Analysis of the Effect of Timing Synchronization Errors on Pilot-aided OFDM Systems

Yasamin Mostofi
Stanford University
Stanford, CA 94305, USA
yasi@wireless.stanford.edu

Donald C. Cox
Stanford University
Stanford, CA 94305, USA
dcox@spark.stanford.edu

Abstract

Timing synchronization errors degrade the performance of an OFDM receiver by introducing Inter-Carrier-Interference and Inter-Symbol-Interference. These errors can occur due to either an erroneous initial synchronization or a change in the power delay profile of the channel. In this paper, mathematical analysis of the effect of timing errors on the performance of an OFDM receiver is provided. Exact expressions for the resulting average Signal to Interference Ratio (SIR) were derived in the presence of timing errors. Then the effect of timing errors on the performance of a pilot-aided channel estimator is analyzed. Expressions for average power of channel estimation error in the presence of timing errors and noise are derived for a frequency selective fading channel. The results show the non-symmetric effect of timing errors on the performance of an OFDM system. Furthermore, they show the super-sensitivity of pilot-aided channel estimator to such errors. Finally simulation results confirm the analysis.

INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) handles delay spread by sending low data rates on narrowband sub-channels in parallel [1]. Timing synchronization errors, however, degrade the performance of an OFDM receiver by introducing Inter-Carrier-Interference (ICI) and Inter-Symbol-Interference (ISI). Several methods have been proposed for timing synchronization in OFDM receivers [2], $[4-7]$. To evaluate and improve the performance of any proposed method, a complete mathematical analysis of the effect of timing errors on an OFDM system is essential. In this paper, we derive key performance measure parameters for an OFDM system in the presence of timing errors. First we derive expressions for the resulting average SIR in the presence of timing errors[1]. Furthermore, by considering both timing synchronization and channel estimation jointly, we provide a complete mathematical analysis of the impact of timing errors on pilot-aided channel estimation. We derive expressions for average power of channel estimation error in the presence of timing errors and noise for a frequency-selective fading channel. Finally, simulation results confirm the mathematical analysis.

[1] In reference [5], authors have provided an approximate formula with limited applications for the resulting SIR. In this paper, we show how to derive the exact expressions.

EFFECT OF TIMING SYNCHRONIZATION ERRORS

Consider an OFDM system in which the available bandwidth is divided into N sub-channels and the guard interval spans G sampling periods. X_i represents the transmitted data in the i^{th} sub-band and is related to the time domain sequence, x_i, as $X_i = \sum_{k=0}^{N-1} x_k e^{-\frac{j2\pi ki}{N}}$. $\vec{x}_{pf}$ and $\vec{y}_{pf}$ contain data points of the transmitted and received cyclic prefix respectively. h_i represents the i^{th} channel tap with Rayleigh fading amplitude and uniformly distributed phase and w_i is AWGN noise. Let $C \leq G$ represent the length of the channel delay normalized by the sampling period[2]. In the absence of timing errors, y_i, the received signal after discarding the cyclic prefix, is as follows: $y_i = \vartheta_i + w_i$ for $0 \leq i \leq N-1$, where $\vartheta_i = \sum_{k=0}^{C} h_k x_{((i-k))_N}$. Consider a case of timing error of m sampling periods. $m > 0$ and $m < 0$ denote timing errors of m to the right and left side respectively.

Case of timing errors to the right $(m > 0)$

In this case, an error of m sampling periods to the right side has occurred. Then, the terms $y_0, y_1, \ldots, y_{m-1}$ are missed and instead m data points of the next OFDM symbol are erroneously selected. The received signal can thus be written as follows:

$$y_i^r = \vartheta_{((i+m))_N} \times \gamma_i^r + s_i + w_i^r \quad 0 \leq i \leq N-1 \qquad (1)$$

where y_i^r is a sample of the received signal for $m > 0$ and $s_i = \begin{cases} 0 & 0 \leq i \leq N-m-1 \\ y_{pf}^{next}(i-N+m) & else \end{cases}$, with $y_{pf}^{next}(i)$ representing the i^{th} sample of the output cyclic prefix of the next OFDM symbol (excluding the effect of AWGN), $\gamma_i^r = \begin{cases} 1 & 0 \leq i \leq N-m-1 \\ 0 & N-m \leq i \leq N-1 \end{cases}$ and w_i^r is a sample of AWGN. Then Y_i^r, the FFT of y_i^r, will be,

$$Y_i^r = \underbrace{\frac{\Gamma_0^r}{N} H_i X_i e^{\frac{j2\pi mi}{N}}}_{I_i^r} + \underbrace{\sum_{k=1}^{N-1} \frac{\Gamma_k^r}{N} H_{((i-k))_N} X_{((i-k))_N} e^{\frac{j2\pi m(i-k)}{N}} + S_i}_{ICI \;\&\; ISI} + W_i^r \qquad (2)$$

where S_i and W_i^r are the FFTs of s_i and w_i^r respec-

[2] Then the channel would have $C+1$ taps.

tively. Γ_i^r, the FFT of γ_i^r, is $\Gamma_i^r = \begin{cases} \frac{1-e^{\frac{j2\pi im}{N}}}{1-e^{-\frac{j2\pi i}{N}}} & i \neq 0 \\ N-m & i=0 \end{cases}$.

I_i^r represents the ICI resulting from multiplication of $\vartheta_{((i+m))_N}$ by γ_i^r in Eq. 1. Let $\sigma_{h_i}^2 = E|h_i^2|$. Since $\overline{X_i X_j^*} = \sigma_X^2 \delta_{i,j}$, $\overline{H_i H_j^*} = \sigma_H^2 \delta_{i,j}$, with $\sigma_H^2 = \sum_{i=0}^{C} \sigma_{h_i}^2$, and $\sum_{k=1}^{N-1} |\Gamma_k^r|^2 = N \sum_{k=0}^{N-1} |\gamma_k^r|^2 - \Gamma_0^2 = m \times (N-m)$. Then the power of I_i^r can be easily calculated to be $\sigma_{I^r}^2 = \frac{(N-m)m}{N^2} \sigma_X^2 \sigma_H^2$. Next, σ_S^2, the power of S_i, is calculated. Since y_{pf}^{next} includes delayed replicas of the current OFDM symbol from the delayed paths, S_i has an ICI term in addition to an ISI term.
$S_i = \sum_{k=0}^{m-1} s_{N-m+k} e^{-\frac{j2\pi i(k-m)}{N}}$, where s_{N-m+k} can be written as follows for $0 \leq k \leq m-1$:

$$s_{N-m+k} = \overbrace{\sum_{k'=k+1}^{C} h_{k'} x_{N-k'+k}}^{ICI} + \overbrace{\sum_{k'=0}^{k} h_{k'} x_{N-G+k-k'}^{next}}^{ISI} \tag{3}$$

where x_i^{next} represents the i^{th} time-domain transmitted data point of the next OFDM symbol. Since x and x^{next} are independent, σ_S^2 will be as follows:

$$\sigma_S^2 = \frac{\sigma_X^2}{N} \sum_{k=0}^{m-1} \sum_{k'=k+1}^{C} \sigma_{h_{k'}}^2 + \frac{\sigma_X^2}{N} \sum_{k=0}^{m-1} \sum_{k'=0}^{k} \sigma_{h_{k'}}^2 = \frac{m \sigma_X^2 \sigma_H^2}{N} \tag{4}$$

Using the definition of I_i^r and S_i,

$$\overline{I_i^r S_i^*} = \frac{1}{N} \sum_{k=0}^{m-1} \sum_{k'=k+1}^{C} \sum_{k''=1}^{N-1} \Gamma_{k''}^r \overline{h_{k'}^* H_{((i-k''))_N}} \times E(x_{N-k'+k}^* X_{((i-k''))_N}) e^{-\frac{j2\pi(mk''-ik)}{N}} \tag{5}$$

Using $\overline{h_{k'}^* H_{((i-k''))_N}} = \sigma_{h_{k'}}^2 e^{-\frac{j2\pi k'(i-k'')}{N}}$ and $\overline{x_{N-k'+k}^* X_{((i-k''))_N}} = \frac{\sigma_X^2}{N} e^{-\frac{j2\pi(k-k')(i-k'')}{N}}$,

$$\begin{aligned} \overline{I_i^r S_i^*} &= \frac{\sigma_X^2}{N^2} \sum_{k=0}^{m-1} \sum_{k'=k+1}^{C} \sigma_{h_{k'}}^2 \sum_{k''=1}^{N-1} \Gamma_{k''}^r e^{\frac{j2\pi k''(k-m)}{N}} \\ &= -\frac{N-m}{N^2} \sigma_X^2 \sum_{k=0}^{m-1} \sum_{k'=k+1}^{C} \sigma_{h_{k'}}^2 \end{aligned} \tag{6}$$

Therefore, the total interference power and average Signal-to-Interference Ratio for $m > 0$ (SIR_{ave}^r) will be as follows:

$$\begin{aligned} \frac{\overline{|I_i^r + S_i|^2}}{\sigma_X^2} &= \frac{(2N-m)m\sigma_H^2}{N^2} - \sum_{k=0}^{m-1} \sum_{k'=k+1}^{C} \frac{2(N-m)\sigma_{h_{k'}}^2}{N^2} \\ SIR_{ave}^r &= \frac{(N-m)^2}{(2N-m)m - 2\frac{N-m}{\sigma_H^2} \sum_{k=0}^{m-1} \sum_{k'=k+1}^{C} \sigma_{h_{k'}}^2} \end{aligned} \tag{7}$$

Case of timing errors to the left $(m < 0)$

In this case, due to the presence of the cyclic prefix, number of data points that are missed is $d = max(C - (G+m), 0)$ which can be less than $-m$. Therefore,

$$y_i^l = \vartheta_{((i+m))_N} \times \gamma_i^l + \psi_i + w_i^l \quad 0 \leq i \leq N-1 \tag{8}$$

Where y_i^l is sample of the received signal for $m < 0$, $\psi_i = \begin{cases} y_{pf}(G+m+i) & 0 \leq i \leq d-1 \\ 0 & d \leq i \leq N-1 \end{cases}$ with $y_{pf}(i)$ representing the i^{th} sample of the output cyclic prefix of the current OFDM symbol (excluding the effect of AWGN), $\gamma_i^l = \begin{cases} 0 & 0 \leq i \leq d-1 \\ 1 & d \leq i \leq N-1 \end{cases}$ and w_i^l is a sample of AWGN.

Similar to the case of $m > 0$, it can be shown that,

$$SIR_{ave}^l = \frac{(N-d)^2}{(2N-d)d - 2\frac{N-d}{\sigma_H^2} \sum_{k=0}^{d-1} \sum_{k'=0}^{G+m+k} \sigma_{h_{k'}}^2} \tag{9}$$

Simulation and Analysis Results for average SIR

The effect of timing errors on the performance of an OFDM system for both $m > 0$ and $m < 0$ cases is simulated. In this simulation $N = 512$ and $G = 52$. The power-delay profile of channel#1 is [0.1214 0.1529 0 0 0.1924 0.1529 0 0.1160 0.0965 0.0766 0.0609 0.0305]. Fig. 1 shows SIR_{ave} resulting from the analysis and simulation for this channel. Since the length of channel#1 spans only 21% of the guard interval, the interference power will be zero ($d = 0$) for $-42 < m < 0$. Furthermore, the level of interference for $m > 0$ and $m < 0$ is different. This non-symmetric effect of the timing errors can be seen from Fig. 1. Moreover, the results of the analysis and simulation match well which confirms the derived expressions.

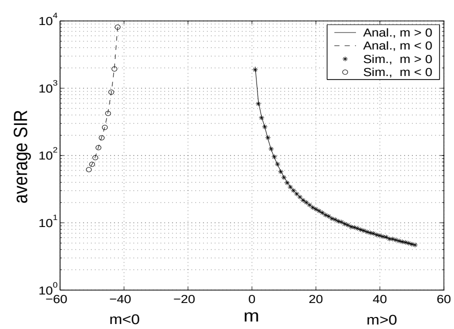

Figure 1. Average SIR vs. m for channel#1

Effect of timing errors on pilot-aided channel estimator, case of $m > 0$

In this section we explore the effect of timing errors on the performance of a pilot-aided channel estimator. Consider the case that $L \geq \nu$ equally spaced pilot tones are inserted among subcarriers where ν represents maximum predicted normalized length of the channel delay spread. Then to estimate the channel at subcarriers in between the pilot tones an IFFT in the base of L, zero padding and an FFT in the base of N should be performed[3] [3]. This section analyzes the effect of timing errors on this channel estimator. Consider the case of $m \geq 0$. Let $H_{eq}(i) = \sum_k h_{eq}(k)e^{-\frac{j2\pi ik}{N}}$ represent the relationship between X_i and Y_i^r. Using Eq. 2, $H_{eq}^r(i) = \frac{\Gamma_0^r}{N} H_i e^{\frac{j2\pi mi}{N}}$. Then, $h_{eq}^r(k) = \frac{\Gamma_0^r}{N} h_{((k+m))_N}$. As can be seen, a timing synchronization error of $m > 0$ introduces a rotation of m sampling periods in the base of N in the equivalent channel. This rotation will result in the expansion of the channel beyond its maximum predicted length. Even one error to the right side will result in an equivalent channel of length $N-1$. This will degrade the performance of the channel estimator, as it assumes an equivalent channel that spans L sampling periods at maximum [2]. To see the effect of timing errors on channel estimation analytically, consider the case that L equally-spaced frequency-domain pilot tones, $X_{pilot}(l_i)$ for $0 \leq i \leq L-1$, are inserted among the sub-carriers where $l_i = \frac{i \times N}{L}$. Then for $0 \leq i \leq L-1$,

$$\hat{H}_{eq}^r(l_i) = \frac{Y_{l_i}^r}{X_{pilot}(l_i)} = H_{eq}^r(l_i) + \frac{I_{l_i}^r + S_{l_i} + W_{l_i}^r}{X_{pilot}(l_i)} \quad (10)$$

Through an IFFT of length L, the estimate of the channel in time-domain would be

$$\hat{h}_{eq}^r(k) = \underbrace{\frac{\Gamma_0^r}{N} h_{((k+m))_L}}_{rotation} + \underbrace{u_k^r}_{Interference} + \underbrace{v_k^r}_{AWGN} \quad (11)$$

Define U_i^r and V_i^r as follows: $U_i^r = \sum_{z=0}^{L-1} \alpha_{i,z} \frac{I_{l_z}^r + S_{l_z}}{X_{pilot}(l_z)}$ & $V_i^r = \sum_{z=0}^{L-1} \alpha_{i,z} \frac{W_{l_z}^r}{X_{pilot}(l_z)}$. Then u^r and v^r will be the IFFTs of U^r and V^r respectively and $\alpha_{i,z} = \frac{1}{L} \sum_{g=0}^{L-1} e^{j2\pi g(\frac{z}{L} - \frac{i}{N})}$. As can be seen from Eq. 11, there are three factors contributing to channel estimation error: effect of rotation, interference and noise. The first factor occurs because the equivalent channel has a rotation in the base of N while the estimated equivalent channel has a rotation in the base of L. Since L is chosen based on ν, it is typically considerably smaller than N. Therefore, the mismatch between the equivalent channel and the estimated equivalent channel can be considerable, solely due to the first factor [2]. It will result in a mismatch in the location of the first m taps of the original channel. Since these taps are typically strong, this can result in a considerable performance degradation of the channel estimator [2]. To analytically assess the contribution of each of the aforementioned factors, an expression for average power of channel estimation error is derived next. Channel estimation error will be,

$$\Delta H_{eq}^r(i) = \underbrace{\sum_{k=0}^{m-1} \beta_{i,k}^r h_k}_{rotation} + \underbrace{U_i^r}_{interference} + \underbrace{V_i^r}_{AWGN} \quad (12)$$

where ΔH_{eq}^r represents the frequency-domain channel estimation error for $m > 0$ and $\beta_{i,k}^r = \frac{N-m}{N} \times e^{\frac{-j2\pi i(k-m)}{N}} \times (1 - e^{-\frac{j2\pi iL}{N}})$. First $\overline{U_i^r h_k^*}$ is calculated.

$$\begin{aligned} \overline{U_i^r h_k^*} &= \sum_{z=0}^{L-1} \alpha_{i,z} \overline{\left[\frac{S_{l_z} h_k^*}{X_{pilot}(l_z)}\right]} \\ &= \left[\sum_{k''=1}^{m-1} \frac{k''}{N} \overline{h_{k''} h_k^*} + \frac{m}{N} \sum_{k''=m}^{C} \overline{h_{k''} h_k^*}\right] \times \\ & \sum_{z=0}^{L-1} \alpha_{i,z} e^{-\frac{j2\pi(k''-m)z}{L}} \end{aligned} \quad (13)$$

Since $0 \leq k \leq m-1$, the second term in the bracket is zero. Therefore,

$$\overline{U_i^r h_k^*} = \begin{cases} \frac{k}{N} \sigma_{h_k}^2 \sum_{z=0}^{L-1} \alpha_{i,z} e^{-\frac{j2\pi(k-m)z}{L}} & k \geq 1 \\ 0 & k = 0 \end{cases} \quad (14)$$

$$\begin{aligned} \sum_{k=0}^{m-1} \beta_{i,k}^{*r} \overline{U_i^r h_k^*} &= \sum_{k=1}^{m-1} \frac{k}{N} \sigma_{h_k}^2 \beta_{i,k}^{*r} \sum_{z=0}^{L-1} \alpha_{i,z} e^{\frac{j2\pi(m-k)z}{L}} \\ &= \sum_{k=1}^{m-1} \frac{k}{NL} \sigma_{h_k}^2 \beta_{i,k}^{*r} \sum_{z'=0}^{L-1} \sum_{z=0}^{L-1} e^{j2\pi\left[\frac{(z'-k+m)z}{L} - \frac{z'i}{N}\right]} \end{aligned} \quad (15)$$

Since $1 \leq z' - k + m \leq L - 2 + m$, therefore, Eq. 15 will be non-zero if $z' - k + m = L$ and $m \geq 2$. Since for any k in the given range of Eq. 15, $0 \leq L + k - m \leq L - 1$ for $m \leq L + 1$, there will always be a z' that will make $z' - k + m = L$. Therefore,

$$\sum_{k=0}^{m-1} \beta_{i,k}^{*r} \overline{U_i^r h_k^*} = \frac{1}{N} \sum_{k=1}^{m-1} k \sigma_{h_k}^2 \beta_{i,k}^{*r} e^{-\frac{j2\pi(L+k-m)i}{N}} \quad (16)$$

Then $\sum_{k=0}^{m-1} \beta_{i,k}^{*r} \overline{U_i^r h_k^*} = \frac{(N-m) \times (e^{-\frac{j2\pi Li}{N}} - 1) \sum_{k=1}^{m-1} k \sigma_{h_k}^2}{N^2}$ for $m \geq 2$ and zero for $m = 1$. Noting that the Gaussian noise term is independent of the first two terms on the right hand side of Eq. 12, the following expression can be derived for channel estimation error:

$$\begin{aligned} \overline{|\Delta H_{eq}^r(i)|^2} &= \sum_{k=0}^{m-1} |\beta_{i,k}^r|^2 \sigma_{h_k}^2 + \sigma_{U_i^r}^2 + \sigma_{V_i^r}^2 + \\ & 2Real(\sum_{k=0}^{m-1} \beta_{i,k}^{*r} \overline{U_i^r h_k^*}) \end{aligned} \quad (17)$$

where $|\beta_{i,k}^r|^2 = 4\frac{(N-m)^2}{N^2} sin^2(\frac{\pi i L}{N})$ and $\sigma_{V_i^r}^2 = \frac{\sigma_W^2}{\sigma_X^2}$. An

[3] There are other (sub-optimum) ways of estimating channel in between the sub-carriers (like linear interpolation). However, the performance of these methods degrades as delay spread increases.

expression for $\sigma^2_{U_i^r}$ is derived next.

$$\sigma^2_{U_i^r} = \sum_{z=0}^{L-1}\sum_{z'=0,z'\neq z}^{L-1} \alpha_{i,z}\alpha^*_{i,z'} E\left[\frac{(I^r_{l_z}+S_{l_z})\times(I^{r*}_{l_{z'}}+S^*_{l_{z'}})}{X_{pilot}(l_z)X^*_{pilot}(l_{z'})}\right] + \sum_{z=0}^{L-1}|\alpha_{i,z}|^2\frac{\overline{|I^r_{l_z}+S_{l_z}|^2}}{\sigma^2_X} \tag{18}$$

It can be easily shown that $E\left[\frac{I^r_{l_z}\times I^{r*}_{l_{z'}}}{X_{pilot}(l_z)X^*_{pilot}(l_{z'})}\right]=0$ for $z\neq z'$. For $g\neq g'$, we can write,

$$\overline{\left[\frac{I^r_g S^*_{g'}}{X_g X^*_{g'}}\right]} = \sum_{k=0}^{m-1}\sum_{k'=k+1}^{C}\sum_{k''=1}^{N-1}\frac{\Gamma^r_{k''}}{N}\overline{H_{((g-k''))_N}h^*_{k'}}\times \overline{\left[\frac{X_{((g-k''))_N}x^*_{N-k'+k}}{X_g X^*_{g'}}\right]} e^{\frac{j2\pi[g'(k-m)+m(g-k'')]}{N}} \tag{19}$$

Since $k''\geq 1$, $\overline{\left[\frac{X_{((g-k''))_N}x^*_{N-k'+k}}{X_g X^*_{g'}}\right]} = \frac{1}{N}\sum_{k'''=0}^{N-1}\overline{\left[\frac{X^*_{k'''}X_{((g-k''))_N}}{X_g X^*_{g'}}\right]} e^{\frac{-j2\pi k'''(k-k')}{N}} = 0$ for $g\neq g'$.
Therefore $E\left[\frac{I^r_g S^*_{g'}}{X_g X^*_{g'}}\right]=0$ for $g\neq g'$. Then,

$$\sigma^2_{U_i^r} = \sum_{z=0}^{L-1}\sum_{z'=0,z'\neq z}^{L-1} \alpha_{i,z}\alpha^*_{i,z'} E\left[\frac{S_{l_z}S^*_{l_{z'}}}{X_{pilot}(l_z)X^*_{pilot}(l_{z'})}\right] + \sum_{z=0}^{L-1}|\alpha_{i,z}|^2\frac{\overline{|I^r_{l_z}+S_{l_z}|^2}}{\sigma^2_X} \tag{20}$$

For an arbitrary k'' and g'' where $k''\neq g''$, the following expression can be written,

$$E\left[\frac{S_{k''}S^*_{g''}}{X_{k''}X^*_{g''}}\right] = E\left[\frac{\sum_{k=0}^{m-1}s_{N-m+k}e^{\frac{-j2\pi k''(k-m)}{N}}\sum_{g=0}^{m-1}s^*_{N-m+g}e^{\frac{j2\pi g''(g-m)}{N}}}{X_{k''}X^*_{g''}}\right] = \sum_{k=0}^{m-1}\sum_{g=0}^{m-1}\sum_{k'=max(k,g)+1}^{C}\frac{\sigma^2_{h_{k'}}}{N^2}e^{\frac{j2\pi(m-k')(k''-g'')}{N}} \tag{21}$$

Therefore, the first term on the right hand side of Eq. 20 can be written as follows:

$$\sum_{z=0}^{L-1}\sum_{z'=0,z'\neq z}^{L-1}\alpha_{i,z}\alpha^*_{i,z'}E\left[\frac{S_{l_z}S^*_{l_{z'}}}{X_{pilot}(l_z)X^*_{pilot}(l_{z'})}\right] = \frac{1}{N^2L^2}\sum_{k=0}^{m-1}\sum_{g=0}^{m-1}\sum_{k'=max(k,g)+1}^{C}\sigma^2_{h_{k'}}\times \sum_{g'=0}^{L-1}\sum_{g''=0}^{L-1}e^{-\frac{j2\pi i(g'-g'')}{N}}\times \left(\sum_{z=0}^{L-1}e^{\frac{j2\pi z(g'-k'+m)}{L}}\sum_{z'=0}^{L-1}e^{-\frac{j2\pi z'(g''-k'+m)}{L}} - \sum_{z=0}^{L-1}e^{\frac{j2\pi z(g'-g'')}{L}}\right) \tag{22}$$

Since $-L < -C+m \leq g'-k'+m \leq L-2+m$, for $m\leq L+1$ then $-L<-C+m\leq g'-k'+m\leq 2L-1$. Therefore, the first two sums ($\sum$) inside the parenthesis will have non-zero values only for $g'-k'+m=L$ and 0. To have $g'-k'+m=0$ and $g'-k'+m=L$, then $0\leq k'-m\leq L-1$ and $-L\leq k'-m\leq -1$ should hold respectively. Therefore, for $-L\leq k'-m\leq L-1$, there will always be a g' in the range of $0\leq g'\leq L-1$.

In Eq. 22, $1\leq k'\leq C$. For any k' in this range, $-L\leq k'-m\leq L-1$ (assuming that $m\leq L+1$, which is a reasonable assumption). Then for any k' of Eq. 22, there will be one and only one g' that would result in $g'-k'+m$ being a multiple of L (here only 0 or L). Therefore,

$$\sum_{z=0}^{L-1}\sum_{z'=0,z'\neq z}^{L-1}\alpha_{i,z}\alpha^*_{i,z'}E\left[\frac{S_{l_z}S^*_{l_{z'}}}{X_{pilot}(l_z)X^*_{pilot}(l_{z'})}\right]=0 \tag{23}$$

Then,

$$\sigma^2_{U_i^r} = \sum_{z=0}^{L-1}|\alpha_{i,z}|^2\frac{\overline{|I^r_{l_z}+S_{l_z}|^2}}{\sigma^2_X} \tag{24}$$

The normalized power of channel estimation error at i^{th} sub-carrier, $Ch^r_{error,norm}(i)$, can then be written as follows:

$$Ch^r_{error,norm}(i) = \frac{\overline{|\Delta H^r_{eq}(i)|^2}}{\overline{|H^r_{eq}(i)|^2}} = \underbrace{4\frac{\sum_{k=0}^{m-1}\sigma^2_{h_k}-\frac{\sum_{k=1}^{m-1}k\sigma^2_{h_k}}{N-m}\varpi(m-2)}{\sum_{k=0}^{C}\sigma^2_{h_k}}sin^2(\frac{\pi iL}{N})}_{factor\#1:rotation} + \underbrace{\frac{1}{SIR^r_{ave}(m)}}_{interference} + \underbrace{\frac{1}{SNR^r_{ave}(m)}}_{noise} \tag{25}$$

where $SNR^r_{ave}(m)=\frac{(N-m)^2\sigma^2_X\sigma^2_H}{N^2\sigma^2_W}$ and $\varpi(z)=1$ for $z\geq 0$ and zero otherwise. Since, with high probability, m is much smaller than N, Eq. 25 can be tightly approximated as long as $\frac{k}{N-m} << 1$ for $1\leq k\leq m-1$. For $m << \frac{N+1}{2}$, $\frac{m-1}{N-m} << 1$. Therefore, Eq. 25 can be tightly approximated as follows:

$$Ch^r_{error,norm}(i) \approx \underbrace{4\Upsilon^r_{\%}(m)sin^2(\frac{\pi iL}{N})}_{factor\#1:rotation} + \underbrace{\frac{1}{SIR^r_{ave}(m)}}_{interference} + \underbrace{\frac{1}{SNR^r_{ave}(m)}}_{noise} \tag{26}$$

where $\Upsilon^r_{\%}(m)=\frac{\sum_{k=0}^{m-1}\sigma^2_{h_k}}{\sum_{k=0}^{C}\sigma^2_{h_k}}$ represents the ratio of the power of the misplaced channel taps to the total power of the channel. Let factor#1 represent the effect of rotation (first term on the right-hand side of Eq. 26). As can be seen, it does not affect those sub-channels carrying pilot tones. However, it results in a considerable increase of error for other sub-carriers particularly for those at $i=z_{odd}\times ceil(\frac{N}{2L})$, where z_{odd} represents odd integers. Examining $Ch_{error,ratio}$ for different values of m and $\Upsilon_{\%}$ in a reasonable SNR environment shows that factor#1 is the dominant factor with high probability [2].

Effect of timing errors on the trigonometric interpolator, case of $m<0$

Similar expressions can be derived for the case of $m < 0$. It can be shown that $H^l_{eq}(i) = \frac{\Gamma^l_0}{N} H_i e^{\frac{j2\pi mi}{N}}$ where $\Gamma^l_0 = \sum_{k=0}^{N-1} \gamma^l_k$. Then, $h^l_{eq}(k) = \frac{\Gamma^l_0}{N} h_{((k+m))_N}$. In contrast to the case for $m > 0$, where even one error to the right resulted in an equivalent channel of length $N-1$, the equivalent channel length for $m < 0$ varies depending on the length of the channel. For instance, for a channel of length $C \leq \nu$, the equivalent channel length will be $C - m$ for $m \leq -1$. Therefore for $C - \nu \leq m \leq -1$, the equivalent length would still be less than or equal to ν, which poses no problem for the channel estimator. Furthermore, the mismatch is in the location of the last m taps of the original channel and these taps typically have the lowest amplitudes. Depending on the length of the channel, these samples may be solely occupied by noise and/or interference. Therefore, it can be seen again that errors to the left side may not cause any performance degradation, depending on the length of the channel delay spread, guard interval and number of pilots. Following the same procedure, an analytical expression can be found for the case of $m < 0$,

$$Ch^l_{error,norm}(i) = \frac{4\sum_{k=L+m}^{L-1} \sigma^2_{h_k}}{\sum_{k=0}^{C} \sigma^2_{h_k}} sin^2(\frac{\pi iL}{N}) - \frac{\sum_{k=L+m}^{G+m+d-1} (k-G-m)\sigma^2_{h_k}}{(N-d)\times\sum_{k=0}^{C} \sigma^2_{h_k}} \varpi(d-2) sin^2(\frac{\pi iL}{N}) + \frac{1}{SIR^l_{ave}(m)} + \frac{1}{SNR^l_{ave}(m)} \quad (27)$$

Performing a similar approximation, it can be shown that,

$$Ch^l_{error,norm}(i) \approx \underbrace{sin^2(\frac{\pi iL}{N})\Upsilon^l_{\%}(m)}_{factor\#1:rotation} + \underbrace{\frac{1}{SIR^l_{ave}(m)}}_{interference} + \underbrace{\frac{1}{SNR^l_{ave}(m)}}_{noise} \quad (28)$$

where $SNR^l_{ave}(m) = \frac{(N-d)^2 \sigma^2_X \sigma^2_H}{N^2 \sigma^2_W}$ and $\Upsilon^l_{\%}(m) = \frac{\sum_{k=L+m}^{L-1} \sigma^2_{h_k}}{\sum_{k=0}^{C} \sigma^2_{h_k}}$ represents the ratio of the power of the misplaced channel taps to the total power of the channel.

Analytical and simulation results of the effect of timing errors on channel estimation

Fig. 2 shows normalized channel estimation error as a function of sub-carrier for channel#2, which has the following power-delay profile: [0.1214 0.1969 0.0987 0.0784 0.1242 0.1969 0.0987 0.0623 0.0197]. System specifications are as follows for this result: $N = 892$ and $G = L = 223$. As can be seen, for both cases of $m > 0$ and $m < 0$, factor#1 (effect of rotation) contributes essentially all of the channel estimation error. Furthermore, it can be seen that in the presence of timing errors, channel estimation error has quite high values. For example $Ch_{error,norm} = 1$ means normalized channel estimation error of 100%. Finally, the agreement of analysis and simulation results can be clearly seen.

CONCLUSIONS

We derived exact expressions for the interference terms (ICI and ISI), their corresponding average powers and the resulting average SIR in the presence of timing errors in an OFDM system. The effect of timing errors on the performance of the pilot-aided channel estimator was then analyzed. Analytical expressions for average power of channel estimation error in the presence of timing errors were derived. Simulation results confirmed the derivations. Furthermore, the results showed super-sensitivity of pilot-aided channel estimators to timing synchronization errors. This super-sensitivity can be exploited to correct for timing errors as is shown in [2].

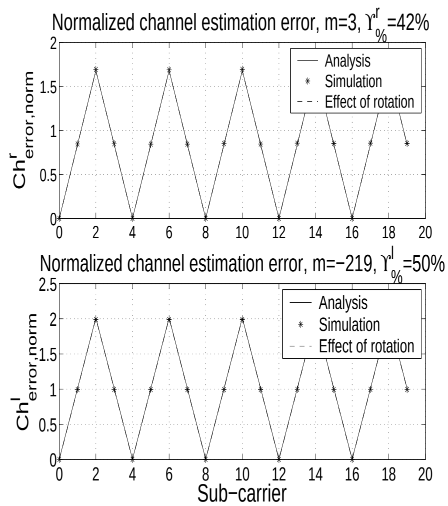

Figure 2. Channel estimation error for channel#2

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