Mathematical Analysis of the Impact of Timing Synchronization Errors on the Performance of an OFDM System

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***Abstract*—This letter addresses the effect of timing synchronization errors that are introduced by an erroneous detection of the start of an orthogonal frequency-division multiplexing (OFDM) symbol.[1] Such errors degrade the performance of an OFDM receiver by introducing intercarrier interference (ICI) and intersymbol interference (ISI). They can occur due to either an erroneous initial frame synchronization or a change in the power delay profile of the channel. In this letter, we provide a mathematical analysis of the effect of timing errors on the performance of an OFDM receiver in a frequency-selective fading environment.[2] We find exact formulas for the power of interference terms and the resulting average signal-to-interference ratio. We further extend the analysis to the subsample level. Our results show the nonsymmetric effect of timing errors on the performance of an OFDM system. Finally, simulation results confirm the analysis.[3]**

***Index Terms*—Orthogonal frequency-division multiplexing (OFDM), signal-to-interference ratio (SIR), timing-synchronization errors.**

I. INTRODUCTION

ORTHOGONAL frequency-division multiplexing (OFDM) divides the given bandwidth into narrow subchannels. It handles delay spread by sending low data rates in parallel on these subchannels [1]. By adding a guard interval to the beginning of each OFDM symbol, the effect of delay spread (provided that there is perfect synchronization) would appear as a multiplication in the frequency domain. Adding the guard interval will also prevent intersymbol interference (ISI).[4] Timing-synchronization errors, however, degrade the performance of an OFDM receiver by introducing intercarrier interference (ICI) and ISI. Several methods have been proposed for timing synchronization in OFDM receivers [2]–[6]. To evaluate and improve the performance of these methods, a comprehensive mathematical analysis of the effect of timing errors and the underlying interference terms is necessary.

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[1]Throughout this letter, the term "timing error" would refer to this type of error.

[2]The analysis presented in this letter is for the case that no equalization technique has been used to mitigate the introduced ICI and ISI.

[3]The results of this letter can be easily extended to address the effect of such errors on DMT modems.

[4]Note that intersymbol interference refers to inter-OFDM symbol-interference.

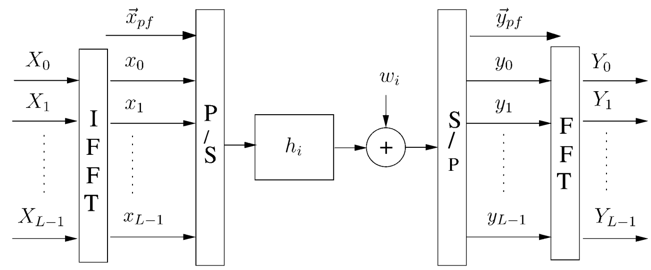

Fig. 1. Discrete baseband equivalent model.

Authors in [5] have provided an *approximated* formula with limited applications for the interference caused by timing errors. It is the goal of this letter to provide an *exact* mathematical analysis of the effect of timing errors, which can be a base for evaluating the performance of different synchronization methods. Furthermore, while most of the work on timing synchronization use sampling-period-level modeling, we also show how to analyze and evaluate the performance on the subsample level.

II. EFFECT OF TIMING SYNCHRONIZATION ERRORS (SAMPLING-PERIOD LEVEL)

Consider an OFDM system, shown in Fig. 1, in which the available bandwidth is divided into L subchannels, and the guard interval spans G sampling periods. Let T represent the length of an OFDM symbol (including the guard interval). Then $T_s = \frac{T}{L+G}$ is the sampling period. In this section, we keep the analysis on the sampling-period level, which translates to the following assumptions:

1) no oversampling is done in the receiver;
2) timing error is a multiple of T_s.

We relax these conditions in Section III, where we extend the analysis to include oversampling.

A. System Model

X_i represents the transmitted data in the ith frequency subband and is related to the time-domain sequence x_i as follows: $X_i = \sum_{k=0}^{L-1} x_k e^{\frac{-j2\pi ki}{L}}$ for $0 \le i \le L-1$. $\vec{x}_{pf}$ and $\vec{y}_{pf}$ represent the transmitted and received cyclic prefixes (CPs), respectively. h_i represents the ith channel tap with Rayleigh fading amplitude and uniformly distributed phase, and w_i is additive white Gaussian noise (AWGN). Let $C \le G$ represent the length of the channel delay spread normalized by the sampling period. Then the channel would have $C+1$ taps. In the absence of timing errors, y_i, the received signal after discarding the CP (see Fig. 1), is as follows:[5]

$$y_i = \overbrace{\sum_{k=0}^{C} h_k x_{((i-k))_L}}^{z_i} + w_i$$

for $0 \leq i \leq L-1$. Consider a case of a timing error of m sampling periods. $0 < m \leq G$ and $-G \leq m < 0$ denote timing errors of m to the right and left sides, respectively.

B. Case of Timing Errors to the Right $(m > 0)$

In this case, an error of m sampling periods to the right side has occurred. Then, the terms $y_0, y_1, \ldots, y_{m-1}$ are missed, and instead, m data points of the next OFDM symbol are erroneously selected. The received signal can thus be written as $v_i^r = z_{((i+m))_L} \times \gamma_i^r + s_i + w_i^r$ for $0 \leq i \leq L-1$, where v_i^r is the received signal for $m > 0$

$$s_i = \begin{cases} 0, & 0 \leq i \leq L-m-1 \\ y_{pf}^{\text{next}}(i-L+m), & \text{else} \end{cases}$$

with $y_{pf}^{\text{next}}(i)$ representing the ith sample of the output CP of the next OFDM symbol (excluding the effect of AWGN)

$$\gamma_i^r = \begin{cases} 1, & 0 \leq i \leq L-m-1 \\ 0, & L-m \leq i \leq L-1 \end{cases}$$

and w_i^r is AWGN. Then V_i^r, the fast Fourier transform (FFT) of v_i^r, will be

$$V_i^r = \frac{\Gamma_0^r}{L} H_i X_i e^{\frac{j2\pi mi}{L}} + \underbrace{\overbrace{\sum_{k=1}^{L-1} \frac{\Gamma_k^r}{L} H_{((i-k))_L} X_{((i-k))_L} e^{\frac{j2\pi m(i-k)}{L}}}^{I_i^r} + S_i}_{\text{ICI \& ISI}} + W_i^r \quad 0 \leq i \leq L-1 \tag{1}$$

where S_i is the FFT of s_i and Γ_i^r, the FFT of γ_i^r, is

$$\Gamma_i^r = \begin{cases} (1-e^{\frac{j2\pi im}{L}})/(1-e^{-\frac{j2\pi i}{L}}), & i \neq 0 \\ L-m, & i = 0. \end{cases}$$

I_i^r represents the ICI resulting from multiplication of $z_{((i+m))_L}$ by γ_i^r in v_i^r.

Theorem 1: The total interference power and the resulting signal-to-interference ratio (SIR) for $m > 0$, SIR_r, will be as follows:

$$\overline{|I_i^r + S_i|^2} = \frac{(2L-m)m}{L^2}\sigma_X^2\sigma_H^2 - 2\frac{L-m}{L^2}\sigma_X^2 \sum_{k=0}^{m-1}\sum_{k'=k+1}^{C} \sigma_{h_{k'}}^2$$
$$\Rightarrow \text{SIR}_r = \frac{(L-m)^2}{(2L-m)m - 2\frac{L-m}{\sigma_H^2}\sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sigma_{h_{k'}}^2} \tag{2}$$

where $\overline{X_i X_j^*} = \sigma_X^2 \delta_{i,j}$, $\sigma_H^2 = \overline{H_k H_k^*} = \sum_{i=0}^{C} \sigma_{h_i}^2$ with $\sigma_{h_i}^2$ representing the power of the ith channel tap.

Proof: Theorem 1 is proved in Appendix A.

C. Case of Timing Errors to the Left $(m < 0)$

In this case, due to the presence of the CP, the number of data points that are missed can be less than $-m$. If the length of the channel delay spread spans C sampling periods, only $d = \max(C-(G+m), 0)$ data points are corrupted, due to the interference from the previous symbol. Therefore, in this case, we will have $v_i^l = z_{((i+m))_L} \times \gamma_i^l + p_i + w_i^l$ for $0 \leq i \leq L-1$, where v_i^l is the received signal for $m < 0$

$$p_i = \begin{cases} y_{pf}(G+m+i), & 0 \leq i \leq d-1 \\ 0, & d \leq i \leq L-1 \end{cases}$$

with $y_{pf}(i)$ representing the ith sample of the output CP of the current OFDM symbol (excluding the effect of AWGN)

$$\gamma_i^l = \begin{cases} 0, & 0 \leq i \leq d-1 \\ 1, & d \leq i \leq L-1 \end{cases}$$

and w_i^l is a sample of AWGN. Then V_i^l, the FFT of v_i^l, will be

$$V_i^l = \frac{\Gamma_0^l}{L} H_i X_i e^{\frac{j2\pi mi}{L}} + \underbrace{\overbrace{\sum_{k=1}^{L-1} \frac{\Gamma_k^l}{L} H_{((i-k))_L} X_{((i-k))_L} e^{\frac{j2\pi m(i-k)}{L}}}^{I_i^l} + P_i + W_i^l}_{\text{ICI \& ISI}} \tag{3}$$

where P_i is the FFT of p_i, and Γ_i^l, the FFT of γ_i^l, is

$$\Gamma_i^l = \begin{cases} (e^{\frac{-j2\pi id}{L}}-1)/(1-e^{\frac{-j2\pi i}{L}}), & i \neq 0 \\ L-d, & i = 0. \end{cases}$$

Theorem 2: The total interference power and the resulting SIR for $m < 0$, SIR_l, will be as follows:

$$\overline{|I_i^l + P_i|^2} = \frac{(2L-d)d}{L^2}\sigma_X^2\sigma_H^2 - 2\frac{L-d}{L^2}\sigma_X^2 \sum_{k=0}^{d-1}\sum_{k'=0}^{G+m+k} \sigma_{h_{k'}}^2$$
$$\Rightarrow \text{SIR}_l = \frac{(L-d)^2}{(2L-d)d - 2\frac{L-d}{\sigma_H^2}\sum_{k=0}^{d-1}\sum_{k'=0}^{G+m+k}\sigma_{h_{k'}}^2}. \tag{4}$$

Proof: Theorem 2 is proved in Appendix B.

D. Simulation Results

We verify the analytical results by simulating the effect of timing errors for both $m > 0$ and $m < 0$. We choose $L = 512$ and $G = 52$ in our simulations. The simulated taps of channel #1 have relative powers of [0.1214 0.1529 0 0 0.1924 0.1529 0 0.1160 0.0965 0.0766 0.0609 0.0305]. Fig. 2 shows SIR resulting from analysis [(2) and (4)] and simulation for this channel (note that we are only interested in integer m in this section). Since the

[5]In this letter, channel is assumed constant over one OFDM symbol. This is the case for time-invariant or slowly varying channels.

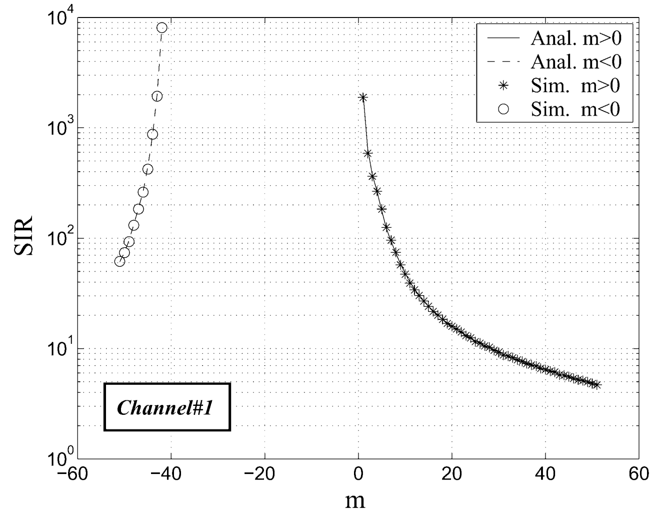

Fig. 2. SIR versus m for channel #1 (integer m).

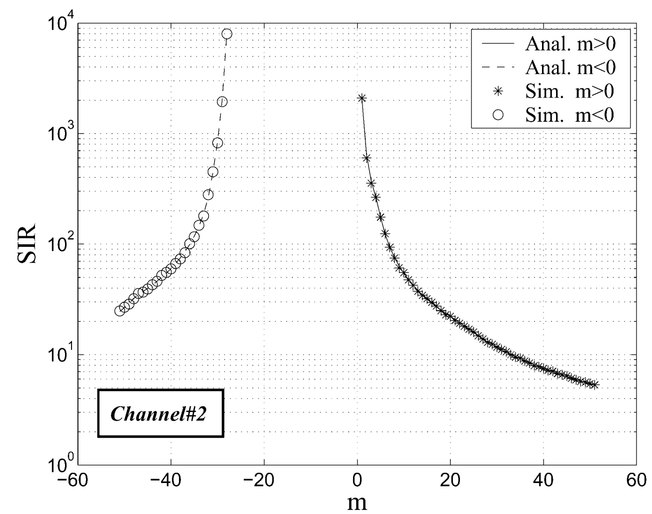

Fig. 3. SIR versus m for channel #2 (integer m).

length of channel #1 spans only 21% of the guard interval, the interference power will be zero $(d = 0)$ for $-42 < m < 0$. Furthermore, the level of interference for $m > 0$ and $m < 0$ cases is different. This nonsymmetric effect of timing errors can be seen from Fig. 2. The results of analysis and simulation match well. To see the effect of $m < 0$ more pronouncedly, we simulate the effect of timing errors for channel #2, which has the same power delay profile as channel #1, but the last five taps of it are delayed such that the total delay spans 48% of the guard interval. Fig. 3 shows the results of simulation and analysis for channel #2. Increasing the delay does not have a considerable effect on $m > 0$. However, for $m < 0$, the interference power is only zero for the reduced range of $-28 < m < 0$ in this case. Exact match of simulation and analysis can be seen from Fig. 3, as well.

III. Extension to the Subsample Level

In practice, the receiver samples the received signal on the subsample level. Furthermore, m may not be a multiple of T_s,

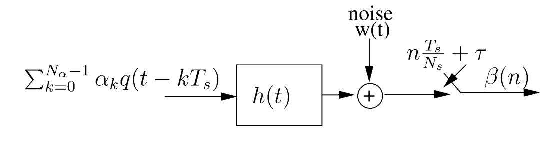

Fig. 4. Transmission in the subsample level.

as was assumed in the previous sections. We extend the analysis to include such scenarios in this section.

A. System Model in the Subsample Level

Let $\alpha(k)$ for $0 \le k \le N_\alpha - 1$ represent consecutive time-domain transmitted data points. $\alpha(k)$, for instance, can represent the time-domain transmitted data points of an OFDM symbol. We keep the analysis general in this subsection, and apply the results to an OFDM system, eventually. Let $q(t)$ and $h(t)$ represent the continuous-time pulse shaper and channel. Fig. 4 represents transmission in the subsample level. The receiver samples the received signal with the frequency of N_s/T_s, and $\tau < T_s/N_s$ represents sampling uncertainty. We take care of part of the uncertainty that is bigger than T_s in Section III-B, as that part would involve samples of the adjacent symbols, as well. We will have

$$\beta(n) = \sum_{k=0}^{N_\alpha-1}\alpha_k\psi\left(n\frac{T_s}{N_s} - kT_s + \tau\right) + w\left(n\frac{T_s}{N_s} + \tau\right)$$
$$0 \le n \le N_s N_\alpha - 1 \qquad (5)$$

where $\psi(t) = q(t) * h(t)$. β can be divided into N_s sets, each consisting of N_α T_s-spaced samples. Let η_i represent the ith set. Then we will have

$$\begin{aligned}\eta_i(k') &= \beta(k'N_s + i)\\ &= \sum_{k=0}^{N_\alpha-1}\alpha_k f_{i,\tau}(k'-k) + w_i(k')\\ &= [\alpha * f_{i,\tau}]_{|k'} + w_i(k')\\ &\qquad 0 \le i \le N_s - 1 \text{ and } 0 \le k' \le N_\alpha - 1 \qquad (6)\end{aligned}$$

where $f_{i,\tau}(k) = \Psi(kT_s + i\frac{T_s}{N_s} + \tau)$ and $w_i(k')$ is a sample of Gaussian noise. This suggests that depending on the chosen set and τ, different channels will be realized in the receiver.

B. Timing Errors in an OFDM System

In this section, we take the timing error of an OFDM system to have the general form of $\Delta t = i_1 T_s + i_2\frac{T_s}{N_s} + \tau$ for $0 \le i_2 \le N_s - 1$ and $\tau < \frac{T_s}{N_s}$. We can see from the previous subsection that depending on i_2 and τ, different equivalent channels $(f_{i_2,\tau}(k))$, with different frequency responses, are realized. i_1, on the other hand, defines the amount of interference in the form of ICI and ISI. We take $q(t) = 0$ for $t < 0$ and $t > T_s$. Then T_s-spaced samples of ψ can be shown to be uncorrelated.[6] Let $\sigma^2_{f_{i_2,\tau}}(k)$ represent the power of the kth sample of the equivalent channel, $f_{i_2,\tau}(k)$, and $\sigma^2_{F_{i_2,\tau}} = \sum_{k=0}^{C}\sigma^2_{f_{i_2,\tau}}(k)$. Equation

[6]As $q(t)$ has most of its energy concentrated in the time duration of $0 \le t \le T_s$, this is a reasonable model. If the energy of $q(t)$ outside the main lobe can not be neglected, then the analysis of the previous sections can be extended to the case of correlated equivalent-channel samples.

TABLE I
ANALYSIS AND SIMULATION FOR NONINTEGER ERRORS

timing error normalized by T_s	4.7500	8.1250	12.3750	20.5000	30.6250	40.8750
Anal.	105.9299	48.2283	25.9376	13.5340	8.3406	5.9354
Sim.	105.7933	48.4414	26.0886	13.4288	8.2944	5.9153

(2) can be easily extended to the following for timing errors to the right ($i_1 > 0$):

$$\mathrm{SIR}_{r,\mathrm{subsample}} = \frac{(L-i_1)^2}{(2L-i_1)i_1 - 2\frac{L-i_1}{\sigma^2_{F_{i_{2,\tau}}}} \sum_{k=0}^{i_1-1} \sum_{k'=k+1}^{C} \sigma^2_{f_{i_{2,\tau}}}(k')} \quad (7)$$

where C, similar to the previous section, represents the normalized length of $f_{i_{2,\tau}}(k)$ (normalized by T_s). The results of $m < 0$ of the previous section (sampling-period level) can be easily extended in a similar manner.

We verify the theoretical results with simulations. We take the following parameters in our simulation: $L = 512$, $G = 52$, $N_s = 4$, channel power delay profile of $e^{-(0.4t/T_s)}$ for $0 \le t \le 12T_s$, and a raised cosine pulse shaper

$$q(t) = \frac{\mathrm{sinc}\left(\frac{t-.5T_s}{.5T_s}\right)\cos\left(\frac{\pi\zeta(t-.5T_s)}{.5T_s}\right)}{1-4\zeta^2\frac{(t-.5T_s)^2}{.25T_s^2}}$$

for $0 \le t \le T_s$, with $\zeta = 0.25$. Table I shows results of simulation and analysis for different noninteger timing errors. As can be seen, mathematical derivations are confirmed by simulation results.

IV. CONCLUSION

In this letter, we provided a mathematical analysis of the effect of timing-synchronization errors on the performance of an OFDM receiver. We found exact expressions for the average SIR in the case of errors to both right and left sides, including the impact of ICI and ISI. We furthermore extended the analysis to the subsample level. The formulas reflected the nonsymmetric effect of these errors on the performance due to the presence of the CP. Finally, our simulation results confirmed the analysis.

APPENDIX A

Using (1), power of I_i^r will be as follows:

$$\begin{aligned}\sigma^2_{I^r} &= \frac{1}{L^2}\sum_{k=1}^{L-1}\sum_{k'=1}^{L-1}\Gamma_k^r\Gamma_{k'}^{r*}\overline{H_{((i-k))_L}H^*_{((i-k'))_L}} \\ &\quad\times\overline{X_{((i-k))_L}X^*_{((i-k'))_L}}e^{\frac{-j2\pi m(k-k')}{L}} \\ &= \frac{(L-m)m}{L^2}\sigma_X^2\sigma_H^2. \end{aligned} \quad (8)$$

In writing the second equality of (8), we used $\sum_{k=1}^{L-1}|\Gamma_k^r|^2 = L\sum_{k=0}^{L-1}|\gamma_k^r|^2 - |\Gamma_0^r|^2 = m \times (L-m)$. Since y_{pf}^{next} includes delayed replicas of the current OFDM symbol through delay spread, S_i has an ICI term, in addition to an ISI term. We have $S_i = \sum_{k=0}^{m-1} s_{L-m+k}e^{\frac{-j2\pi i(k-m)}{L}}$, where s_{L-m+k} can be written as follows:

$$\begin{aligned} S_{L-m+k} &= y_{pf}^{\mathrm{next}}(k) \\ &= \overbrace{\sum_{k'=k+1}^{C} h_{k'}x_{L-k'+k}}^{\mathrm{ICI}} + \overbrace{\sum_{k'=0}^{k} h_{k'}x^{\mathrm{next}}_{L-G+k-k'}}^{\mathrm{ISI}} \\ & 0 \le k \le m-1 \end{aligned} \quad (9)$$

where x_i^{next} represents the ith time-domain transmitted data point of the next OFDM symbol. Let σ_S^2 represent the power of S_i. Using (9) and noting that x and x^{next} are independent, will result in

$$\sigma_S^2 = \frac{\sigma_X^2}{L}\sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sigma^2_{h_{k'}} + \frac{\sigma_X^2}{L}\sum_{k=0}^{m-1}\sum_{k'=0}^{k}\sigma^2_{h_{k'}} = \frac{m\sigma_X^2\sigma_H^2}{L}.$$

The cross-interference term will be as follows, using $\overline{h^*_{k'}H_{((i-k''))_L}} = \sigma^2_{h_{k'}}e^{\frac{-j2\pi k'(i-k'')}{L}}$ and $\overline{x^*_{L-k'+k}X_{((i-k''))_L}} = \frac{\sigma_X^2}{L}e^{\frac{-j2\pi(k-k')(i-k'')}{L}}$:

$$\begin{aligned}\overline{I_i^rS_i^*} &= \frac{1}{L}\sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sum_{k''=1}^{L-1}\Gamma^r_{k''}\overline{h^*_{k'}H_{((i-k''))_L}} \\ &\quad\times\overline{x^*_{L-k'+k}X_{((i-k''))_L}}e^{\frac{-j2\pi(mk''-ik)}{L}} \\ &= \frac{\sigma_X^2}{L^2}\sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sigma^2_{h_{k'}}\sum_{k''=1}^{L-1}\Gamma^r_{k''}e^{\frac{j2\pi k''(k-m)}{L}} \\ &= \frac{\sigma_X^2}{L^2}\sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sigma^2_{h_{k'}}\left(L\gamma^r_{((k-m))_L} - \Gamma_0^r\right) \\ &= -\frac{L-m}{L^2}\sigma_X^2\sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sigma^2_{h_{k'}}. \end{aligned} \quad (10)$$

Adding all the interference terms will result in (2).

APPENDIX B

Similar to the $m > 0$ case, the power of I_i^l will be $\sigma^2_{I^l} = \frac{(L-d)d}{L^2}\sigma_X^2\sigma_H^2$. p_k, the inverse FFT of P_k, can be written as

$$p_k = \sum_{k'=0}^{G+m+k} h_{k'}x_{L+m+k-k'} + \sum_{k'=G+m+k+1}^{C} h_{k'}x^{\mathrm{prev}}_{L-k'+G+m+k}$$

for $0 \leq k \leq d-1$, where x_i^{prev} represents the ith time-domain transmitted data point of the previous OFDM symbol. We will have

$$\sigma_P^2 = \frac{\sigma_X^2}{L} \sum_{k=0}^{d-1} \sum_{k'=0}^{G+m+k} \sigma_{h_{k'}}^2 + \frac{\sigma_X^2}{L} \sum_{k=0}^{d-1} \sum_{k'=G+m+k+1}^{C} \sigma_{h_{k'}}^2 = \frac{(d\sigma_X^2 \sigma_H^2)}{L}.$$

Similar to the case of $m > 0$, $\overline{I_i^l P_i^*}$ will be

$$\overline{I_i^l P_i^*} = \frac{-(L-d)}{L^2} \sigma_X^2 \sum_{k=0}^{d-1} \sum_{k'=0}^{G+m+k} \sigma_{h_{k'}}^2.$$

This results in the total interference power and SIR_l of (4).

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