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Kalman filtering over wireless fading channels—How to handle packet drop[‡]

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SUMMARY

In this paper we consider estimation of dynamical systems over wireless fading communication channels using a Kalman filter. We show the impact of the stochastic communication noise on the estimation process. We furthermore show how noisy packets should be handled in the receiver. More specifically, we illustrate the impact of the availability of a cross-layer information path on the optimum receiver design. In the absence of a cross-layer information path, it was shown that packet drop should be designed to balance information loss and communication noise in order to optimize the performance. In the presence of a cross-layer path, we show that keeping all the packets will minimize the average estimation error covariance. We also derive the stability condition in the presence of noisy packets and show that it is independent of the shape of the communication noise variance or availability of a cross-layer information path.

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1. INTRODUCTION

Recently there has been considerable interest in cooperative sensing and control. Advances in technology have resulted in an abundance of cheap embedded units equipped with sensing, processing, communication and actuation capabilities. This has resulted in a wide range of sensor network applications [1, 2]. Such applications bring together different aspects of estimation, communication and control, necessitating non-traditional and cross-disciplinary approaches.

In such cooperative network applications, sensing and estimation/control may be assigned to different agents due to the heterogeneity of the agents. For instance, there may be cases where one agent does the sensing and sends its measurements to another agent, which will be in charge

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of estimation and will possibly produce a control command that will be sent back to the first agent. Communication plays a key role in the overall performance of cooperative estimation and control since both sensor measurements and control commands may be transmitted over wireless links. Among the unreliabilities introduced by digital wireless transmission, impact of quantization on estimation and control over a communication link has been studied extensively, where the minimum rate required for stabilization, quantizer design, practical stability and rate/convergence time trade-offs were addressed [3–7]. To address the inadequacy of the classical definition of capacity for networked control applications, anytime capacity was introduced and utilized for stabilization of linear systems [8]. Disturbance rejection and the corresponding required extra rate were considered in [9].

While characterizing the effect of quantization noise has received considerable attention, the impact of channel unreliability, such as fading, on control over a wireless link has not been studied extensively. Fading is the dominant performance degradation factor, making the impact of quantization negligible. For mobile applications, it can result in noisy reception. The receiver can then decide to either keep the received packet or drop it. The criteria for making this decision vary depending on the application. Data networks, for example, are not as sensitive to delays since the application is not real time. The receiver, therefore, can afford to drop erroneous packets and wait for retransmission. The amount of tolerable bit error rate is therefore set on the order of 10^{-8}, which is considerably low [10]. Voice applications such as cellular networks, on the other hand, are sensitive to delays. In every transmitted bit stream, there are key bits embedded for synchronization and other crucial tasks. If these bits get corrupted, the receiver drops the transmitted stream. However, once these bits are received accurately, the rest of the bit error rate is either corrected through channel coding or tolerated [11] since there is no time for retransmission. The level of tolerable bit error rate is therefore set considerably higher, on the order of 10^{-3} [10].

Estimation and control of dynamical systems over wireless links is an emerging application for which new communication design paradigms should be developed. Control applications are typically delay sensitive as we may be racing against the dynamics of the system under observation (such as a moving target). While these applications are, in this sense, more similar to voice applications, current literature on networked control systems assume a strategy for handling the received data that is more suitable for data networks by dropping any erroneous received data. Along this line, impact of packet drop on networked control applications has been studied. Micheli *et al.*, investigated impact of packet loss on estimation by considering random sampling of a dynamical system [12]. This is followed by the work of Sinopoli *et al.*, which derived bounds for the maximum tolerable probability of packet loss to maintain stability [13]. Liu *et al.*, extended that work to the case of two sensors [14]. The framework adopted in the current work in literature; however, does not allow for extracting as much information as possible from the received data and can therefore result in poor performance, excessive delays or waste of transmission power. A characterization of the tolerable level of bit error rate for networked control applications is therefore missing, which we will address. In this paper, we consider a mobile sensor that is observing a dynamical system. It transmits its observation over a wireless link to a remote node that is in charge of estimation using a Kalman. We are interested in studying the impact of unreliability introduced by multipath fading channels on estimation of the dynamical system over a wireless link. Instead of applying data network design principles to such delay-sensitive applications, we are interested in finding new design paradigms. Inspired by delay-sensitive voice applications, we take a fundamentally different approach and formulation, which will allow us to provide the right abstraction for modeling the impact of stochastic communication noise in these systems. We then

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features \ applications	Voice	Data	Networked control
application requirements	delay sensitive	not real-time	delay sensitive
when should received data be dropped?	if key bits are corrupted	if any bit error is perceived	addressed in this paper
specified tolerable bit error rate	10^{-3}	10^{-8}	

Figure 1. Different applications have different constraints and require different communication strategies.

explore the role of a cross-layer information path and its impact on the optimum design. Figure 1 shows a comparison of the communication requirements of different applications.

We conclude this section with an overview of the paper. In Section 2 we formulate the problem and provide the abstractions necessary to model noisy packets and packet drop mechanism. In Section 3, we consider the impact of a cross-layer path and the knowledge of the link quality on estimation over a wireless link, where we develop receiver design principles for optimizing the stability and performance. We prove that the receiver should keep all the packets to optimize the performance in the presence of a cross-layer information path. We furthermore show that if maximizing the stability range is the only concern, the receiver should keep all the packets independent of the quality of the link or availability of a cross-layer path. In Section 4 we discuss further extensions of our work. We conclude in Section 5. The paper complements our previous work [15], where estimation over a fading channel was considered in the absence of a cross-layer path.

2. SYSTEM MODEL

Consider a mobile sensor observing a system with the following linear dynamics:

$$\begin{aligned} x[k+1] &= Ax[k]+w[k] \\ y[k] &= Cx[k]+v[k] \end{aligned} \tag{1}$$

where $x[k]\in\mathbb{R}^N$ and $y[k]\in\mathbb{R}^M$ represent the state and observation, respectively. $w[k]\in\mathbb{R}^N$ and $v[k]\in\mathbb{R}^M$ represent zero-mean Gaussian process and observation noise vectors with covariances of $Q\succcurlyeq 0$ and $R\succ 0$, respectively. Table I contains a list of key variables used throughout this paper and their definitions. In this paper, we take $M=N$ and C invertible to focus on the impact of communication noise and leave the case where C is not invertible to the section on Further Extensions. We are interested in estimating unstable dynamics and therefore we consider cases where matrix A has at least one eigenvalue outside the unit circle.[§] The sensor then transmits its observation over a wireless fading channel to a remote node, which is in charge of estimation. This is shown in Figure 2. In practice, there can be several scenarios where there is an incentive to

[§]The concepts introduced in this paper are also applicable for a stable system.

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Table I. List of key variables.

x	state	Ξ	function relating σ_n^2 to Υ
A	dynamical system parameter	μ	probability of packet drop
w	process noise	G	function relating μ to Υ
Q	process noise covariance	Υ_{T}	Signal to Noise Ratio threshold
y	observation	μ_{ave}	average packet loss probability
C	observation parameter	$\sigma_{n,\mathrm{ave}}^2$	average communication noise variance
v	observation noise	$\sigma_{n,\mathrm{norm}}^2$	normalized average communication noise variance
R	observation noise covariance	N_{p}	number of symbols per packet
$\hat{y}$	received observation	Δ	quantization step size
$\hat{x}$	receiver estimate of state	h	baseband equivalent channel
P	estimation error covariance	$\rho_{\max}$	spectral radius of A
n	communication noise	ρ_i	ith eigenvalue of A
σ_n^2	communication noise variance	χ	probability density function of Υ
Υ	instantaneous received Signal to Noise Ratio	$\mathbb{E}(.)$	average operator
Υ_{ave}	average received Signal to Noise Ratio		

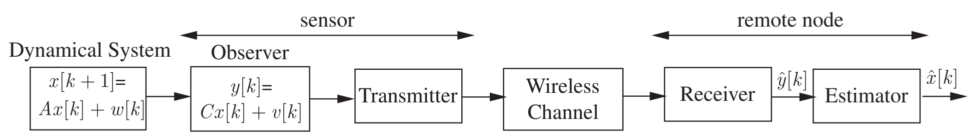

Figure 2. System model.

do the estimation at another node. For instance, in some applications, there may be an abundance of cheap nodes with low computation power and only a few powerful nodes. Therefore, it will be more efficient to have the nodes with low computation power sense and transmit their observations to a more powerful node that will produce an estimate based on the gathered information. In this paper, the term 'sensor' refers to the node that is in charge of observation whereas the term 'remote node' denotes the remote node that is in charge of estimation based on the received observation. Since estimation of dynamical systems over mobile links has not been extensively studied before, we keep our analysis general by considering mobile channels.

2.1. *Physical layer*: *a brief review of wireless communication* [*15–17*]

In this part we briefly summarize how to model the impact of a time-varying fading wireless communication channel on the observation. Readers are also referred to [15] for a similar overview. The sensor quantizes the observation, $y[k]$, transforms it into a packet of bits and transmits it over a fading channel. The remote node will receive a noisy version of the transmitted data due to bit flip. Let $\hat{y}[k]$ represent the received signal. $\hat{y}[k]$ is what the second node assumes the kth transmitted observation was. Let $n[k]$ represent the difference between the transmitted observation and the received one

$$n[k]=y[k]-\hat{y}[k] \tag{2}$$

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where $n[k] = n_c[k] + n_q[k]$. In this paper, we refer to $n[k]$ as communication noise. It consists of two parts, *link noise* (n_c): noise due to the quality of the communication link and *quantization noise* (n_q). For fading channels, the impact of link noise typically dominates the impact of quantization noise [17]. However, while estimation/control in the presence of quantization noise has received considerable attention, impact of fading on estimation and control has mainly remained unexplored. Therefore, in this paper we will mainly focus on the impact of the link noise.

2.1.1. Multipath fading channel. One of the major performance degradation factors of a mobile communication environment is multipath fading. 'Multipath' is a term used to describe multiple paths that a radio wave may follow between the transmitter and the receiver. Waves that are received in phase reinforce each other producing a stronger signal, while those that are received out of phase produce a weaker signal. Small changes in the transmission paths caused by movements of the receiver or transmitter can change the phase relationship of the two signals, introducing a rapidly time-varying fading channel. This is in addition to the distance-dependent attenuation factor. Signal attenuation and fading can result in bit error rate, i.e. some of the transmitted bits will be flipped. This will result in an erroneous reception of the transmitted packets, i.e. $n_c[k] \neq 0$. Correlation characteristics of fading channels depend on several parameters such as the transmission environment, speed of the mobile unit and frequency of operation. For instance, for a mobile node that moves at 25 mph and communicates at 1 GHz, channel will be uncorrelated after 13.5 ms using Jakes model [17]. Therefore, as long as the time interval between consecutive transmissions is larger than 13.5 ms in a networked control setup, channel can be considered uncorrelated from one transmission to the next. This time interval corresponds to observing a dynamical system at 74 Hz. As the mobile speed or frequency of communication increases, the channel gets uncorrelated even faster. Therefore, in this paper we take the channel to be uncorrelated from one transmission to the next, as it will be the case for several networked control applications, and leave the case of correlated channel to Section 4 on Further Extensions. There is even more incentive for using such a model. We have shown in [18] that even for a correlated channel, the link noise ($n_c[k]$), which is the parameter that reflects the impact of channel on the estimation process, becomes uncorrelated from one transmission to the next. In this paper the emphasis is on exploring the impact of the link noise, not the quantization noise, on estimation over fading channels. This is particularly important since link noise is the dominant factor compared with the quantization noise for fading channels.¶

2.1.2. Channel Signal-to-Noise Ratio. A fundamental parameter that characterizes the performance of a communication channel is the received Signal-to-Noise Ratio. Received Signal-to-Noise Ratio is defined as the ratio of the received signal power divided by the receiver thermal noise power. Let $\Upsilon[k]$ represent the instantaneous received Signal-to-Noise Ratio at kth transmission. Then we will have

$$\Upsilon[k] = \frac{|h[k]|^2 \sigma_s^2}{\sigma_T^2} \tag{3}$$

¶Quantization noise may or may not be correlated from one transmission to the next. However, its impact will be negligible for fading channels.

where $h[k]\in\mathbb{C}$ represents the time-varying fading coefficient of the baseband equivalent channel during the transmission of $x[k]$. If the channel changes during one transmission, $h[k]$ will represent the average of the channel during the transmission of the kth observation. $\sigma_{\mathrm{s}}^2=\mathbb{E}(|s|^2)$ is the transmitted signal power and $\sigma_{\mathrm{T}}^2=\mathbb{E}(|n_{\mathrm{thermal}}|^2)$ is the power of the receiver thermal noise. $\Upsilon[k]$ determines how well the transmitted bits of the kth transmission can be retrieved. As the sensor moves, the remote node will experience different channels and therefore different received Signal-to-Noise Ratios. In a given area, $\Upsilon[k]$ can be considered a stationary stochastic process with Υ_{ave} representing its average. The distribution of $\Upsilon[k]$ is a function of the transmission environment and the level of mobility of the sensor. In this paper we do not make any assumption on the probability distribution of Υ. Only when we want to provide an example, we will take Υ to be exponentially distributed, which is a common model for outdoor fading channels with no line-of-sight paths. Under the assumption that channel becomes uncorrelated from one transmission to the next, $\Upsilon[k]$ becomes uncorrelated from one transmission to the next as well.

2.1.3. Communication noise variance. Let $\sigma_n^2[k]$ represent the variance of $n[k]$ at kth transmission. $\sigma_n^2[k]$ is a function of $\Upsilon[k]$

$$\sigma_n^2[k]=\mathbb{E}(n^2[k]|\Upsilon[k])=\Xi(\Upsilon[k]) \tag{4}$$

where Ξ is a non-increasing function that depends on the transmitter and receiver design principles, such as modulation and coding, as well as the transmission environment. To keep our analysis general, in this paper we do not make any assumption on Ξ. Figure 3(a) shows one example of the

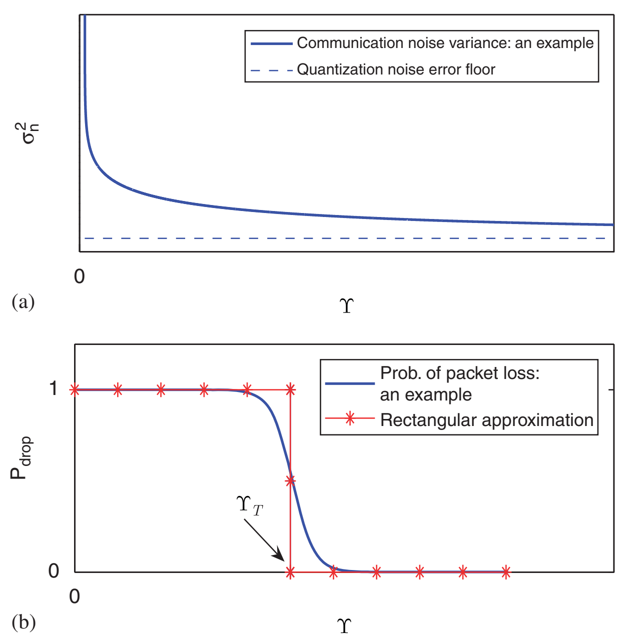

Figure 3. Examples of the communication noise variance and probability of packet loss as functions of Signal-to-Noise Ratio.

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communication noise variance, σ_n^2. In this example, as Υ goes to ∞, σ_n^2 reaches the quantization noise error floor. It should be noted that the communication noise has a time-varying variance.

2.1.4. Packet drop probability. Depending on the receiver design, there can be a packet drop mechanism deployed in the receiver. Let $\mu[k]$ represent the probability that the receiver drops the kth packet. $\mu[k]$ can also be represented as a function of $\Upsilon[k]:\mu[k]=G(\Upsilon[k])$, where function G is a non-increasing function. Figure 3(b) shows a sample μ as a function of Υ (solid line). It should be noted that the receiver may not decide on dropping packets directly based on the instantaneous received Signal-to-Noise Ratio. However, since any other utilized measure is a function of $\Upsilon[k]$, we find it useful to express μ as a function of this fundamental parameter. G is also a function of the receiver and transmitter technologies. Functions Ξ and G provide the abstraction necessary to model the impact of the physical layer in the higher application layer. Experimental results have shown G to be well approximated as follows [19]:

$$\mu[k]=\begin{cases}0, & \Upsilon[k]\geqslant\Upsilon_{\mathrm{T}}\\ 1 & \text{else}\end{cases} \tag{5}$$

This means that the receiver keeps those packets with the received instantaneous Signal-to-Noise Ratio above a designated threshold Υ_{T}. This approximation is shown in Figure 3(b) (star line) and is the model we will use in this paper.

2.2. Estimation at the remote node

The remote node estimates the state based on the received observation using a Kalman filter [20]. Let $\hat{x}[k]$ denote the estimate of $x[k]$ at the remote node. Then $P[k]$ represents the corresponding estimation error covariance matrix given $\Upsilon[k-1], \Upsilon[k-2], \ldots, \Upsilon[0]$:

$$P[k]=\mathbb{E}[(x[k]-\hat{x}[k])(x[k]-\hat{x}[k])^{\mathrm{T}}]_{|\Upsilon[k-1],\Upsilon[k-2],\ldots,\Upsilon[0]} \tag{6}$$

This is different from the traditional form of Kalman filtering since $P[k]$ is a function of channel statistics through $\Upsilon[k-1], \Upsilon[k-2], \ldots, \Upsilon[0]$. For instance, to obtain $\mathbb{E}(P[k])$, $P[k]$ should be averaged over the joint distribution of $\Upsilon[k-1], \Upsilon[k-2], \ldots, \Upsilon[0]$.

2.3. Cross-layer information path

A cross-layer information path refers to a path from the physical layer to the application layer that carries information on the quality of the link (Signal-to-Noise Ratio or communication noise variance). In other words, the physical layer can let the application layer know, using a cross-layer path, how much it trusts the accuracy of each received packet. In [15], optimum handling of packet drop was considered in the absence of knowledge of link quality. In this paper we will consider scenarios where such a path is available at the receiver supported by the architecture. We show how the knowledge of link quality can impact estimation of a dynamical system over a wireless link.

2.4. Scenario 1: ideal communication noise [13]

Current work in literature mainly applies data network design principles to networked control applications by assuming that the receiver drops packets that contain any amount of error. Then those packets that are kept in the receiver are considered noise free. We refer to this assumption on

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the communication noise as 'ideal noise' throughout the paper. Similarly we refer to this design strategy, which applies data-network type protocols, as 'scenario 1'. Such an assumption translates to the following recursion for the estimation error covariance:

$$P[k+1]=AP[k]A^{\mathrm{T}}-AP[k]C^{\mathrm{T}}(R+CP[k]C^{\mathrm{T}}+S_1(\Upsilon[k]))^{-1}CP[k]A^{\mathrm{T}}+Q \tag{7}$$

where

$$S_1[k]=\begin{cases}0, & \Upsilon[k]\geqslant\Upsilon_{\mathrm{T}}\\ \infty & \text{otherwise}\end{cases} \tag{8}$$

For a fixed probability of packet drop, authors in [13] found the following stability condition for the system model introduced earlier in this section:

$$\mu_{\text{scenario1}}<\rho_{\max}^{-2} \tag{9}$$

where $\rho_{\max}$ represents spectral radius of matrix A.

3. ESTIMATION OF A DYNAMICAL SYSTEM OVER A WIRELESS LINK

In Section 2.4 we saw that the current work in the literature assumed a receiver that drops those packets that contain any amount of error. For non real-time applications like data networks, the receiver can afford to drop erroneous packets and wait for retransmission. Considering packets to be noise free once they are kept in the receiver, therefore, is a reasonable model for these applications. However, estimation of a rapidly changing dynamical system is delay sensitive. Dropping erroneous packets can result in loss of information, can reduce the useful transmission rate and can lead to instability. Therefore, the receiver cannot afford to wait for receiving noise-free packets.

In this section, we will consider the impact of stochastic communication noise and will derive receiver design theories for real-time estimation over wireless links. To keep the analysis general, we will not make any assumption on the communication noise variance or Signal-to-Noise Ratio distribution. Instead of finding a globally optimum design, we will find optimum designs given constraints and limitations of a receiver. More specifically, we consider the following two cases:

(1) *Scenario 2*: The receiver cannot provide a cross-layer path.
(2) *Scenario 3*: The receiver is equipped with a cross-layer path that can constantly update the application layer with information on link quality.

Scenario 2 was considered in [15]. In this part we first briefly summarize the results of [15], which will provide a benchmark and will be used in the mathematical derivations of scenario 3. Our main focus will then be scenario 3. *Scenario 3* basically refers to the case where the information on the link quality is available and can be utilized in the Kalman filter. We will see the impact of this information on the optimum handling of the received packets.

3.1. Scenario 2: case of no cross-layer path [*15*]

In order to provide a base for comparison, in this part we briefly summarize the optimum design for the case where the receiver does not support a constant cross-layer path. Then the application layer (i.e. the Kalman filter) does not have any knowledge of the quality of the communication link.

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The details of the derivations of this part can be found in [15]. To ease mathematical derivation of this scenario, we assume that the observation noise is negligible compared with the communication noise.‖ The estimation using a Kalman filter will then be as follows:

$$\hat{x}[k+1]=\begin{cases} A\hat{x}[k] & \text{if } k\text{th packet is dropped} \\ AC^{-1}y[k] & \text{if } k\text{th packet is kept} \end{cases} \tag{10}$$

The estimation error will be as follows using Equation (1):

$$x[k+1]-\hat{x}[k+1]=\begin{cases} A(x[k]-\hat{x}[k])+w[k] & \text{if } k\text{th packet is dropped} \\ w[k]-AC^{-1}v[k] & \text{if } k\text{th packet is kept} \end{cases} \tag{11}$$

This will result in the following recursion for the estimation error covariance:

$$P[k+1]=AP[k]A^{\mathrm{T}}+Q-\frac{AP[k]A^{\mathrm{T}}-\sigma_n^2(\Upsilon[k])A(C^{\mathrm{T}}C)^{-1}A^{\mathrm{T}}}{S_2[k]} \tag{12}$$

where σ_n^2 is the communication noise variance as defined in Section 2 and

$$S_2[k]=\begin{cases} 1, & \Upsilon[k]\geqslant\Upsilon_{\mathrm{T}} \\ \infty & \text{otherwise} \end{cases} \tag{13}$$

As the mobile node moves in a given area, it will experience different Signal-to-Noise Ratios. Averaging Equation (12) over $\Upsilon[k], \Upsilon[k-1], \ldots, \Upsilon[0]$ will result in the following recursion for average estimation error covariance (note that the channel is taken uncorrelated from one transmission to the next as discussed in Section 2):

$$\mathbb{E}(P[k+1])=\mu_{\mathrm{ave}}(\Upsilon_{\mathrm{T}})A\mathbb{E}(P[k])A^{\mathrm{T}}+Q+\sigma^2_{n,\mathrm{ave}}(\Upsilon_{\mathrm{T}})A(C^{\mathrm{T}}C)^{-1}A^{\mathrm{T}} \tag{14}$$

μ_{ave} and $\sigma^2_{n,\mathrm{ave}}$ represent average probability of packet loss (spatial averaging) and average communication noise variance that entered the estimation process, respectively:

$$\mu_{\mathrm{ave}}(\Upsilon_{\mathrm{T}})=\mathbb{E}(\mu)=\int_0^{\Upsilon_{\mathrm{T}}}\chi(\Upsilon)\,\mathrm{d}\Upsilon \tag{15}$$

and

$$\sigma^2_{n,\mathrm{ave}}(\Upsilon_{\mathrm{T}})=\int_{\Upsilon_{\mathrm{T}}}^{\infty}\sigma_n^2(\Upsilon)\chi(\Upsilon)\,\mathrm{d}\Upsilon \tag{16}$$

where χ represents probability density function of Υ.

Lemma 1 (*see Kailath et al.* [*20*])
Consider the following Lyapunov equation with Θ Hermitian

$$\Sigma=\Pi\Sigma\Pi^{\mathrm{T}}+\Theta \tag{17}$$

‖The analysis can be similarly carried out under the condition that the knowledge of observation noise covariance, R, is not available in the estimator. Then $\sigma_n^2 I_N$ should be replaced by $\sigma_n^2 I_N+R$ throughout the analysis.

The following holds:

(a) If Π is a stable matrix (spectral radius less that one), Σ will be unique and Hermitian and can be expressed as follows:

$$\Sigma = \sum_{i=0}^{\infty} \Pi^i \Theta (\Pi^{\mathrm{T}})^i \tag{18}$$

(b) if $\{\Pi, \Theta^{1/2}\}$ is controllable and $\Theta \succcurlyeq 0$, then Σ will be Hermitian, unique and positive definite iff Π is stable.

3.1.1. Stability.

Definition 1
We consider the estimation process stable as long as average estimation error covariance stays bounded.

Using Lemma 1(b), it can be easily seen from Equation (14) that the stability condition will be as follows:

$$\mu_{\text{ave,scenario 2}} < \rho_{\max}^{-2} \tag{19}$$

where $\rho_{\max}$ represents the spectral radius of matrix A. Intuitively, the instability of the estimation process can also be thought of in terms of the communication link. An unstable estimation process means that the rate of the changes of the dynamical system, or equivalently the incoming information rate to the communication channel, is higher than the outgoing rate of the channel.

3.1.2. Optimum performance. Figure 4 shows examples of average probability of packet drop and average communication noise power that entered the estimation process. We can see that increasing Υ_{T}, on one hand, will increase the average probability of drop and therefore the information loss rate. On the other hand, it will decrease the amount of noise that enters the estimation process. Therefore, there should be an optimum Υ_{T} (optimum way of dropping packets) that will minimize the asymptotic average estimation error covariance for this case. If Υ_{T} is too low, the receiver will keep most of the packets but the estimation will be too noisy. On the other hand, if Υ_{T} is too high, the receiver will be strict about the quality of the packets that it will keep. This reduces the amount of communication noise that enters the estimation process but will result in high packet loss rate and therefore information loss rate. Then the optimum Υ_{T} will be the one that provides a balance between information loss and communication noise.

The asymptotic average estimation error covariance will be as follows as long as the stability condition of Equation (19) holds:

$$\mathbb{E}(P[\infty]) = \mu_{\text{ave}}(\Upsilon_{\mathrm{T}}) A \mathbb{E}(P[\infty]) A^{\mathrm{T}} + \sigma^2_{n,\text{ave}}(\Upsilon_{\mathrm{T}}) A (C^{\mathrm{T}} C)^{-1} A^{\mathrm{T}} + Q \quad \text{for } \mu_{\text{ave}}(\Upsilon_{\mathrm{T}}) < \rho_{\max}^{-2} \tag{20}$$

Let $\Upsilon_{\mathrm{T}_1,\text{opt}}$ represent the optimum way of dropping packets, which will minimize the spectral norm of the asymptotic average estimation error covariance matrix

$$\Upsilon_{\mathrm{T}_1,\text{opt}} = \arg\min \|\mathbb{E}(P[\infty, \Upsilon_{\mathrm{T}}])\| \tag{21}$$

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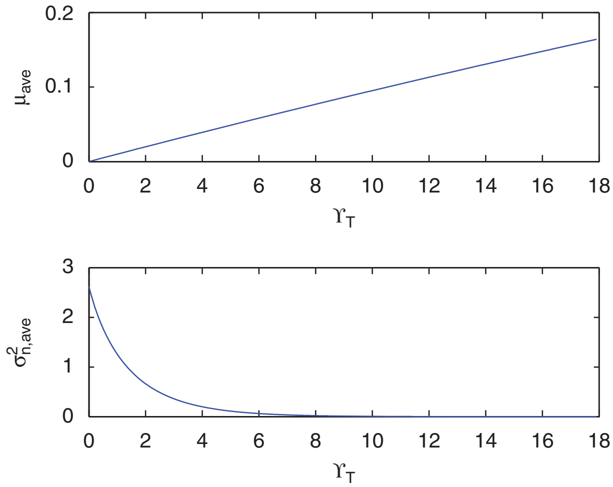

Figure 4. Examples of the average probability of packet drop and average communication noise variance. As Υ_T increases, average probability of packet drop increases whereas average communication noise power that enters the estimation process decreases.

Let $\Upsilon_{T_2,\text{opt}}$ represent the optimum way of dropping packets, which will minimize the determinant of the asymptotic average estimation error covariance:

$$\Upsilon_{T_2,\text{opt}} = \arg\min \det \mathbb{E}(P[\infty, \Upsilon_T]) \tag{22}$$

Theorem 1 (*Balance of information loss and communication noise (see* [15] *for the proof)*)
Consider the system model of Figure 2, with $C = \varsigma I_N$, $Q = qI_N$ and $A = A_s$, where A_s is a symmetric matrix, i.e. $A_s = A_s^T$ and I_N represents an $N \times N$ identity matrix. Consider a receiver that is equipped with a packet drop mechanism described by Equation (5) and does not support a cross-layer path. Then $\Upsilon_{T_1,\text{opt}}$ will be as follows:

$$\Upsilon_{T_1,\text{opt}} = \begin{cases} \Upsilon^*_{T_1}, & \Upsilon^*_{T_1} \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

where $\Upsilon^*_{T_1}$ is the unique solution to the following equation:

$$\underbrace{\mu_{\text{ave}}(\Upsilon^*_{T_1})}_{\text{information loss}} + \underbrace{\sigma^2_{n,\text{norm}}(\Upsilon^*_{T_1})}_{\text{communication noise}} + \frac{\varsigma^2 q}{\rho^2_{\max}\sigma^2_n(\Upsilon = \Upsilon^*_{T_1})} = \rho^{-2}_{\max} \tag{24}$$

where $\sigma^2_{n,\text{norm}}$ refers to the normalized average communication noise variance

$$\sigma^2_{n,\text{norm}}(\Upsilon^*_{T_1}) = \frac{\sigma^2_{n,\text{ave}}(\Upsilon^*_{T_1})}{\sigma^2_n(\Upsilon = \Upsilon^*_{T_1})}$$

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and $\Upsilon_{\mathrm{T}_2,\mathrm{opt}}$ will be as follows:

$$\Upsilon_{\mathrm{T}_2,\mathrm{opt}} = \begin{cases} \Upsilon^*_{\mathrm{T}_2}, & \Upsilon^*_{\mathrm{T}_2} \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

where $\Upsilon^*_{\mathrm{T}_2}$ is the unique solution to the following equation:

$$\sum_{i=1}^{N} \frac{\rho_i^2}{1-\rho_i^2 \mu_{\mathrm{ave}}(\Upsilon^*_{\mathrm{T}_2})} = \sum_{i=1}^{N} \frac{1}{\sigma^2_{n,\mathrm{norm}}(\Upsilon^*_{\mathrm{T}_2}) + \dfrac{q\varsigma^2}{\sigma_n^2(\Upsilon=\Upsilon^*_{\mathrm{T}_2})\rho_i^2}} \tag{26}$$

where $\rho_1, \rho_2, \ldots, \rho_N$ represent eigenvalues of matrix A, where $|\rho_1| \geqslant |\rho_2| \geqslant \cdots \geqslant |\rho_N|$ and $\rho_{\max} = |\rho_1|$.

Equation (24) (and Equation (26)) may not have a positive solution if process noise is the dominant noise compared with the communication noise. In such cases, the receiver should keep all the packets as communication noise is not the bottleneck [15].

Theorem 1 confirms that dropping all the erroneous packets will not minimize the estimation error covariance and that the optimum receiver would allow some amount of communication noise in the estimation process in order to avoid high information loss rate.

To see the impact of operating at the optimum Υ_T, Figure 5 shows $\|\mathbb{E}(P[\infty])\|$ as a function of Υ_T and for different levels of average Signal-to-Noise Ratio, Υ_{ave}. For this example, Signal-to-Noise Ratio, Υ, is taken to have an exponential distribution and the communication noise variance is taken as follows: $\sigma_n^2(\Upsilon) = \alpha + \delta \times \Omega(\sqrt{\Upsilon})$, where $\Omega(d) = 1/\sqrt{2\pi} \int_d^\infty \mathrm{e}^{-t^2/2}\,\mathrm{d}t$ for an arbitrary d.

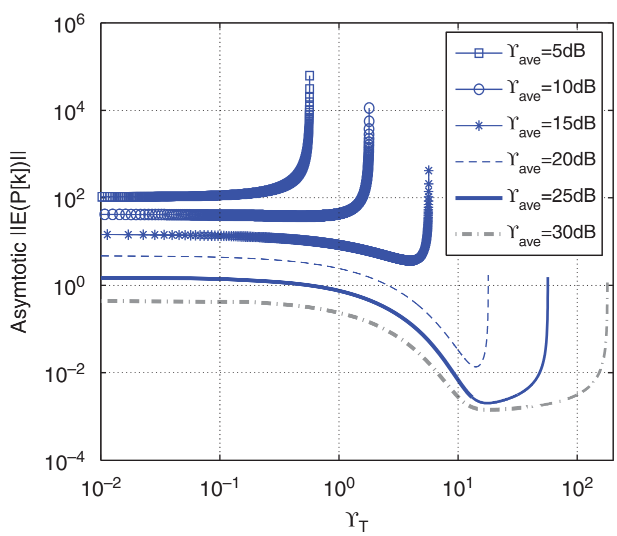

Figure 5. Scenario 2: minimums of the curves indicating optimum packet drop mechanism in the absence of a cross-layer information path.

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This is the variance of the communication noise for a binary modulation system that utilizes gray coding [18]. The following parameters are chosen for this example:

$$A=\begin{pmatrix} 2 & 0.3 & 0.45 \\ 0.4 & 0.2 & 0.5 \\ 1.5 & 0.6 & 0.34 \end{pmatrix}, \quad Q=qI_3, \quad C=\varsigma I_3, \quad q=0.001, \quad \varsigma=2, \quad N_p=10 \text{ and } \Delta=0.0391$$

This results in $\alpha=1.27\times 10^{-4}$ and $\delta=533.3$. It can be seen from Figure 5 that if Υ_{T} is too low, estimation performance degrades due to excessive communication noise. On the other hand, having Υ_{T} too high will result in loss of information, which will degrade the performance. The optimum Υ_{T} (as predicted by Theorem 1) provides the necessary balance between loss of information and communication noise, reaching the minimums of the estimation error curves. It can be seen that dropping packets properly can improve the performance considerably. As Υ_{T} increases, the estimation will approach the instability regions, predicted by Equation (19) due to high information loss. The existing approaches for Kalman filtering over wireless links [12, 13] assume that the receiver drops any erroneous packets. This implies a considerably high Υ_{T} in Figure 5, which can increase the probability of instability and poor performance (as the figure indicates). By keeping some of the erroneous packets through the optimization of Υ_{T}, we can improve the performance of the estimation over wireless links considerably.

3.2. Scenario 3: *impact of a cross-layer information path*

In this section we show the impact of the availability of the knowledge of the link quality on the optimum packet drop design. Consider a scenario where the receiver can support a constant cross-layer path. This means that the Kalman filter will have access to and can utilize the knowledge of the communication noise variance. We will have the following recursion for the estimation error covariance:

$$P[k+1]=AP[k]A^{\mathrm{T}}-AP[k]C^{\mathrm{T}}(\sigma_z^2(\Upsilon[k])+CP[k]C^{\mathrm{T}})^{-1}CP[k]A^{\mathrm{T}}+Q \tag{27}$$

where

$$\sigma_z^2(\Upsilon[k])=\begin{cases} \sigma_n^2(\Upsilon[k])I_N+R, & \Upsilon[k]\geqslant\Upsilon_{\mathrm{T}} \\ \infty & \text{otherwise} \end{cases} \tag{28}$$

3.2.1. Stability. Matrix convexity (*see Boyd and Vandenberghe* [21]): Let f represent a symmetric matrix-valued function, $f:\mathbb{R}^{N\times N}\to\mathbb{R}^{M\times M}$. Function f is convex with respect to matrix inequality if

$$f(\theta\Pi_1+(1-\theta)\Pi_2)\preccurlyeq\theta f(\Pi_1)+(1-\theta)f(\Pi_2) \tag{29}$$

for arbitrary $\Pi_1\in$ and $\Pi_2\in\mathbb{R}^{N\times N}$ and $\theta\in[0,1]$.

Lemma 2

Consider Π_1, Π_2, Π_3 and $\Pi_4\in\mathbb{R}^{N\times N}$. The following can be confirmed:

If Π_1 and Π_2 are positive definite, then $\Pi_1\preccurlyeq\Pi_2$ if and only if $\Pi_2^{-1}\preccurlyeq\Pi_1^{-1}$ (see [22]).

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Lemma 3

Let Π_1 and $\Pi_2 \in \mathbb{R}^{N\times N}$ represent symmetric positive-definite matrices:

(a) Let $f:\mathbb{R}^{N\times N}\rightarrow\mathbb{R}^{N\times N}$ represent inverse of Π: $f(\Pi)=\Pi^{-1}$. f is convex with respect to matrix inequality (see [23]).
(b) if $f:\mathbb{R}^{N\times N}\rightarrow\mathbb{R}^{M\times M}$ is convex as a function of Π_1, it can be easily confirmed that $f(\Pi_1+\Pi_2)$ is convex for a constant $\Pi_2\in\mathbb{R}^{N\times N}$.
(c) if $f:\mathbb{R}^{N\times N}\rightarrow\mathbb{R}^{N\times N}$ is convex as a function of Π_1, it can be easily shown that $\Psi^{\mathrm{T}} f(\Pi_1)\Psi$ is convex for an arbitrary matrix $\Psi\in\mathbb{R}^{N\times M}$.

Lemma 4

Let Π_1 and $\Pi_2\in\mathbb{R}^{N\times N}$ represent symmetric positive-definite matrices. Let $f:\mathbb{R}^{N\times N}\rightarrow\mathbb{R}^{N\times N}$ represent the following function $f(\Pi_1)=\Pi_1(\Pi_2+\Pi_1)^{-1}\Pi_1$. f is a convex function of Π_1.

Proof

$$\begin{aligned} f(\Pi_1) &= \Pi_1(\Pi_2+\Pi_1)^{-1}\Pi_1 \\ &= [I_N-\Pi_2(\Pi_2+\Pi_1)^{-1}]\Pi_1 \\ &= \Pi_1-\Pi_2+\Pi_2(\Pi_2+\Pi_1)^{-1}\Pi_2 \end{aligned} \tag{30}$$

Using Lemma 3, it can be easily seen that f is a convex function of Π_1. □

The following two lemmas relate stability region of scenario 3 to those of scenarios 1 and 2.

Lemma 5

The stability region of scenario 1 includes that of scenario 3:

$$\mu_{\text{ave,c,scenario 1}} \geqslant \mu_{\text{ave,c,scenario 3}} \tag{31}$$

where $\mu_{\text{ave,c}}$ represents the maximum tolerable average probability of packet loss for stability.

Proof

Consider a special case of scenario 1, where $R=0$. Let scenarios 1 and 3 have the same packet drop threshold. Let $P_1[k]$ and $P_3[k]$ represent the estimation error covariance matrices of scenario 1 with $R=0$ and scenario 3, respectively. Using Equation (7) with $R=0$, we will have

$$\mathbb{E}(P_1[k+1]) = \mu_{\text{ave}} A\mathbb{E}(P_1[k])A^{\mathrm{T}}+Q \tag{32}$$

Consider $S_1[k]$ as it was defined in Equation (8). We will have

$$\sigma_z^2[k]\succcurlyeq S_1[k] \Rightarrow \sigma_z^2[k]+CP_3[k]C^{\mathrm{T}}\succcurlyeq S_1[k]+CP_3[k]C^{\mathrm{T}} \tag{33}$$

Using Lemma 2,

$$AP_3[k]C^{\mathrm{T}}(\sigma_z^2[k]+CP_3[k]C^{\mathrm{T}})^{-1}CP_3[k]A^{\mathrm{T}}\preccurlyeq AP_3[k]C^{\mathrm{T}}(S_1[k]+CP_3[k]C^{\mathrm{T}})^{-1}CP_3[k]A^{\mathrm{T}} \tag{34}$$

Therefore,

$$\begin{aligned} &P_3[k+1]\succcurlyeq AP_3[k]A^{\mathrm{T}}-AP_3[k]C^{\mathrm{T}}(S_1[k]+CP_3[k]C^{\mathrm{T}})^{-1}CP_3[k]A^{\mathrm{T}}+Q \\ &\Rightarrow \mathbb{E}(P_3[k+1])\succcurlyeq \mu_{\text{ave}} A\mathbb{E}(P_3[k])A^{\mathrm{T}}+Q \end{aligned} \tag{35}$$

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which results in the following:

$$\text{if } \mathbb{E}(P_3[k]) \succcurlyeq \mathbb{E}(P_1[k]) \Rightarrow \mathbb{E}(P_3[k+1]) \succcurlyeq \mathbb{E}(P_1[k+1]) \tag{36}$$

Therefore, the stability region of scenario 1 includes that of scenario 3. □

Lemma 6
The stability region of scenario 3 includes that of scenario 2:

$$\mu_{\text{ave,c,scenario 3}} \geqslant \mu_{\text{ave,c,scenario 2}} \tag{37}$$

Proof
Let $P_2[k]$ represent the estimation error covariance of scenario 2 for an $R \neq 0$. Then no knowledge of R is available in the estimator for scenario 2 (see footnote of Section 3, part A). Using Equation (12), $\mathbb{E}(P_2[k+1])$ will be as follows:

$$\mathbb{E}(P_2[k+1]) = \mu_{\text{ave}} A \mathbb{E}(P_2[k]) A^{\mathrm{T}} + Q + A C^{-1} \Sigma C^{-1\mathrm{T}} A^{\mathrm{T}} \tag{38}$$

where $\Sigma = \sigma^2_{n,\text{ave}} I_N + (1-\mu_{\text{ave}})R$. Let $P_3[k]$ represent the estimation error covariance of scenario 3, as indicated by Equation (27). We will have

$$\begin{aligned}\mathbb{E}(P_3[k+1]|P_3[k]) = {} & (1-\mu_{\text{ave}})\mathbb{E}(P_3[k+1]|P_3[k], \Upsilon[k] > \Upsilon_{\mathrm{T}}) \\ & + \mu_{\text{ave}} \mathbb{E}(P_3[k+1]|P_3[k], \Upsilon[k] \leqslant \Upsilon_{\mathrm{T}})\end{aligned} \tag{39}$$

Using Lemma 3, it can be easily confirmed that $P_3[k+1]$ is a concave function of $\sigma^2_z[k]$ in Equation (27). Therefore, using conditional Jensen's inequality, we will have

$$\begin{aligned}\mathbb{E}(P_3[k+1]|P_3[k], \Upsilon[k] > \Upsilon_{\mathrm{T}}) \preccurlyeq {} & A P_3[k] A^{\mathrm{T}} + Q - A P_3[k] C^{\mathrm{T}} (\mathbb{E}(\sigma^2_z[k]|\Upsilon[k] > \Upsilon_{\mathrm{T}}) \\ & + C P_3[k] C^{\mathrm{T}})^{-1} C P_3[k] A^{\mathrm{T}}\end{aligned} \tag{40}$$

Therefore,

$$\mathbb{E}(P_3[k+1]|P_3[k]) \preccurlyeq A P_3[k] A^{\mathrm{T}} + Q - (1-\mu_{\text{ave}}) f(P_3[k]) \tag{41}$$

where $f : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ is as follows: $f(P_3[k]) = A P_3[k] C^{\mathrm{T}} (\mathbb{E}(\sigma^2_z[k]|\Upsilon[k] > \Upsilon_{\mathrm{T}}) + C P_3[k] C^{\mathrm{T}})^{-1}$ $C P_3[k] A^{\mathrm{T}}$. We will have

$$f(P_3[k]) = A P_3[k] (P_3[k] + C^{-1} \mathbb{E}(\sigma^2_z[k]|\Upsilon[k] > \Upsilon_{\mathrm{T}}) C^{-1\mathrm{T}})^{-1} P_3[k] A^{\mathrm{T}} \tag{42}$$

It can be seen, using Lemma 4, that f is a convex function of $P_3[k]$. Therefore, by applying Jensen's inequality,

$$\begin{aligned}\mathbb{E}(P_3[k+1]) \preccurlyeq {} & A \mathbb{E}(P_3[k]) A^{\mathrm{T}} + Q - (1-\mu_{\text{ave}}) A \mathbb{E}(P_3[k]) C^{\mathrm{T}} [\mathbb{E}(\sigma^2_z[k]|\Upsilon[k] > \Upsilon_{\mathrm{T}}) \\ & + C \mathbb{E}(P_3[k]) C^{\mathrm{T}}]^{-1} C \mathbb{E}(P_3[k]) A^{\mathrm{T}}\end{aligned} \tag{43}$$

Noting that $\mathbb{E}(\sigma^2_z(\Upsilon[k])|\Upsilon[k] > \Upsilon_{\mathrm{T}}) = \Sigma/(1-\mu_{\text{ave}})$, it can be confirmed, after a few lines of derivations using Equations (38) and (43), that

$$\text{if } \mathbb{E}(P_3[k]) \preccurlyeq \mathbb{E}(P_2[k]) \Rightarrow \mathbb{E}(P_3[k+1]) \preccurlyeq \mathbb{E}(P_2[k+1]) \tag{44}$$

Therefore, the stability region of scenario 3 includes that of scenario 2. □

Theorem 2
Consider the system model of Figure 2. Consider a receiver that is equipped with a packet drop mechanism described by Equation (5) but can support a cross-layer path. Then the estimation will be stable as long as the following holds:

$$\mu_{\text{ave,scenario } 3} < \rho_{\max}^{-2} \tag{45}$$

Proof
Lemmas 5 and 6 showed that

$$\mu_{\text{ave,c,scenario } 2} \leqslant \mu_{\text{ave,c,scenario } 3} \leqslant \mu_{\text{ave,c,scenario } 1} \tag{46}$$

Noting that scenarios 1 and 2 have the same stability regions proves Theorem 2. □

Theorem 2 shows that availability of a cross-layer path does not impact the stability region. This suggests, similar to scenario 2, that keeping all the packets will maximize the stability range.

3.2.2. Optimum performance.

Theorem 3
Consider the system model of Figure 2. Consider a receiver that is equipped with a packet drop mechanism described by Equation (5) but can support a cross-layer path. Keeping all the packets, i.e. $\Upsilon_{\text{T}} = 0$, will minimize the average estimation error covariance.

Proof
Let $P[k]$ represent the estimation error covariance of a receiver that is equipped with a cross-layer path, as indicated by Equation (27). $P[k]$ can be written as follows using the same formulation utilized in the derivation of Equation (30):

$$\begin{aligned} P[k+1] &= AP[k]A^{\text{T}} + Q - AP[k](P[k] + C^{-1}\sigma_z^2[k]C^{-1\text{T}})^{-1}P[k]A^{\text{T}} \\ &= Q + A\Pi[k]A^{\text{T}} - A\Pi[k](P[k] + \Pi[k])^{-1}\Pi[k]A^{\text{T}} \end{aligned} \tag{47}$$

where $\Pi[k] = C^{-1}\sigma_z^2[k]C^{-1\text{T}}$. Let P_1 and P_2 represent estimation error covariance matrices of two estimators using Υ_{T1} and Υ_{T2}, where $\Upsilon_{\text{T1}} < \Upsilon_{\text{T2}}$. Then for any received Signal-to-Noise Ratio at time step k, $\Upsilon[k]$, we will have

$$\sigma_{z,1}^2(\Upsilon[k]) \preccurlyeq \sigma_{z,2}^2(\Upsilon[k]) \Rightarrow \Pi_1[k] \preccurlyeq \Pi_2[k] \tag{48}$$

where $\sigma_{z,1}^2$ and $\sigma_{z,2}^2$ are as defined in Equation (28) for these two estimators. Assume that $P_1[0] = P_2[0]$. It is easy to see that $P_1[1] \preccurlyeq P_2[1]$ for any $\Upsilon[0]$. Using Lemma 2, we will have the following for any given $\Upsilon[0], \Upsilon[1], \ldots, \Upsilon[k]$:

$$\begin{aligned} \text{if } P_1[k] &\preccurlyeq P_2[k] \Rightarrow -(P_1[k] + \Pi_1[k])^{-1} \preccurlyeq -(P_2[k] + \Pi_2[k])^{-1} \Rightarrow -\Pi_1[k](P_1[k] + \Pi_1[k])^{-1}\Pi_1[k] \\ &\preccurlyeq -\Pi_2[k](P_2[k] + \Pi_2[k])^{-1}\Pi_2[k] \Rightarrow A\Pi_1[k]A^{\text{T}} - A\Pi_1[k](P_1[k] + \Pi_1[k])^{-1}\Pi_1[k]A^{\text{T}} \\ &\preccurlyeq A\Pi_2[k]A^{\text{T}} - A\Pi_2[k](P_2[k] + \Pi_2[k])^{-1}\Pi_2[k]A^{\text{T}} \Rightarrow P_1[k+1] \\ &\preccurlyeq P_2[k+1] \end{aligned} \tag{49}$$

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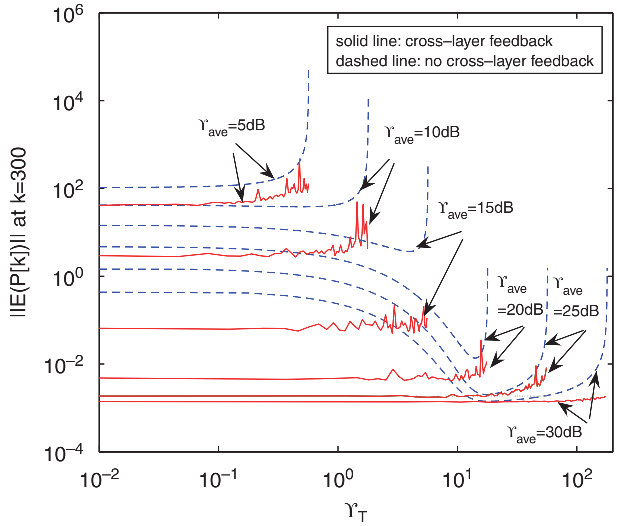

Figure 6. Effect of a cross-layer information path: compare scenarios 2 and 3.

This shows that using a lower threshold will result in a lower estimation error covariance. Therefore, keeping all the packets, i.e. $\Upsilon_T = 0$, will minimize the estimation error covariance (and its average over the distribution of Υ). □

We can see that keeping all the packets not only prevents instability but also minimizes estimation error covariance in the presence of a cross-layer path.

To see the impact of a cross-layer path, Figure 6 shows spectral norm of the average estimation error covariance after 300 time steps for the system parameters of Figure 5 and for both scenarios 2 and 3. By comparing the corresponding curves for these cases, it can be seen that a cross-layer path can improve the performance considerably even when compared with operating at the optimum Υ_T of scenario 2. Furthermore, it can be seen that keeping more packets will reduce the norm of the estimation error covariance for scenario 3. In general, scenario 3 is more robust to the changes in Υ_T due to the availability of a cross-layer path, as can be seen from Figure 6. Finally, the stability condition of scenario 3 is confirmed to be the same as predicted by Theorem 2.

3.2.3. Analytical performance evaluation: *an example.* In this part we are interested in finding an analytical expression characterizing the optimum performance in the presence of a cross-layer information path, which will be achieved when all the packets are kept in the receiver. While finding a general expression is beyond the scope of this paper, in this section we will derive it for a special case. Consider a single-input–single-output system that is equipped with a cross-layer information path with $\Upsilon_T = 0$. Consider the following communication noise variance**:

$$\Xi(\Upsilon[k]) = \frac{\beta}{\Upsilon[k]} \tag{50}$$

**See [24] for conditions that result in such a variance.

for $\beta \geqslant 0$, where $\Upsilon[k]$ is exponentially distributed (a common model in outdoor environments). To focus on the communication noise, in this section we take $A=a>1$, $C=1$, $R=0$ and $Q=0$. We are interested in finding an analytical expression for the average estimation error variance, $\mathbb{E}(P[k])$. Since cross-layer path is available in the receiver, the Kalman filter has the knowledge of the communication noise variance. Inserting the aforementioned parameters and channel noise model in Equation (27) with $\Upsilon_{\mathrm{T}}=0$, will result in the following recursion for $P[k]$:

$$P[k+1]=\frac{a^2\beta P[k]}{\beta+\Upsilon[k]P[k]} \tag{51}$$

Lemma 7
Let γ be an exponentially distributed random variable with $\varpi=1/\mathbb{E}(\gamma)$. Then we will have the following for arbitrary κ, $\varrho>0$ and $d>0$:

$$\mathbb{E}\left(\frac{\kappa}{\varrho+\mathrm{d}\gamma}\right)=\frac{\kappa\varpi}{d}\mathrm{e}^{\varpi\varrho/d}\mathrm{Expint}\left(\frac{\varpi\varrho}{d}\right) \tag{52}$$

where 'Expint' represents exponential integral: $\mathrm{Expint}(\phi)=\int_{\phi}^{\infty}\mathrm{e}^{-t}/t\,\mathrm{d}t$.

Proof

$$\begin{aligned}\mathbb{E}\left(\frac{\kappa}{\varrho+\mathrm{d}\gamma}\right)&=\kappa\varpi\int_0^{\infty}\frac{\mathrm{e}^{-\varpi\gamma}}{\varrho+d\times\gamma}\mathrm{d}\gamma\\&=\frac{\kappa\varpi}{d}\mathrm{e}^{\varpi\varrho/d}\int_{\varrho}^{\infty}\frac{\mathrm{e}^{-\varpi z/d}}{z}\,\mathrm{d}z\\&=\frac{\kappa\varpi}{d}\mathrm{e}^{\varpi\varrho/d}\mathrm{Expint}\left(\frac{\varpi\varrho}{d}\right)\end{aligned} \tag{53}$$

□

Lemma 8
Let γ be an exponentially distributed random variable with $\varpi=1/\mathbb{E}(\gamma)$. Let $\Pi(\varphi)=\mathrm{e}^{\varphi}\mathrm{Expint}(\varphi)$ for an arbitrary φ, and d and ϱ represent positive scalars where $d\leqslant\varpi$. Then we will have

$$\mathbb{E}[\Pi(\mathrm{d}\gamma+\varrho)]=\frac{\varpi}{\varpi-d}\Pi(\varrho)-\frac{\varpi}{\varpi-d}\Pi\left(\frac{\varpi\varrho}{d}\right) \tag{54}$$

Proof

$$\begin{aligned}\mathbb{E}[\Pi(\mathrm{d}\gamma+\varrho)]&=\mathbb{E}[\mathrm{e}^{\mathrm{d}\gamma+\varrho}\mathrm{Expint}(\mathrm{d}\gamma+\varrho)]\\&=\int_0^{\infty}\varpi\mathrm{e}^{(d-\varpi)\gamma+\varrho}\mathrm{Expint}(\mathrm{d}\gamma+\varrho)\,\mathrm{d}\gamma\\&=\frac{\varpi}{d-\varpi}\mathrm{e}^{(d-\varpi)\gamma+\varrho}\mathrm{Expint}(\mathrm{d}\gamma+\varrho)\big|_{\gamma=0}^{\gamma=\infty}+\frac{\mathrm{d}\varpi}{d-\varpi}\int_0^{\infty}\frac{\mathrm{e}^{-\gamma\varpi}}{\mathrm{d}\gamma+\varrho}\mathrm{d}\gamma\end{aligned}$$

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$$= -\frac{\varpi}{d-\varpi}\mathrm{e}^{\varrho}\mathrm{Expint}(\varrho) + \frac{\varpi}{d-\varpi}\mathrm{e}^{\varpi\varrho/d}\mathrm{Expint}\left(\frac{\varpi\varrho}{d}\right)$$

$$= \frac{\varpi}{\varpi-d}\Pi(\varrho) - \frac{\varpi}{\varpi-d}\Pi\left(\frac{\varpi\varrho}{d}\right) \tag{55}$$

□

Theorem 4

Consider the system model of Figure 2, a receiver that keeps all the packets and is equipped with a cross-layer path, the noise variance of Equation (50) and an exponentially distributed Signal-to-Noise Ratio. Then the average estimation error variance will be as follows for $N=1$, $A=a>1$, $R=0$, $Q=0$ and $C=1$:

$$\mathbb{E}(P[k+1]) = \sum_{i=0}^{k} B[i,k]\mathrm{e}^{\varpi\beta/(a^{2i}P[0])}\mathrm{Expint}\left(\frac{\varpi\beta}{a^{2i}P[0]}\right) \tag{56}$$

where $\varpi = \Upsilon_{\text{ave}}^{-1}$. $B[i,k]$ for $0\leqslant i\leqslant k$ is calculated using $B[i,k-1]$ for $0\leqslant i\leqslant k-1$ as follows:

$$B[i,k] = \begin{cases} -\displaystyle\sum_{z=0}^{k-1}\frac{B[z,k-1]}{\xi_{z+1}}, & i=0 \\ \dfrac{B[i-1,k-1]}{\xi_i}, & i\neq 0 \end{cases} \tag{57}$$

where $\xi_i = 1-1/a^{2i}$ and $B[0,0]=a^2\varpi\beta$. This means that $\mathbb{E}(P[k+1])$ can be found recursively using the coefficients of $\mathbb{E}(P[k])$.

Proof

Let $\mathbb{E}(P[k+1,i])$ refer to the case that $P[k+1]$ is averaged over $\Upsilon[k]$, $\Upsilon[k-1],\ldots,\Upsilon[k-i]$ and $\mathbb{E}(P[k+1]) = \mathbb{E}(P[k+1,k])$. We can take averages over $\Upsilon[i]$s for $0\leqslant i\leqslant k$ one at a time since they are assumed independent (see Section 2). Averaging Equation (51) over $\Upsilon[k]$, using Lemma 7 with $\varrho=\beta$, $d=P[k]$ and $\kappa=a^2\beta P[k]$, will result in the following:

$$\begin{aligned}\mathbb{E}(P[k+1,0]) &= a^2\varpi\beta\Pi\left(\frac{\varpi\beta}{P[k]}\right)\\ &= a^2\varpi\beta\Pi\left(\frac{\varpi(\beta+\Upsilon[k-1]P[k-1])}{a^2P[k-1]}\right)\end{aligned} \tag{58}$$

where $\Pi(.)$ is as defined in Lemma 8. By inserting $\varrho=\varpi\beta/(a^{2i}q)$ and $d=\varpi/a^{2i}$ in Equation (54), we will have the following for a $\vartheta>0$ and an exponentially distributed random variable Υ with $\varpi=1/\mathbb{E}(\Upsilon)$:

$$E\left[\Pi\left(\frac{\varpi(\beta+\vartheta\Upsilon)}{a^{2i}\vartheta}\right)\right] = \frac{\Pi\left(\dfrac{\varpi\beta}{a^{2i}\vartheta}\right)}{1-\dfrac{1}{a^{2i}}} - \frac{\Pi\left(\dfrac{\varpi\beta}{\vartheta}\right)}{1-\dfrac{1}{a^{2i}}}, \quad i\geqslant 1, \quad |a|>1 \tag{59}$$

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By applying Equation (59) with $i=1$ and $\vartheta=P[k-1]$, we will have

$$\mathbb{E}(P[k+1,1])=\frac{a^2\varpi\beta\Pi\left(\dfrac{\varpi\beta}{a^2P[k-1]}\right)}{1-\dfrac{1}{a^2}}-\frac{a^2\varpi\beta\Pi\left(\dfrac{\varpi\beta}{P[k-1]}\right)}{1-\dfrac{1}{a^2}} \tag{60}$$

It can be seen from Equations (58) and (60) that we will have the following after $m+1$ steps of averaging:

$$\mathbb{E}(P[k+1,m])=\sum_{z=0}^{m}B[z,m]\Pi\left(\frac{\varpi\beta}{a^{2z}P[k-m]}\right) \tag{61}$$

where $B[0,0]=a^2\varpi\beta$. The goal is to find $B[z,m]$ for $m=k$. Let

$$O_k[z,m]=\Pi\left(\frac{\varpi\beta}{a^{2z}P[k-m]}\right)$$

Then,

$$\mathbb{E}(P[k+1,m])=\sum_{z=0}^{m}B[z,m]O_k[z,m] \tag{62}$$

Substituting $P[k-m]$ as a function of $P[k-m-1]$ and averaging over $\Upsilon[k-m-1]$ (using Equation (59)) will result in the following for $-1\leqslant m\leqslant k-1$:

$$\begin{aligned}\mathbb{E}(P[k+1,m+1]) &= \sum_{z=0}^{m}\frac{B[z,m]}{\xi_{z+1}}O_k[z+1,m+1]-\sum_{z=0}^{m}\frac{B[z,m]}{\xi_{z+1}}O_k[0,m+1]\\ &= \sum_{i=1}^{m+1}\frac{B[i-1,m]}{\xi_i}O_k[i,m+1]-\left[\sum_{z=0}^{m}\frac{B[z,m]}{\xi_{z+1}}\right]O_k[0,m+1]\\ &= \sum_{i=0}^{m+1}B[i,m+1]O_k[i,m+1]\end{aligned} \tag{63}$$

where $\xi_z=1-1/a^{2z}$ and the last equality is written using Equation (62). Therefore for $0\leqslant i\leqslant m+1$,

$$B[i,m+1]=\begin{cases}-\displaystyle\sum_{z=0}^{m}\frac{B[z,m]}{\xi_{z+1}}, & i=0\\ \dfrac{B[i-1,m]}{\xi_i}, & i\neq 0\end{cases} \tag{64}$$

Then $\mathbb{E}(P[k+1])=\mathbb{E}(P[k+1,k])$ will be as written in Equation (56), where $B[i,k]$ for $0\leqslant i\leqslant k$ is calculated using $B[i,k-1]$ for $0\leqslant i\leqslant k-1$ using[††] Equation (57). □

[††]A similar equation can be derived for cases where $|a|<1$.

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4. FURTHER EXTENSIONS

In this paper we derived new design paradigms for estimating dynamical systems over wireless links. There are several possible extensions for this work. For instance, we made two assumptions in our derivations: channel gets uncorrelated from one transmission to the next and matrix C is invertible. Here we discuss scenarios where these are not the case. We also summarize other possible extensions of our work.

4.1. Correlated channel

As we discussed in Section 2, channel coherence time, time between consecutive transmissions and the makeup of the environment are among key factors that determine if (and to what extent) the channel stays correlated from one transmission to the next. In a rich scatterer environment with no LOS path, channel correlation function can be represented by a zero-order Bessel function as follows [17]:

$$\mathbb{E}(h(t)h^*(t-t_1)) = \mathbb{E}(|h(t)|^2) J_0\left(\frac{2\pi s_{\text{mob}} F_{\text{c}}}{c} t_1\right) \tag{65}$$

where J_0 represents zero-order Bessel function, F_{c} is the frequency of operation, s_{mob} is the speed of the mobile unit and c is the speed of light. To see the impact of such correlation characteristics on the estimation performance, we look at a case where channel stays correlated even after three transmissions with the correlation coefficient of 0.3 after three consecutive transmissions. Figure 7 shows the spectral norm of the average estimation error covariance of this system (obtained through simulations) for a single-input–single-output system with $A=2$, $C=1$, $\Upsilon_{\text{ave}}=20\,\text{dB}$, $F_{\text{c}}=1\,\text{GHz}$ and $s_{\text{mob}}=25\,\text{mph}$. The time between consecutive transmissions is taken to be $10\,\text{ms}$ in this example. Comparing the correlated case with the uncorrelated one shows that this amount of channel correlation has negligible impact on the performance and optimum threshold.

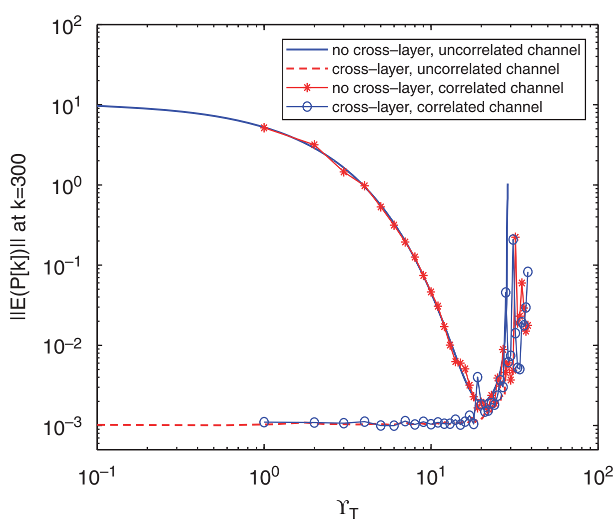

Figure 7. Impact of channel correlation on the performance.

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If the mobile node is in a deep fade for a period of time, then channel can be considerably correlated from one transmission to the next. In case of such error bursts; however, maintaining an acceptable performance can be difficult while in deep fade independent of the receiver design. Then it will be more important to have a fast recovery once out of deep fade. Analyzing the impact of a general channel correlation characteristic on estimation over wireless links in general, and on the optimum design in particular, is one possible extension of the work presented in this paper. In communication literature, channel time variation for correlated channels is sometimes modeled with a linear dynamical system. Such an approach is compatible with estimation and control formulation and can be utilized to incorporate the impact of channel correlation in estimation and control of linear dynamical systems over wireless links.

4.2. Non-invertible C matrix

In general, deriving analytical expressions for performance evaluation, stability analysis and optimum packet drop threshold is more challenging if C is not invertible. It is important to consider the impact of such cases on the optimum design as, depending on the application, the system may be under-estimated for instance. The concepts introduced in this paper, such as balance of information loss and communication noise or impact of a cross-layer path on the optimum design, are all applicable for any general C matrix. Thus, deriving mathematical foundations of such general cases is among possible extensions of this work. For instance consider the system parameters of Figure 5 but with $C=[1\ \ 2\ \ 3]$ at $\Upsilon_{\text{ave}}=20\,\text{dB}$. The optimum packet drop threshold for this under-estimated system is found to be $\Upsilon_{\text{T,opt}}=12+\varepsilon$ where $-1<\varepsilon<1$ through simulations. By comparing it with the corresponding case in Figure 5 (the optimum is 14 in this case), it can be seen that the optimum packet drop criteria of the two systems are close for this example. To draw any general conclusion, however, further analytical investigations of non-invertible C cases are necessary and are among possible future directions.

4.3. Other extensions

There are other possible avenues for extending the current framework. The theoretical framework can be extended to embrace more general scenarios. For instance, we considered one transmitter and one receiver in this paper. There may be cases where a number of nodes share the bandwidth to perform networked estimation. The analysis and results of this paper can be extended to such scenarios by replacing Signal-to-Noise Ratio by signal to interference and noise ratio. In this paper we considered the impact of a wireless fading channel on the performance of a Kalman filter. Intuitively, the derived design strategies should be applicable when considering the performance of a controller. Proving this analytically is among possible extensions of this work. Also finding the optimum controller in the presence of a wireless fading channel is an important issue that needs to be addressed.

5. SUMMARY

In this paper we considered estimation over mobile communication channels using a Kalman filter. We showed that the communication protocols suitable for other already existing applications like data networks may not be entirely applicable for estimation and control of a rapidly changing dynamical system. We derived stability conditions and investigated the performance of different

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receiver designs. We proved that in order to maximize the stability range, the receiver should keep all the packets independent of the quality of the link or availability of a cross-layer path. In the presence of a cross-layer path, we proved that this design will also optimize the performance. We furthermore derived an analytical expression for the estimation error covariance of Kalman filtering over a wireless link in the presence of a cross-layer path.

REFERENCES

1. Chong C, Kumar S. Sensor networks: evolution, opportunities and challenges. *Proceedings of the IEEE* 2003; **91**(8):1247–1256.
2. Sinopoli B, Sharp C, Schenato L, Schaffert S, Sastry S. Distributed control applications within sensor networks. *Proceedings of the IEEE* 2003; **91**(8):1235–1246.
3. Wong W, Brockett R. Systems with finite communication bandwidth constraints—part I: state estimation problems. *IEEE Transactions on Automatic Control* 1997; **42**(9):1294–1299.
4. Wong W, Brockett R. Systems with finite communication bandwidth constraints—II: stabilization with limited information feedback. *IEEE Transactions on Automatic Control* 1999; **44**(5):1294–1298.
5. Tatikonda S. Control under communication constraints. *Ph.D. Dissertation*. MIT: Cambridge, MA, 2000.
6. Elia N, Mitter S. Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control* 2001; **46**(9):1384–1400.
7. Faganini F, Zampieri S. Stability analysis and synthesis for scalar linear systems with a quantized feedback. *IEEE Transactions on Automatic Control* 2003; **48**(9):1569–1584.
8. Sahai A, Mitter S. The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link, part I: scalar systems. *IEEE Transactions on Information Theory* 2006; **52**(8):3369–3395.
9. Martins N, Dahleh M. Fundamental limitations of performance in the presence of finite capacity feedback. *Proceedings of 2005 American Control Conference*, Portland, OR, vol. 1, 2005; 79–86.
10. Goldsmith A. *Wireless Communications*. Cambridge University Press: Cambridge, MA, 2005.
11. Goodman D. *Wireless Personal Communications Systems*. The Addison-Wesley Wireless Communications Series. Addison-Wesley: Reading, MA, 1997.
12. Micheli M, Jordan MI. Random sampling of a continuous-time dynamical systems. *Proceedings of 15th International Symposium on the Mathematical Theory of Networks and Systems* (*MTNS*), University of Notre Dame, South Bend, IN, 2002.
13. Sinopoli B, Schenato L, Franceschetti M, Poolla K, Jordan M, Sastry S. Kalman filtering with intermittent observations. *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, HI, vol. 1, 9–12 December 2003; 701–708.
14. Liu X, Goldsmith AJ. Kalman filtering with partial observation losses. *Forty-third IEEE Conference on Decision and Control*, Atlantis, Bahamas, 2004.
15. Mostofi Y, Murray R. To drop or not to drop: design principles for Kalman filtering over wireless fading channels. *IEEE Transactions on Automatic Control* 2008; in press.
16. Rappaport TS. *Wireless Communications*, *Principles and Practice*. Prentice-Hall: Englewood Cliffs, NJ, 1999.
17. Jakes W. *Microwave Mobile Communications*. IEEE Press: New York, 1974.
18. Mostofi Y, Murray R. Effect of time-varying fading channels on the control performance of a mobile sensor node. *Proceedings of IEEE 1st International Conference on Sensor and Adhoc Communications and Networks* (*SECON*), Santa Clara, California, October 2004.
19. Son D, Krishnamachari B, Heidemann J. Experimental analysis of concurrent packet transmissions in low-power wireless networks. *USC-ISI Technical Report ISI-TR-2005-609*, November 2005.
20. Kailath T, Sayed AH, Hassibi B. *Linear Estimation*. Prentice-Hall Information and System Sciences Series. Prentice-Hall: Englewood Cliffs, NJ, 2002.
21. Boyd S, Vandenberghe L. *Convex Optimization*. Cambridge University Press: Cambridge, MA, 2004.
22. Horn R, Johnson C. *Matrix Analysis*. Cambridge University Press: Cambridge, MA, 1999.
23. Marshall AW, Olkin I. *Inequalities*: *Theory of Majorization and its Applications*. Academic Press: New York, 1979.
24. Laneman JN, Martinian E, Wornell GW, Apostolopoulos JG, Wee SJ. Comparing application- and physical-layer approaches to diversity on wireless channels. *Proceedings of the IEEE International Conference on Communications* (*ICC*), Anchorage, AK, May 2003.

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