Binary Consensus Over Fading Channels: A Best Affine Estimation Approach

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Abstract—In this paper we consider a cooperative network that is trying to reach binary consensus over fading channels. We first characterize the impact of fading on network consensus by upper bounding the second largest eigenvalue of the underlying probability transition matrix in fading environments. Using the information of link qualities, we then propose a novel consensus-seeking protocol based on the best affine estimation of network state. We characterize the performance of our proposed strategy mathematically. We derive an approximated expression for the second largest eigenvalue in order to characterize the convergence rate. Our results show the impact of fading on network consensus. They furthermore indicate that the proposed technique can improve the consensus performance considerably.

Index Terms—Binary consensus, Fading channels, Best Affine Estimation (BAE)

I. INTRODUCTION

In recent years, there has been significant interest in cooperative sensing, estimation/detection and control. Such problems arise in many different areas such as environmental monitoring, surveillance and security, smart homes and factories, target tracking and military systems. Distributed detection and estimation for multi-sensor data fusion has, in particular, received considerable attention. In such problems, sensors transmit their observations to a fusion center which fuses all the received data in order to provide a better networked performance. While such central fusion problems have been explored considerably, the problem of a cooperative network trying to reach consensus over unreliable links has only started to receive attention recently.

Consensus problems arise when the agents need to reach an agreement on the value of a parameter and can be categorized into two main groups: *Estimation Consensus* and *Detection Consensus*. *Estimation consensus* refers to the problems where the parameter of interest can take values over an infinite set or an unknown finite set. These problems received considerable attention over the past few years. In particular, control community have applied tools from algebraic graph theory and advanced matrix analysis to characterize estimation consensus problems over graphs that are not fully connected [1]-[9]. [10] provides a comprehensive survey of the literature on such consensus problems. The impact of communication link qualities such as noise and fading, however, has received less attention in the control literature. Recently, there has been considerable interest in estimation consensus over unreliable links from signal processing and communication community. In [11],

the authors develop distributed consensus-seeking algorithms for estimation of deterministic signals. The uncertainty in the exchanged information was also considered in [12] by using a Kalman filtering approach. In [13], the authors consider the average consensus problem under quantization constraints.

Detection Consensus, on the other hand, refers to the problems in which the parameter of interest takes values from a finite known set. Then the update protocol that each agent will utilize becomes non-linear. We referred to a subset of detection consensus problems where the network is trying to reach an agreement over a parameter that can only have two values as *binary consensus* [14]. For instance, networked detection of fire falls into this category. While estimation consensus problems have received considerable attention, detection consensus problems have mainly remained unexplored. In [15], the authors consider convergence in a detection consensus setup over perfect channels, with repeated sensing and known probabilistic sensing models.

In [14] and [16], a different binary consensus problem is considered, where the nodes start with an initial decision regarding the occurrence of an event. Through repeated communication, the goal for every node is to reach the majority of the initial votes, without knowing anything about the sensing qualities. [16] considered and characterized phase transition of such a binary consensus problem in the presence of a uniformly distributed communication noise. Since the probability density function of this noise is bounded, there exists a transition point beyond which consensus will be guaranteed in this case [16]. In most applications the agents will communicate their values wirelessly and they will experience Gaussian receiver noise as opposed to a uniformly distributed noise. Furthermore, the transmitted information can be corrupted by fading. In [14], a binary consensus problem was considered over AWGN channels, where the goal of the network was for all the nodes to reach the majority of the initial decisions. Since the noise is not bounded, there is no transition point beyond which consensus is guaranteed in this case. Instead, a probabilistic approach was utilized to characterize the asymptotic and transient behavior of the network. In [17], a soft information processing approach was introduced to improve the performance of consensus over AWGN channels.

In this paper, we consider binary consensus over fading channels, where the goal of every node is to reach the

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majority of the initial votes. We extend the analysis of [14] to characterize consensus behavior over fading channels. To improve the performance and robustness of network cooperation, we propose novel consensus-seeking protocols that utilize information of link qualities and noise variances. We show that our proposed approach can improve consensus performance drastically. We characterize this mathematically by deriving a tight approximation for the second largest eigenvalue of the underlying transition probability matrix in fading environments.

II. PROBLEM FORMULATION

Consider a cooperative network of M sensors that are trying to reach consensus over the occurrence of an event. Each agent has its own initial decision, based on its one-time sensing. The goal of the network is for each node to reach a decision that is equal to the majority of the initial votes. For instance, in a cooperative fire detection scenario, each node has an initial opinion as to if there is a fire or not. However, as a network they may act only based on the majority vote. Therefore, it is desirable that every node reaches the majority of the initial votes without a group leader. As it may happen in realistic scenarios, the nodes may not have any information on the sensing quality of themselves or others. Therefore, in this paper, the main goal is that each node reaches the majority of the initial votes. Considering sensing quality of the nodes is among possible extensions of this work.

In order to achieve this, each node will transmit its current decision to other nodes. The transmissions occur over fading channels and are furthermore corrupted by the receiver noise. Each node will then revise its current vote based on the received information. This process will go on for a while. We say that *accurate consensus* is achieved if each agent reaches the majority of the initial votes [14]. *Memoryless consensus* then denotes the case where the network reaches consensus but it is not the majority of the initial values. For instance, if 70% of the nodes start voting 1, but they end up all voting zero, we have memoryless consensus, which is not desirable.

Let $b_i(0) \in \{0, 1\}$ represent the initial vote of the ith node, at time step $k = 0$, where $b_i = 1$ indicates that the ith agent decides that the event occurred whereas $b_i = 0$ denotes otherwise. Each agent will send its binary vote (only one bit of information) to the rest over Rayleigh fading channels.[1] Let $r_{j,i}(k)$ represent the fading coefficient of the link from node j to node i. We assume that the agents do not move or move slowly such that $r_{j,i}$s can be considered constant during consensus process. The receiver will then learn $r_{j,i}$ and use it in the detection process. We furthermore assume that the graph of the network is fully connected, i.e. every agent can talk to the rest. However, since the transmitted information is corrupted by noise and fading, no agent has all the information. The work of this paper could be extended to not fully-connected graphs using the analysis of [18]. Let

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[1]We assume an FDMA- or TDMA-based MAC approach. If CDMA is used to differentiate different receptions, same analysis can be performed by adding interference terms.

$n_{j,i}(k)$ represent the receiver noise at the kth time step in the transmission from the jth node to the ith one. $n_{j,i}(k)$ is zero-mean Gaussian with variance of $\sigma_{j,i}^2$. Let $b_{j,i}(k)$ represent the reception of the ith node from the transmission of the jth one at kth time step. We have the following,

$$b_{j,i}(k) = r_{j,i}(k)b_j(k) + n_{j,i}(k) \text{ for } 1 \le i, j \le M, \quad (1)$$

where $n_{i,i}(k) = 0$ and $r_{i,i}(k) = 1$.

III. BINARY CONSENSUS OVER FADING CHANNELS - UNKNOWN $\sigma_{j,i}$

We first consider the case where $\sigma_{j,i}$ is not known. Following the same procedure of [14], we have,

$$b_i(k+1) = \text{Dec}\left(\frac{1}{M}\left(b_i(k) + \sum_{j=1, j \neq i}^{M} \frac{b_{j,i}(k)}{r_{j,i}(k)}\right)\right)$$

$$= \text{Dec}\left(\frac{S(k)}{M} + \frac{1}{M}\sum_{j=1, j \neq i}^{M} \frac{n_{j,i}(k)}{r_{j,i}(k)}\right), \quad (2)$$

where $\text{Dec}(x) = \begin{cases} 1 & x \ge 0.5 \\ 0 & x < 0.5 \end{cases}$ and $S(k) = \sum_{i=1}^{M} b_i(k)$ represents the state of the network. Without loss of generality, we assume that if equal number of nodes are voting one and zero, then it is desirable for the whole network to vote one. Let $\kappa_{i,m}$ represent the probability that the ith agent votes one given that the current state is m. We will have $\kappa_{i,m} = \text{Prob}\left[b_i(k+1) = 1 | S(k) = m\right] = Q(\frac{\frac{1}{2} - \frac{m}{M}}{\sigma_i})$, where $n_i(k) = \frac{1}{M}\sum_{j=1, j \neq i}^{M} \frac{n_{j,i}(k)}{r_{j,i}(k)}$ with the variance of $\sigma_i^2 = \frac{1}{M^2}\sum_{j=1, j \neq i}^{M} \frac{1}{2\text{SNR}_{j,i}}$ and $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-u^2/2} du$. We take $\text{SNR}_{j,i} = \frac{0.5 r_{j,i}^2}{\sigma_{j,i}^2}$ since we do not know the probability with which each node votes zero or one (therefore we take it to be half). We show in the next section how each node can learn such probabilities. Let $\Pi(k) = \begin{bmatrix} \text{Prob}[S(k) = 0] \cdots \text{Prob}[S(k) = M] \end{bmatrix}^T$. We will have

$$\Pi(k+1) = P^T \Pi(k), \quad (3)$$

where $P_{m,n} = \text{Prob}[S(k+1) = n | S(k) = m]$. Matrix P is row stochastic and positive (element-wise). Let λ_0, $\lambda_1, \ldots, \lambda_M$ represent the eigenvalues of P, where $|\lambda_0| \ge |\lambda_1| \ge \ldots \ge |\lambda_M|$. Using Perron's theorem, we will have (assuming $\sigma_i \neq 0$, $\forall i$) $\lambda_0 = 1$, $|\lambda_i| < 1$ for $1 \le i \le M$ and $\lim_{k\to\infty}(P^T)^k \to xy^T$ where $x = P^T x$, $y = Py$, and $x^T y = 1$ [19]. Furthermore, it can be easily confirmed that $\kappa_{i,M-m} = 1 - \kappa_{i,m}$ for $0 \le i \le M$. From the asymptotic properties of matrix P, it can be seen that the network will lose its memory of the initial state asymptotically, which is undesirable. Fig. 1 shows a network of 4 nodes that is trying to reach consensus over fading channels, where 3 out of 4 nodes vote one initially. Then it is desirable that all four nodes vote one through communication. The figure shows $E\left[\text{prob}[S(k) = 4]\right]$, average probability of accurate consensus (average over fading). It can be seen that at the earlier iterations, the probability of accurate consensus increases. However, after a while communication is not beneficial anymore as
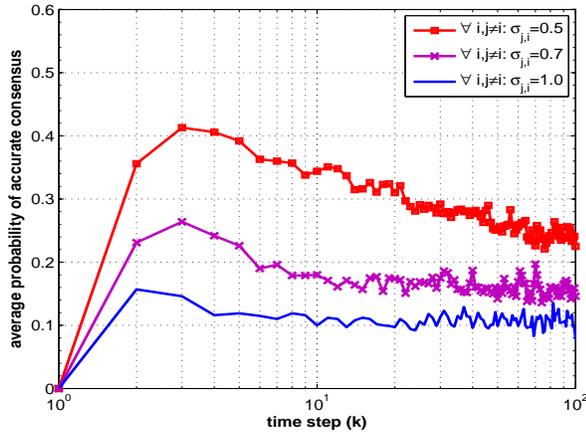
Fig. 1. Average probability of accurate consensus - $E[r_{j,i}^2] = 1, \forall i, j \neq i$

it results in error propagation in the network, a decrease in the probability of accurate consensus and an eventual memoryless consensus. Furthermore, by comparing this figure with Fig. 1 of [14], it can be seen that fading furthermore ruins the performance drastically by reducing the maximum probability of accurate consensus. Then the second largest eigenvalue of P, λ_1, becomes an important factor in characterizing the transient behavior. λ_1 plays a key role in determining how fast the network is approaching its steady-state. The closer the second eigenvalue is to the unit circle, the network will be in consensus for a longer period of time. In the limit, if $\forall i$, $\sigma_i = 0$, we will have $\lambda_1 = 1$. In [14], binary consensus over AWGN channels was considered, when knowledge of $\sigma_{j,i}$ is not available. To understand the transient behavior, an expression for the second largest eigenvalue known as *Fiedler eigenvalue* was derived. We next extend that analysis to the case of fading.

Assumption 1: For large enough σ_is, we have the following approximation based on the linearization of the Q function: $\kappa_{i,m,\text{approx}} = \frac{m}{M} + (1 - \frac{2m}{M})\kappa_{i,0}$ for $0 \leq m \leq M$. See [14] for more details.

Theorem 1: (Second largest eigenvalue) Let P_{approx} and $\lambda_{1,\text{approx}}$ represent the approximation of matrix P and its second largest eigenvalue under Assumption 1 respectively. We will have $\lambda_{1,\text{approx}} = 1 - \frac{2}{M} \sum_{i=1}^{M} Q(\frac{1}{2\sigma_i})$.

Proof: The theorem can be easily proved by extending the analysis of [14] and [18] to embrace fading. ∎

Theorem 1 shows that the higher σ_is are, the smaller the second largest eigenvalue will be. λ_1 is a function of the fading coefficients, $r_{j,i}$s, and therefore will change for different node configurations. Finding a closed form expression for the average of λ_1 over fading, however, is considerably challenging. Instead, we characterize a tight upper bound on it next.

Theorem 2: (Average second largest eigenvalue) Take $\text{SNR}_{j,i}$s to be i.i.d exponential random variables with $\mu = \overline{\text{SNR}_{j,i}}$ representing the average. We will have $\overline{\lambda}_{1,\text{approx}} \leq \sqrt{\frac{0.25M^2\mu/(M-1)}{1+0.25M^2\mu/(M-1)}}$.

Proof: From the definition of σ_i, we have $\frac{1}{2\sigma_i} = \sqrt{\frac{0.5M^2}{\sum_{j=1,j\neq i}^{M} \frac{1}{\text{SNR}_{j,i}}}} \leq \sqrt{0.5M^2\text{SNR}_{i,\min}}$, where $\text{SNR}_{i,\min} = $

$\min(\{\text{SNR}_{j,i}|1 \leq j \leq M, j \neq i\})$. Then, from Theorem 1, $\lambda_{1,\text{approx}} \leq 1 - \frac{2}{M} \sum_{i=1}^{M} Q(\sqrt{0.5M^2\text{SNR}_{i,\min}})$. The minimum of $M - 1$ i.i.d exponential random variables, $\text{SNR}_{i,\min}$, can be easily shown to have an exponential distribution with $\frac{\mu}{M-1}$ representing its average. For an exponentially distributed random variable u, with the average $\overline{u}$, we have $\overline{Q(\sqrt{au})} = \frac{1}{2}(1 - \sqrt{\frac{0.5a\overline{u}}{1+0.5a\overline{u}}})$ for an arbitrary $a > 0$. Therefore, $\overline{\lambda}_{1,\text{approx}} \leq \sqrt{\frac{0.25M^2\mu/(M-1)}{1+0.25M^2\mu/(M-1)}}$. ∎

Fig. 2 shows the average of the 2nd largest eigenvalue, $\overline{\lambda}_{1,\text{approx}}$, and its upper bound derived in Theorem 2. For comparison, the 2nd largest eigenvalue of the average dynamics (average over fading), $\overline{P}$, is also plotted. It can be seen that the curves are close to each other and that the upper bound of Theorem 2 is considerably tight.
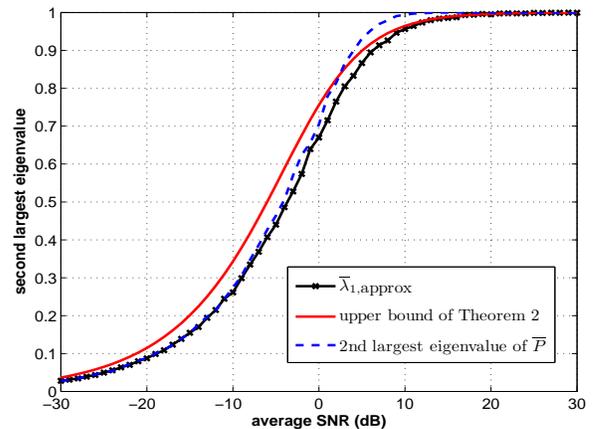

Fig. 2. Characterization of the 2nd largest eigenvalue in fading environments

IV. BINARY CONSENSUS OVER FADING CHANNELS - KNOWN $\sigma_{j,i}$

A. A consensus-seeking protocol based on the Best Affine Estimation (BAE)

If the knowledge of $\sigma_{j,i}$s is available at the receiver, it could be used in the decision-making process to improve the performance. In this part we consider such cases and propose to build decision-making functions based on a Best Affine Estimation (BAE) approach. Let $\phi_i(k)$ represent the sum of the votes of all the nodes except for the ith node: $\phi_i(k) = \sum_{j=1,j\neq i}^{M} b_j(k)$. Then the ith node tries to estimate $\phi_i(k)$ by using the best affine unbiased function of the received information:

$$\hat{\phi}_i(k) = \sum_{j=1,j\neq i}^{M} \alpha_{j,i}(k)b_{j,i}(k) + \beta_i(k), \quad (4)$$

where $\hat{\phi}_i(k)$ is the ith node's estimate of $\phi_i(k)$. To ensure an unbiased estimator, we should have

$$E[\hat{\phi}_i(k)] = E[\phi_i(k)]$$
$$\Rightarrow \sum_{j=1,j\neq i}^{M} \alpha_{j,i}(k)E[b_{j,i}(k)] + \beta_i(k) = \sum_{j=1,j\neq i}^{M} E[b_j(k)]. \quad (5)$$

Then we will have

$$\alpha_{j,i}(k) = \arg\ \min\ E[(\hat{\phi}_i(k) - \phi_i(k))^2],$$

$$\text{subject to } \beta_i(k) = \sum_{j=1,j\neq i}^{M} (1 - \alpha_{j,i}(k)r_{j,i}(k))q_j(k), \quad (6)$$

where $q_j(k) = E[b_j(k)]$. We have,

$$E[(\hat{\phi}_i(k) - \phi_i(k))^2]$$

$$= E[(\sum_{j=1,j\neq i}^{M} (\alpha_{j,i}(k)b_{j,i}(k) - b_j(k)) + \beta_i(k))^2]$$

$$= E[(\sum_{j=1,j\neq i}^{M} \alpha_{j,i}(k)b_{j,i}(k) - b_j(k))^2]$$

$$- 2\beta_i(k) \sum_{j=1,j\neq i}^{M} (1 - r_{j,i}(k)\alpha_{j,i}(k))q_j(k) + \beta_i^2(k)$$

$$= E[(\sum_{j=1,j\neq i}^{M} \alpha_{j,i}(k)b_{j,i}(k) - b_j(k))^2] - \beta_i^2(k). \quad (7)$$

To facilitate mathematical derivations, we will assume that different node values are uncorrelated, i.e. $E[b_j(k)b_l(k)] = 0$ for $j \neq l$.[2] By noting that $E[n_{j,i}(k)n_{l,i}(k)] = 0$ for $j \neq l$ and $E[b_j(k)] = E[b_j^2(k)] = q_j(k)$, we will have the following for $j \neq l$:

$$E[(\alpha_{j,i}(k)b_{j,i}(k) - b_j(k))^2]$$

$$= (\alpha_{j,i}(k)r_{j,i}(k) - 1)^2 q_j(k) + \alpha_{j,i}^2(k)\sigma_{j,i}^2 \quad \text{and} \quad (8)$$

$$E[(\alpha_{j,i}(k)b_{j,i}(k) - b_j(k))(\alpha_{l,i}(k)b_{l,i}(k) - b_l(k))]$$

$$= (\alpha_{j,i}(k)r_{j,i}(k) - 1)(\alpha_{l,i}(k)r_{l,i}(k) - 1)q_j(k)q_l(k).$$

After a straight forward derivation, we will have

$$E[(\hat{\phi}_i(k) - \phi_i(k))^2] \quad (9)$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{M} [(1 - \alpha_{j,i}(k)r_{j,i}(k))^2 (q_j(k) - q_j^2(k)) + \alpha_{j,i}^2(k)\sigma_{j,i}^2].$$

By noting that Eq. 9 is a convex function of $\alpha_{j,i}(k)$, we have

$$\alpha_{j,i}(k) = \frac{r_{j,i}(k)}{r_{j,i}^2(k) + \frac{\sigma_{j,i}^2}{q_j(k)(1-q_j(k))}} = \frac{1/r_{j,i}(k)}{1 + \frac{1}{(1-q_j(k))\text{SNR}_{j,i}(k)}},$$

$$\beta_i(k) = \sum_{j=1,j\neq i}^{M} \beta_{j,i}(k), \quad (10)$$

where $\text{SNR}_{j,i}(k) = \frac{q_j(k)r_{j,i}^2(k)}{\sigma_{j,i}^2}$ and $\beta_{j,i}(k) = (1 - \alpha_{j,i}(k)r_{j,i}(k))q_j(k)$. Therefore, the ith node will update its decision as follows:

$$b_i(k+1) = \text{Dec}\left(\frac{1}{M}[b_i(k) + \hat{\phi}_i(k)]\right) =$$

$$\text{Dec}\left(\frac{1}{M}\left[b_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{M} (\alpha_{j,i}(k)b_{j,i}(k) + \beta_{j,i}(k))\right]\right). \quad (11)$$

[2]In general, $b_j(k)$ and $b_l(k)$ will be correlated. However, assessing the correlation and incorporating it in the estimation process is infeasible.

It can be seen that Eq. 11 assumes that the knowledge of $q_j(k)$ is available at the receiver. If the ith node does not have an estimate of $q_j(k)$, it will assume that $q_j(k) = \frac{1}{2}$. We refer to this case as *basic BAE*. Then, *learning BAE* refers to the case where $q_j(k)$ is statistically learned in the receiver. In order to do so, node i will pass $b_{j,i}(k)$ through a hard decision function to estimate the number of times that b_j becomes one in a given time interval. Fig. 3 summarizes the steps involved in the learning BAE scheme, where $q_{j,i}(k)$ represents ith node's estimate of $q_j(k)$. The BAE processing block refers to the aforementioned process of estimating $b_j(k)$ based on $b_{j,i}(k)$ and $q_{j,i}(k)$.
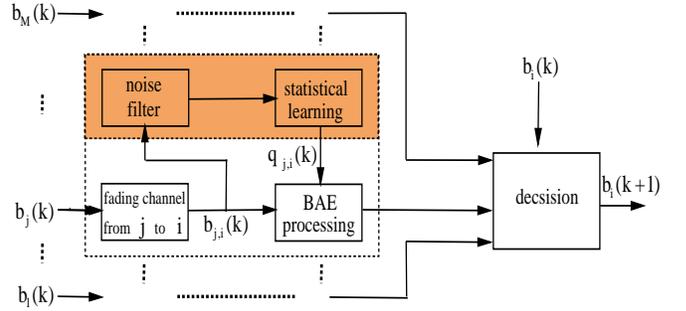

Fig. 3. An illustration of consensus over fading channels with BAE approach – the shaded area is used when statistical learning is deployed and the rest demonstrates basic BAE

In [17], a soft information processing approach is proposed based on the best nonlinear estimation of $\sum_{j=1,j\neq i}^{M} b_j(k)$, which equals to $\sum_{j=1,j\neq i}^{M} E[b_j(k)|b_{j,i}(k)]$. It can be shown that our proposed BAE approach is the projection of this best nonlinear estimation to the linear space.

B. State Transition Matrix and the Second Largest Eigenvalue

Next we will characterize the dynamics of consensus based on Eq. 11. In this case, $S(k)$ is no longer the sufficient information to represent the state of the network. Instead, we define $D(k) = [b_1(k)\ b_2(k)\cdots b_M(k)]$ as the state of the network at kth time step. Let $\Psi(k)$ represent a $2^M \times 1$ vector that contains the probabilities of being in different states:

$$\Psi(k) = \begin{bmatrix} \text{Prob}[D(k) = [00\cdots 0]] \to S(k) = 0 \\ \text{Prob}[D(k) = [00\cdots 1]] \\ \vdots \\ \text{Prob}[D(k) = [10\cdots 0]] \\ \vdots \\ \text{Prob}[D(k) = [11\cdots 1]] \to S(k) = M \end{bmatrix}, \quad (12)$$

where $S(k)$ is the sum of all the votes as defined in Section III. In Eq. 12, without loss of generality, possible states are ordered such that $S(k)$ increases. Within each group where $S(k)$ is constant, the states are ordered increasingly. Then, $\Psi_m(k) = \text{Prob}[D(k) = \chi_m]$ for $0 \leq m \leq 2^M - 1$, where χ_m is the mth state chosen from the ordered list. We will have

$$\Psi(k+1) = T^T \Psi(k), \quad (13)$$

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where $T = [T_{m,n}]$ represents a $2^M \times 2^M$ state transition matrix and $T_{m,n} = \text{Prob}[D(k+1) = \chi_n|D(k) = \chi_m]$, for $0 \le m, n \le 2^M - 1$. Matrix T is row stochastic and positive (element-wise). Similar to Section III, all the eigenvalues of T are inside the unit circle: $|\lambda_i| < 1$ for $1 \le i \le 2^M - 1$ except for $\lambda_0 = 1$ (for $\sigma_i \ne 0$). Using Perron's theorem, $\lim_{k\to\infty}(T^T)^k \to xy^T$ where $x = T^T x$, $y = Ty$, and $x^T y = 1$ [19]. In order to compare the performance of the proposed BAE with the case that the knowledge of $\sigma_{j,i}$ is not available, we characterize the second largest eigenvalue of the basic BAE case (in this case, the nodes do not estimate $q_j(k)$ and assume that it is 0.5). Deriving an expression for the second largest eigenvalue of T, however is considerably challenging. Therefore, instead of first deriving an expression for λ_1 of T and then averaging it over fading, we consider the dynamics after averaging the fused votes over fading to facilitate the mathematical derivations of this section. Our simulation results indicate that the eigenvalue derived in this manner approximates the average of the 2nd largest eigenvalue of T tightly. In Eq. 11, $\hat{\phi}_i(k)$ represents the output of the fusion of the incoming votes at the ith node. By averaging it over fading, we will have

$$\widetilde{b}_i(k+1) = \text{Dec}\Big(\frac{1}{M}\Big[\widetilde{b}_i(k) + E_{\text{fading}}[\hat{\phi}_i(k)]\Big]\Big) =$$

$$\text{Dec}\left(\frac{1}{M}\Big[\widetilde{b}_i(k) + \sum_{\substack{j=1\\j\ne i}}^{M} \widetilde{\alpha}_{j,i}\widetilde{b}_j(k) + \widetilde{\beta}_i + \sum_{\substack{j=1\\j\ne i}}^{M} \widetilde{\gamma}_{j,i}z_{j,i}(k)\Big]\right),\tag{14}$$

where $E_{\text{fading}}[.]$ denotes averaging over fading (as opposed to noise) and $\widetilde{b}_i(k+1)$ is the next vote of node i based on averaging the fused data over fading. We have $\widetilde{\alpha}_{j,i} = E_{\text{SNR}_{j,i}}\Big[\frac{1}{1+\frac{2}{\text{SNR}_{j,i}}}\Big]$, $\widetilde{\beta}_i = \sum_{j=1,j\ne i}^{M} \frac{1-\widetilde{\alpha}_{j,i}}{2}$, $\widetilde{\gamma}_{j,i} = E_{\text{SNR}_{j,i}}\Big[\frac{1}{\sqrt{2\text{SNR}_{j,i}}}\frac{1}{1+\frac{2}{\text{SNR}_{j,i}}}\Big]$, and $z_{j,i}(k)$s are i.i.d zero-mean Gaussian random variables with variance of one.

Given the current state of χ_m, the probability of node i voting one will be as follows based on the new dynamics of Eq. 14:

$$\widetilde{\kappa}_{i,m} = \text{Prob}[\widetilde{b}_i(k+1) = 1|\widetilde{D}(k) = \chi_m]$$
$$= Q\left(\frac{0.5 - \frac{1}{M}[\widetilde{b}_i^m + \sum_{j=1,j\ne i}^{M} \widetilde{\alpha}_{j,i}\widetilde{b}_j^m + \widetilde{\beta}_i]}{\widetilde{\sigma}_i}\right),\tag{15}$$

where $\widetilde{\sigma}_i^2 = \frac{\sum_{j=1,j\ne i}^{M} \widetilde{\gamma}_{j,i}^2}{M^2}$ and $\chi_m = [\widetilde{b}_1^m \cdots \widetilde{b}_M^m]$. Given $\widetilde{D}(k)$, $\widetilde{b}_i(k+1)$s become independent:

$$\text{Prob}\Big[\widetilde{D}(k+1) = [w_1 \cdots w_M]|\widetilde{D}(k)\Big]$$
$$= \prod_{i=1}^{M} \text{Prob}[\widetilde{b}_i(k+1) = w_i|\widetilde{D}(k)],$$

where $w_i \in \{0,1\}$ for $1 \le i \le M$. We will have,

$$\widetilde{\Psi}(k+1) = \widetilde{T}^T \widetilde{\Psi}(k),\tag{16}$$

where $\widetilde{T} = [\widetilde{T}_{m,n}]$ represents a $2^M \times 2^M$ state transition matrix based on the average dynamics and $\widetilde{T}_{m,n} = \text{Prob}[\widetilde{D}(k+1) = \chi_n|\widetilde{D}(k) = \chi_m]$, for $0 \le m, n \le 2^M - 1$.

Assumption 2: For large enough $\widetilde{\sigma}_i$, we will have the following approximation based on the linearization of the Q function: $\widetilde{\kappa}_{i,m} \approx \frac{\widetilde{S}_{i,m}}{M} + (1 - \frac{2\widetilde{S}_{i,m}}{M})\widetilde{\kappa}_{i,0}$,where $\widetilde{S}_{i,m} = \frac{M(\widetilde{b}_i^m + \sum_{j=1,j\ne i}^{M} \widetilde{\alpha}_{j,i}\widetilde{b}_j^m)}{1+\sum_{j=1,j\ne i}^{M} \widetilde{\alpha}_{j,i}}$.

Theorem 3: Assume that $\text{SNR}_{j,i}$s are i.i.d. exponential random variables with $\mu = \overline{\text{SNR}}_{j,i}$. We have $\widetilde{\alpha}_{j,i} = \widetilde{\alpha}$, $\widetilde{\beta}_{j,i} = \widetilde{\beta}$, $\widetilde{\gamma}_{j,i} = \widetilde{\gamma}$ and $\widetilde{\kappa}_{i,0} = \widetilde{\kappa}_0$. Let $\widetilde{T}_{\text{approx}}$ and $\widetilde{\lambda}_{1,\text{approx}}$ represent the approximation of matrix $\widetilde{T}$ and its second largest eigenvalue under Assumption 2 respectively. We will have $\widetilde{\lambda}_{1,\text{approx}} = 1 - 2Q\Big(\frac{0.5 - \frac{\widetilde{\beta}}{M}}{\frac{\sqrt{M-1}}{M}\widetilde{\gamma}}\Big)$.

Proof: For any $0 \le m \le 2^M - 1$ and $\widetilde{S}(k) = \sum_{i=1}^{M} \widetilde{b}_i(k)$, we have

$$\sum_{j=0}^{M} j\text{Prob}[\widetilde{S}(k+1) = j|\widetilde{D}(k) = \chi_m]$$
$$= E[\widetilde{S}(k+1)|\widetilde{D}(k) = \chi_m] = \sum_{i=1}^{M} \widetilde{\kappa}_{i,m}.\tag{17}$$

By using Assumption 2 and the fact that $\sum_{i=1}^{M} \widetilde{S}_{i,m} = M\sum_{i=1}^{M} \widetilde{b}_i^m$, we can show $\sum_{j=0}^{M}(\frac{M}{2} - j)\text{Prob}[\widetilde{S}(k+1) = j|\widetilde{D}(k) = \chi_m] = (1 - 2\widetilde{\kappa}_0)(\frac{M}{2} - \sum_{i=1}^{M} \widetilde{b}_i^m)$. Let ζ represent a $2^M \times 1$ vector, where $\zeta_d = \frac{M}{2} - \text{sum}(\chi_d)$ and $\xi = \widetilde{T}_{\text{approx}}\zeta$. Then, we have $\xi(d) = \sum_{j=0}^{M}(\frac{M}{2} - j)\text{Prob}[\widetilde{S}(k+1) = j|\widetilde{D}(k) = \chi_d] = \zeta_d(1 - 2\widetilde{\kappa}_0)$. Therefore, $1 - 2Q\Big(\frac{0.5 - \frac{\widetilde{\beta}}{M}}{\frac{\sqrt{M-1}}{M}\widetilde{\gamma}}\Big)$ is an eigenvalue of $\widetilde{T}_{\text{approx}}$. It can furthermore be shown that the eigenvalues of $\widetilde{T}_{\text{approx}}$ have the same properties of the eigenvalues of $\widetilde{T}$. Therefore, this is the second largest eigenvalue of $\widetilde{T}_{\text{approx}}$. The closer $\widetilde{\lambda}_{1,\text{approx}}$ is to 1, the better consensus performance will be, which is confirmed by Theorem 3. ∎

V. SIMULATION AND COMPARISON

Fig. 4 shows the performance of the proposed BAE approach where all the channels experience the same noise variance ($\sigma_{j,i}^2 = 1$) and the average power of fading coefficients is equal to one ($E[r_{j,i}^2] = 1$). The figure shows the dynamics of the learning BAE case (where $q_j(k)$s are estimated as described in Section IV-A), basic BAE case and the case where the fused data was averaged over fading as described in Section IV-B. The performance of the case where knowledge of $\sigma_{j,i}$ is not available is also shown for comparison (Section III). It can be seen that using the knowledge of noise variances can improve the performance drastically. Still the basic BAE has an undesirable asymptotic behavior as the probability of accurate consensus starts to decrease after a while. By incorporating statistical learning of $q_j(k)$, it can be seen that learning BAE can improve the performance drastically and can solve the asymptotic memoryless consensus problem as also denoted in [17]. It can also be seen that the simplification of the previous section, in which we first averaged over the fused

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data, results in a dynamics that tightly approximates the true case.
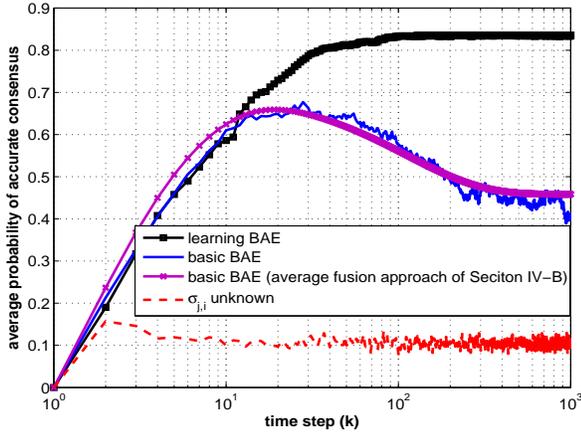

Fig. 4. Average probability of accurate consensus - $\forall i, j \neq i$ $\sigma_{j,i}^2 = 1$ and $E[r_{j,i}^2] = 1$

To see how well Theorem 3 approximates λ_1, Fig. 5 shows $\widetilde{\lambda}_{1,\text{approx}}$, average of λ_1 of matrix T and the second largest eigenvalue of $\overline{T}$ (all for the basic BAE case). It can be seen that the approximation of Section IV-B works well, i.e. the $\widetilde{\lambda}_{1,\text{approx}}$ curve approximates $\overline{\lambda}_1$ well. At higher average SNR, as average SNR decreases, λ_1 decreases. However, at lower average SNR, λ_1 starts increasing. This is due to the fact that the BAE approach weighs the received information based on link qualities. Therefore, at considerably low average SNR, the received information is almost ignored, which results in λ_1 approaching one. This makes the proposed approach more robust to the receiver noise. If the knowledge of $\sigma_{j,i}$ is not used, as is the case in Section III, λ_1 becomes a non-decreasing function (see Fig. 2).
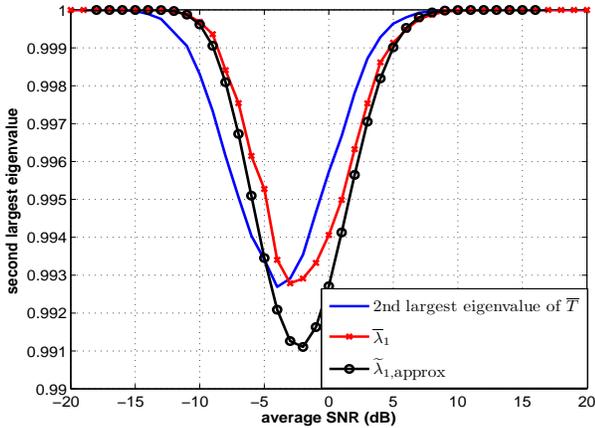

Fig. 5. Characterization of the 2nd largest eigenvalue in fading environments – basic BAE case

VI. Conclusions

In this paper we considered binary consensus over fading channels. We characterized the impact of fading on the performance by deriving a tight upper bound for the second largest eigenvalue of the probability transition matrix. We then proposed novel consensus-seeking strategies based on the Best

Affine Estimation (BAE) of network state. We characterized the dynamics of the network that deploys the proposed strategy. Furthermore, we derived a tight approximation for the second largest eigenvalue of the transition matrix. Our results showed the impact of fading on binary consensus. They also demonstrated the superior performance of the proposed BAE approach.

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