{2}+1} \end{bmatrix}, \text{ which results}$$

in $\Sigma\zeta = \lambda\zeta$, where $\zeta = \begin{bmatrix} \zeta_0 & \zeta_1 & \cdots & \zeta_{\frac{M}{2}-1} \end{bmatrix}^T$ with $\zeta_i = \beta_i - \beta_{M-i}$ for $0 \le i \le \frac{M}{2}-1$ and $\Sigma_{j,i} = P_{i,j} - P_{M-i,j}$ for $0 \le i,j \le \frac{M}{2}-1$. Therefore, the eigenvalues of Σ are eigenvalues of P. Let Σ_{approx} represent the Σ generated from P_{approx}. From Lemma 6, we will have $\Sigma_{\text{approx}} = \Gamma$, with $1 - 2\kappa_0$ as one of its eigenvalues. Therefore, $1 - 2\kappa_0$ is one of the eigenvalues of matrix P_{approx}. Furthermore, it is the second largest eigenvalue since as σ_n goes to zero, this eigenvalue goes towards one, which only happens to the second largest eigenvalue (see Lemma 7). Therefore, $\lambda_{1,\text{approx}} = 1 - 2\kappa_0 = 1 - 2Q\left(\frac{1}{2\sigma_M}\right)$. ∎

To see how well this approximation works, Fig. 2 shows the second largest eigenvalue and its approximation as a function of σ_n and for $M = 4$, $M = 7$ and $M = 16$. As can be seen, the approximation works well especially for smaller M and larger σ_n. For instance for $M = 16$, we see a small mismatch for smaller values of σ_n. This is as expected since the $Q(.)$ function approximation of Assumption 1 works better the larger σ_M is. Nevertheless, the proposed approximation can be considerably useful in understanding the behavior of group consensus in terms of probability of reaching and staying in consensus.
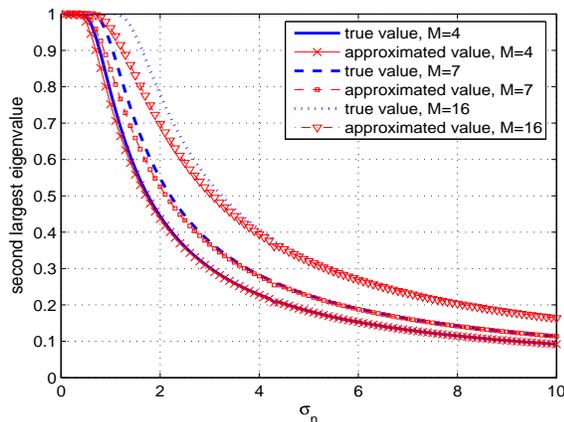

Fig. 2. Approximation of the second largest eigenvalue

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we considered a binary consensus problem in the presence of Gaussian communication noise. We took a probabilistic approach since in the presence of Gaussian communication noise, there will be no transition point beyond which convergence is guaranteed. We showed that for any amount of communication noise, the network's steady-state behavior is undesirable since it can only reach a memoryless consensus, i.e. a consensus that is independent of the initial state. However, the network can still be in accurate consensus for a long period of time (good enough for practical purposes), before it heads towards a memoryless steady-state, depending on its second largest eigenvalue. To characterize this, we derived an approximated expression for the second largest eigenvalue of the transition probability matrix of the network. The approximation matched the true value considerably well and showed how the communication noise variance and size of the network affect consensus behavior. We considered a fully connected network and the same communication noise variance for all the links in this paper. We are currently working on relaxing these assumptions. We are also extending our results to include both communication and sensing errors as well as time-varying link qualities.

REFERENCES

[1] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," IEEE Transactions on Automatic Control, vol. 50, no. 5, pp. 655-661, 2005.

[2] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," IEEE Transactions on Automatic Con- trol, vol. 49, no. 9, pp. 1520-33, 2004.

[3] Y. Hatano and M. Mesbahi, "Agreement over random networks," IEEE Transactions on Automatic Control, vol. 50, no. 11, pp. 1867-1872, 2005.

[4] M. Mesbahi, "State-dependent graphs," Proceedings of 42nd IEEE conference on Decision and Control, 2003.

[5] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem - part 1: the synchronous case," SIAM Journal on Control and Optimization, 2006.

[6] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," IEEE Transactions on Automatic Control, vol. 48, no. 6, 2003.

[7] J. Cortes, S. Martinez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," IEEE Transactions on Automatic Control, vol. 51, no. 8, pp. 1289-1298, 2006.

[8] V. D. Blondel, J. M. Hendrick, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," in IEEE Conf. on Decision and Control and European Control Conference, pp. 2996-3000, Dec. 2005.

[9] D. P. Spanos, R. Olfati-Saber, and R. M. Murray, "Approximate distributed Kalman filtering in sensor networks with quantifiable performance," in the proceedings of 4th international conference on Information Processing in Sensor Networks (IPSN), pp. 133-139, 2005.

[10] W. Ren, R. Beard and E. Atkins, "A Survey of Consensus Problems in Multi-agent Coordination," Proceedings of 2005 American Control Conference, 2005.

[11] W. Ren, R. Beard, D. Kingston, "Multi-agent Kalman consensus with relative uncertainty," Proceedings of 2005 American Control Conference, 2005.

[12] J. Liu, V. Yadav, H. Sehgal, J. Olson, H. Liu, and N. Elia, "Phase Transitions on Fixed Connected Graphs and Random Graphs in the Presence of Noise," in the proceedings of 44th IEEE Conference on Decision and Control, pp. 2996-3000, 2005.

[13] John G. Proakis, *Digital Communication*. McGraw Hill Series in Electrical and Computer Engineering, 3rd edition, 1995.

[14] William Jakes, *Microwave Mobile Communications*. IEEE Press, 1974.

[15] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press 1999.