# Performance Analysis of Transmit Beamforming for MISO Systems with Imperfect Feedback

Yogananda Isukapalli, *Student Member*, *IEEE*, Ramesh Annavajjala, *Member*, *IEEE*, and Bhaskar D. Rao, *Fellow*, *IEEE*

***Abstract*—In this paper, we analyze the performance of transmit beamforming on multiple-antenna Rayleigh fading channels with imperfect channel feedback. We characterize the feedback imperfections in terms of noisy channel estimation, feedback delay, and finite-rate channel quantization. We develop a general framework, that is valid for an arbitrary two-dimensional linear modulation, to capture the aforementioned imperfections and derive the symbol and bit error probability expressions for both $M$-PSK and $M$-ary rectangular QAM constellations with Gray code mapping. We show that the proposed analytical formulation is valid for a frequency-domain duplexing system with/without finite-rate channel quantization and a time-domain duplexing system. We validate the accuracy of the analysis through simulations, and assess the relative effects of channel estimation inaccuracy, feedback delay, and finite-rate quantization on the symbol and bit error performances for various constellations.**

*Index Terms*: MISO systems, transmit beamforming, noisy channel estimate, feedback delay, channel quantization.

## I. Introduction

In a multiple input and single output (MISO) system, if the channel state information (CSI) is available at the transmitter, one can achieve both the diversity and the array gains with transmit beamforming via maximal ratio transmission (MRT) [1]. Transmit beamforming for MISO systems is an active area of research. In [2], the authors consider efficient use of CSI for transmit beamforming. Performance of MRT with maximal ratio combining is analyzed in [3] for MIMO channels with binary signaling. An information theoretic approach to transmit beamforming with imperfect feedback is presented in [4] and [5]. With MRT and BPSK modulation, the effect of delay with perfect channel estimation (PCE) at the receiver is investigated in [6], whereas [7] studies the effect of imperfect channel estimation (ICE) without feedback delay. In [8], the authors extend the analysis of [6] accounting for the effects of ICE. The effect of feedback delay and feedback errors on the receiver signal-to-noise ratio (SNR) is investigated in [9] for phase-only feedback, whereas [10] analyzes the bit error probability (BEP) degradation with BPSK due to feedback errors with selection and co-phasing feedback schemes. Finally, the design and analysis of transmit beamforming under finite-rate constraints are presented in [11]-[14].

While the aforementioned works considered either BPSK (or QPSK) with ICE and perfect quantization [6]-[10], or PCE with finite-rate quantization [11]-[14], combined effects of various channel imperfections for general modulations is not yet investigated. This contribution is targeted to fill in this important void. In this work, we present a general framework for the performance analysis with imperfect CSI feedback. The channel imperfections are characterized in terms of noisy channel estimation, quantization of CSI, and feedback delay. This formulation is shown to be applicable for any linear two-dimensional modulation schemes. Our analytical framework encompasses three popular MISO system models, namely, frequency-domain duplexing (FDD), FDD with finite-rate quantization of CSI (FDDQ), and time-domain duplexing (TDD). We analyze average symbol error probability (SEP) and BEP of $M$-PSK and $M$-ary rectangular QAM with Gray code mapping.

The rest of this paper is organized as follows. In Section II, we present the modeling of imperfect feedback. The decision variable (DV) at the receiver is also given in Section II. The average SEP and BEP expressions for $M$-PSK and rectangular $M$-QAM are derived in Section III. Numerical and simulation results are presented in Section IV. We conclude this paper in Section V.

Notation: Small and upper case bold letters indicate vector and matrix respectively. $E(.)$, $(.)^H$, $|.|$, $\bar{(.)}$, and $\|.\|$ denote expectation, Hermitian, absolute value, complex conjugate, and norm respectively. $\mathbf{x} \sim \mathcal{NC}(\mu, \mathbf{\Sigma})$ indicates a circularly symmetric complex Gaussian (CSCG) random variable with mean $\mu$ and covariance $\mathbf{\Sigma}$. Some key variables: $t$, number of transmit antennas, $\rho_d$, delay-only correlation coefficient, $\rho_e$, estimation-only correlation coefficient, $B$, number of feedback bits.

## II. Modeling of Imperfect Feedback

We consider a MISO system with $t$ antennas at the base station (BS) and one antenna at the mobile station (MS). The channel is modeled as a frequency-flat Rayleigh fading channel. The channel at time $k$ is $\mathbf{h}[k] = [h_1[k], h_2[k], \ldots, h_t[k]]^T$, $\mathbf{h} \sim \mathcal{NC}(\mathbf{0}, \Omega\mathbf{I})$. The transmitted two-dimensional modulation symbol at time $k$ is denoted by $s_m[k]$, $E[|s_m[k]|^2] = E_s$. Let

Yogananda Isukapalli and Bhaskar D. Rao are with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92092 (e-mail: yoga@ucsd.edu, brao@ece.ucsd.edu). Ramesh Annavajjala is with the Mitsubishi Electric Research Laboratories, Cambridge, MA 02139 (e-mail:ramesh.annavajjala@gmail.com). This research was supported in part by the U. S. Army Research Office under the Multi-University Research Initiative (MURI) grant-W911NF-04-1-0224.

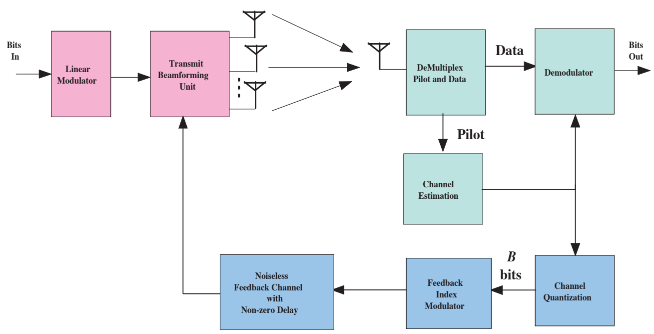


Fig. 1. Block Diagram of a transmit beamforming MISO system with imperfect feedback. Feedback imperfections include inaccurate channel estimation, channel quantization and feedback delay.

$\mathbf{w}[k]$ be the unit norm beamforming vector (BV) at the BS at time $k$. Then, the received signal at the MS at time $k$ is

$$y[k] = \mathbf{h}^H[k]\mathbf{w}[k]s_m[k] + \eta[k], \tag{1}$$

where $\eta[k] \sim \mathcal{NC}\left(0, \sigma_n^2\right)$. We now dwell in detail on three popular channel feedback approaches and derive the DV at the input of the demodulator.

### A. FDD System

Let $\widehat{\mathbf{h}}[k]$ be the channel estimate at the MS at time $k$, $\widehat{\mathbf{h}} \sim \mathcal{NC}\left(\mathbf{0}, \Lambda\mathbf{I}\right)$. The MS feeds back the normalized version of $\widehat{\mathbf{h}}[k]$ to the BS. Assuming a feedback delay of $D$, the normalized version of the delayed estimate forms the BV at the transmitter [11]-[14]

$$\mathbf{w}[k] = \frac{\widehat{\mathbf{h}}[k-D]}{\|\widehat{\mathbf{h}}[k-D]\|}\,. \tag{2}$$

Assuming that $\mathbf{h}[k]$ and $\widehat{\mathbf{h}}[k-D]$ are jointly Gaussian, we can relate them as [15]

$$\mathbf{h}[k] = \rho\sqrt{\frac{\Omega}{\Lambda}}\,\widehat{\mathbf{h}}[k-D] + \sqrt{\left(1-|\rho|^2\right)\Omega}\,\varepsilon[k-D], \tag{3}$$

where $\varepsilon \sim \mathcal{NC}\left(\mathbf{0}, \mathbf{I}\right)$ and $\rho$ is the complex correlation coefficient, $\rho = E\left[h_i[k]\widehat{h}_i^*[k-D]\right]/\sqrt{\Omega\Lambda}$, $i = 1, \ldots, t$. Then, using (3), the received signal (1) can be written as

$$y[k] = \bar{\rho}\sqrt{\frac{\Omega}{\Lambda}}\,\|\widehat{\mathbf{h}}[k-D]\|\,s_m[k] + \zeta[k], \tag{4}$$

conditioned on $|s_m[k]|$, $\zeta \sim \mathcal{NC}\left(0, \sigma_s^2\right)$, where $\sigma_s^2 = \sigma_n^2 + |s_m[k]|^2\Omega(1-|\rho|^2)$. Since the MS knows $\|\widehat{\mathbf{h}}[k-D]\|$, the DV is given by (a DV based approach to performance evaluation can be found in [16])

$$r[k] \triangleq \frac{y[k]}{\|\widehat{\mathbf{h}}[k-D]\|} = \bar{\rho}\sqrt{\frac{\Omega}{\Lambda}}\,s_m[k] + \tilde{\zeta}[k], \tag{5}$$

where, conditioned on $|s_m[k]|$ and $\|\widehat{\mathbf{h}}[k-D]\|$, $\tilde{\zeta} \sim \mathcal{NC}\left(0, \sigma_h^2\right)$, $\sigma_h^2 = \sigma_s^2/\beta\Lambda$, where $\beta \triangleq \|\widehat{\mathbf{h}}[k-D]\|^2/\Lambda$. Clearly, $\beta$ is gamma distributed with the probability density function (pdf)

$$f_\beta(x) = \frac{e^{-x}x^{t-1}}{\Gamma(t)} \quad x \geq 0,$$

where $\Gamma(n)$ is the standard Gamma function. From (5), we notice that the effect of ICE and delay on the DV is that of scaling the transmitted symbol $s_m[k]$ by *an unknown* complex number $\bar{\rho}\sqrt{\Omega/\Lambda}$ and introducing *symbol dependent non-Gaussian noise* $\tilde{\zeta}[k]$ (note that as mentioned earlier $\tilde{\zeta}[k]$ is conditionally Gaussian).

We now describe the structure of the correlation coefficient, $\rho$, due to delay and estimation and show how it relates to delay-only correlation coefficient, $\rho_d$, and estimation-error-only correlation coefficient, $\rho_e$. With delay and no estimation errors, we have

$$h_i[k] = \rho_d h_i[k-D] + \sqrt{(1-|\rho_d|^2)\Omega}\,\nu_i[k], \tag{6}$$

where $\rho_d = E\left[h_i[k]h_i^*[k-D]\right]/\Omega$, $\nu_i \sim \mathcal{NC}\left(0,1\right)$ and is uncorrelated with $h_i$ . Estimation errors and no delay scenario allows us to write

$$h_i[k] = \rho_e\sqrt{\frac{\Omega}{\Lambda}}\,\widehat{h}_i[k] + \sqrt{\left(1-|\rho_e|^2\right)\Omega}\,\tilde{\nu}_i[k], \tag{7}$$

where $\rho_e = E\left[h_i[k]\widehat{h}_i^*[k]\right]/\sqrt{\Omega\Lambda}$, $\tilde{\nu}_i \sim \mathcal{NC}\left(0,1\right)$ and is uncorrelated with $\widehat{h}_i$ . Note that this assumption is well justified [17] for many practical estimation techniques such as additive channel estimation, MMSE channel estimation, and channel estimation derived from pilot-symbol assisted modulation. Using (6) and (7) in the definition of $\rho$, we arrive at

$$\rho = \rho_e\frac{E\left[\widehat{h}_i[k]\widehat{h}_i^*[k-D]\right]}{\Lambda} = \rho_e\rho_d. \tag{8}$$

i.e., *the combined correlation coefficient is equal to the product of individual correlation coefficients.*

### B. FDD with Finite-Rate Feedback (FDDQ) System

As illustrated in Fig. 1, in FDDQ, the MS estimates the channel, and quantizes it into one of $C = 2^B$ code words. The index, which is represented by $B$ bits, of the code word corresponding to the channel estimate is fed back to the BS. In this section, we assume that the feedback channel is error free [11]-[14]. Let

$$\widehat{\mathbf{v}}[k-D] = \frac{\widehat{\mathbf{h}}[k-D]}{\|\widehat{\mathbf{h}}[k-D]\|}\,,$$

and

$$\mathbf{w}[k] = \tilde{\mathbf{v}}[k-D] = \mathcal{Q}\left[\widehat{\mathbf{v}}[k-D]\right],$$

where $\mathcal{Q}[\cdot]$ is the vector quantization operator. The received signal can now be written as

$$y[k] = \bar{\rho}\sqrt{\frac{\Omega}{\Lambda}}\,\|\widehat{\mathbf{h}}[k-D]\|\vartheta s_m[k] + \upsilon[k], \tag{9}$$

where

$$\vartheta \triangleq \widehat{\mathbf{v}}^H[k-D]\tilde{\mathbf{v}}[k-D].$$

Since the MS knows $\|\widehat{\mathbf{h}}[k-D]\|$ and $\vartheta$, the DV can be formed as

$$r[k] \triangleq \frac{y[k]}{\|\widehat{\mathbf{h}}[k-D]\|\vartheta} = \bar{\rho}\sqrt{\frac{\Omega}{\Lambda}}\,s_m[k] + \tilde{\upsilon}[k], \tag{10}$$

conditioned on $|s_m[k]|$, $\beta$ and $\tilde{\Delta}$, $\tilde{\upsilon} \sim \mathcal{NC}\left(0, \sigma_s^2/\beta\tilde{\Delta}\Lambda\right)$. Here, $\tilde{\Delta} \triangleq |\vartheta|^2$. Since finding the exact pdf of $\tilde{\Delta}$ is rather difficult, [13] upper bounded $\tilde{\Delta}$ (i.e., lower bounded the average error performance) by a r.v $\Delta$, whose pdf is given by

$$f_\Delta(x) = 2^B(t-1)(1-x)^{t-2}, \quad 1-\psi < x < 1, \qquad (11)$$

where $\psi = 2^{-B/(t-1)}$. Independently, in [14] the authors showed that (11) is a very accurate approximation to the true pdf of $\tilde{\Delta}$. Note that when $B \to \infty$, we have $\psi \to 0$ and (11) reduces to $f_\Delta(x) = \delta(x-1)$ [13], [14]. In what follows, we use $\Delta$ in place of $\tilde{\Delta}$. More details on the construction of codebook and the proof for $f_\Delta(x)$ can be found in [14].

### C. TDD System

In a TDD system, the BS can estimate the channel using channel reciprocity. Accounting for a delay $D$ between the channel estimation and its actual use, the transmit BV is given by (2). We assume that the actual channel is jointly Gaussian with the delayed version of the estimated channel. The received signal $y[k]$ for this system is exactly the same as (4). However, for the demodulation of $s_m[k]$, the receiver needs $\|\widehat{\mathbf{h}}[k-D]\|$, which has to be fedback by MS. This feedback requirement in TDD system is counterintuitive to the traditional argument that feedback is not required for TDD systems. Assuming ideal knowledge of $\|\widehat{\mathbf{h}}[k-D]\|$ at the receiver, the DV is given by (5).

## III. Error Probability Analysis

In this section, we analyze the average SEP and BEP performances of $M$-PSK and rectangular $M$-QAM modulation with Gray code symbol mapping. Upon observing (5) and (10), the DV at the demodulation input can be expressed in a parametric form as

$$r[k] = \kappa s_m[k] + \xi[k] = r_I[k] + jr_Q[k], \qquad (12)$$

where

$$\kappa = \bar{\rho}\sqrt{\frac{\Omega}{\Lambda}} \triangleq \mu_I + j\mu_Q,$$

and $\xi[k]$, conditioned on $|s_m[k]|$, $\beta$ and $\Delta$, is a CSCG r.v with variance $\mathcal{F}(|s_m[k]|)/(\beta\Delta)$, where

$$\mathcal{F}(x) = \frac{\sigma_n^2 + (1-|\rho|^2)x^2\Omega}{\Lambda}.$$

It is important to note that, due to the presence of signal dependant noise together with the *unknown constant* $\kappa$, it is not possible to borrow the existing error probability expressions [18], [19], and extend them to the present case of ICE, delay and quantization. This motivates us to derive the error probability expressions *ab initio* using the DV given by (10). Key steps in the derivation of average BEP and SEP are given below.

### A. $M$-ary PSK Constellation

Since for $M$-PSK, $|s_m[k]|$ is not a function of the index $m$, let $\mho \triangleq \mathcal{F}(|s_m[k]|) = (\sigma_n^2 + (1-|\rho|^2)E_s\Omega)/\Lambda$. For $M$-PSK, the DV of interest is the phase angle, $\Theta$, of $r[k]$. The cdf of $\Theta$, conditioned on $\beta$ and $\Delta$, when $\theta_m = 2m\pi/M$ is the transmitted phase, can be expressed as [20] (shown at the top of next page in (13)). In (13) $\omega_1 < \omega_2$, $\phi_\rho$ is the phase angle of $\bar{\rho}$, and

$$\lambda = \frac{|\kappa|^2 E_s}{\mho} = \frac{|\rho|^2\gamma}{1+(1-|\rho|^2)\gamma},$$

where $\gamma = \Omega E_s/\sigma_n^2$, is the average received SNR per symbol with ICE. The function $F_{\Phi|\beta\Delta}(\theta;\lambda)$ in (13) is given by (14) (shown in the next page). Due to the discontinuity of $F_{\Phi|\beta\Delta}(\theta;\cdot)$ at $\theta = 0$, we have to use $F_{\Phi|\beta\Delta}(\omega_1 = 0;\cdot) = -1/2$ and $F_{\Phi|\beta\Delta}(\omega_2 = 0;\cdot) = 1/2$ [20].

*1) Average Symbol Error Probability:* Upon using the conditional cdf (13) of $\Theta$ with $\omega_1 = \theta_m - \pi/M$ and $\omega_2 = \theta_m + \pi/M$ and subtracting the result from unity, we can obtain the average SEP as (15) (shown in the next page). (15) is valid for $|\phi_\rho| < \pi/M$ (when $|\phi_\rho| \geq \pi/M$ (15), should be subtracted from unity). In (15),

$$\phi^- = \pi/M - \phi_\rho \quad \text{and} \quad \phi^+ = \pi/M + \phi_\rho\,.$$

To reduce the analytical complications, it is important to take the average of (15) first over $\beta$ then over $\Delta$. By averaging (15) over $\beta$, we arrive at (16) (shown in the next page). For FDD and TDD schemes, the average SEP is given by

$$\mathrm{P}_{S,PSK,FDD/TDD} = \tilde{P}_{S,PSK}(1).$$

For FDDQ, the average SEP can be obtained by averaging (16) over the pdf of $\Delta$ given in (11), and the final expression is given by (17) (shown on page.5). In (17),

$$\mathfrak{z}^- = \sqrt{\lambda\sin^2(\phi^-)}, \quad \mathfrak{z}^+ = \sqrt{\lambda\sin^2(\phi^+)},$$

and $\overline{\mathcal{G}}(\cdot,\cdot,\cdot,\cdot,\cdot)$ is given by (49) in the Appendix. With $\rho = 1$, $\mathrm{P}_{s,PSK}$ coincides with the expression in [13].

*2) Average Bit Error Probability:* We now proceed to derive the average BEP. Similar to [21], we define $\mathcal{P}(k;\beta\Delta)$ as the probability of the received signal phase, $\Theta$, falling in a wedge of width $2\pi/M$ centered around the $k$th symbol point $k = 1,\ldots,M-1$, conditioned on $\beta$ and $\Delta$, when $S_0 = \sqrt{E_s}$ is the transmitted signal. With the help of (13) and $\theta_m = 0$, $\mathcal{P}(k;\beta\Delta)$ can be expressed as

$$\mathcal{P}(k;\beta\Delta) = \mathrm{Prob}\left(\theta_k - \pi/M \leq \Theta \leq \theta_k + \pi/M; \lambda|\beta\Delta\right). \qquad (18)$$

To proceed further, let $|\phi_\rho - \theta_k| > \pi/M$. Following section (III-A1), $\tilde{\mathcal{P}}(k;\Delta) = E[\mathcal{P}(k,\beta\Delta)]$ is given by (19) (shown on page.5). In (19),

$$\theta_k^+ = \theta_k + \phi^-, \quad \text{and} \quad \theta_k^- = \theta_k - \phi^+.$$

Using (49), the average $\overline{\mathcal{P}}(k) = E[\tilde{\mathcal{P}}(k;\Delta)]$ of (19) over $\Delta$ can be expressed as (20) (shown on page.5). Note that (20) is applicable to FDDQ only. For FDD and TDD systems, $\overline{\mathcal{P}}(k) = \tilde{\mathcal{P}}(k;1)$. In (20),

$$\ell_k^- = \sqrt{\lambda\sin^2(\theta_k^+)} \quad \text{and} \quad \ell_k^+ = \sqrt{\lambda\sin^2(\theta_k^-)}.$$

$$\text{Prob}(\omega_1 \le \Theta \le \omega_2; \lambda|\beta\Delta) = \begin{cases} F_{\Theta|\beta\Delta}(\omega_2 - \theta_m - \phi_\rho; \lambda) + F_{\Theta|\beta\Delta}(\omega_1 - \theta_m - \phi_\rho; \lambda) + 1, & \text{if } \omega_1 < \theta_m + \phi_\rho < \omega_2 \\ F_{\Theta|\beta\Delta}(\omega_2 - \theta_m - \phi_\rho; \lambda) - F_{\Theta|\beta\Delta}(\omega_1 - \theta_m - \phi_\rho; \lambda), & \text{if } \omega_1 > \theta_m + \phi_\rho \text{ or } \omega_2 < \theta_m + \phi_\rho \end{cases} \quad (13)$$

$$F_{\Phi|\beta\Delta}(\theta; \lambda) = -\frac{\text{sgn}(\theta)}{2\pi} \int_0^{\pi-|\theta|} \exp\left(-\lambda\beta\Delta \frac{\sin^2\theta}{\sin^2 x}\right) dx, \qquad -\pi < \theta < \pi \quad (14)$$

$$P_{S,PSK}(\beta\Delta) = \frac{\text{sgn}(\phi^-)}{2\pi} \int_0^{\pi-|\phi^-|} e^{-\frac{\lambda\beta\Delta\sin^2(\phi^-)}{\sin^2\theta}} d\theta + \frac{\text{sgn}(\phi^+)}{2\pi} \int_0^{\pi-|\phi^+|} e^{-\frac{\lambda\beta\Delta\sin^2(\phi^+)}{\sin^2\theta}} d\theta \quad (15)$$

$$\tilde{P}_{S,PSK}(\Delta) = \frac{\text{sgn}(\phi^-)}{2\pi} \int_0^{\pi-|\phi^-|} \left(\frac{\sin^2\theta}{\sin^2\theta + \lambda\Delta\sin^2(\phi^-)}\right)^t d\theta + \frac{\text{sgn}(\phi^+)}{2\pi} \int_0^{\pi-|\phi^+|} \left(\frac{\sin^2\theta}{\sin^2\theta + \lambda\Delta\sin^2(\phi^+)}\right)^t d\theta \quad (16)$$

Using (20), the average BEP for Gray coded $M$-PSK is

$$\text{P}_{\text{b},PSK} = \frac{1}{\log_2(M)} \sum_{k=1}^{M-1} d(k)\overline{\mathcal{P}}(k), \quad (21)$$

where $d(k)$ is the weight spectrum of Gray code, derived in [21]. With $M = 2$, $\Delta = 1$, $\phi_\rho = 0$ and $\rho_e = 1$ (i.e., $\rho = \rho_d$ for a delayed feedback case), (21), with the help of (19), coincides with the results presented in [6]. We also note that the average BEP expressions in [6] can be derived in a very simple way using the methodology presented here.

### B. $M$-ary Rectangular QAM Constellation

Let $s_m[k] = s_{m,x}[k] + js_{m,y}[k]$, $m = 0, 1, \ldots, M-1$, $x = 0, 1, \ldots, M_1 - 1$, $y = 0, 1, \ldots, M_2 - 1$, where $s_{m,x}[k] = a_{m,x}[k]d$, and $s_{m,y}[k] = a_{m,y}[k]d$, where $a_{m,x}[k] = -(M_1 - 1) + 2x$ and $a_{m,y}[k] = -(M_2 - 1) + 2y$. For simplicity, let us define the parameter

$$\gamma_{x,y} \triangleq \frac{2d^2}{\mathcal{F}(|s_m[k]|)} = \frac{2d^2\Lambda}{\sigma_n^2 + \Omega(1 - |\rho|^2)|s_m[k]|^2}.$$

*1) Average Symbol Error Probability:* Let us denote by $\mathcal{P}_{C,x,y}(\beta\Delta)$ the probability of correctly receiving $s_{m,x}[k] + js_{m,y}[k]$, conditioned on $\beta$ and $\Delta$. For $x = 1, 2 \ldots, M_1 - 2$, $y = 1, 2, \ldots, M_2 - 2$, $\mathcal{P}_{C,x,y}(\beta\Delta)$ can be expressed as (22) (shown in the next page). In (22)

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-u^2/2) du,$$

and

$$\mathsf{t}_1(x, y) = (a_{m,x}[k] - 1 - a_{m,x}[k]\mu_I + a_{m,y}[k]\mu_Q)\sqrt{\gamma_{x,y}},$$

$$\mathsf{t}_2(x, y) = (a_{m,x}[k] + 1 - a_{m,x}[k]\mu_I + a_{m,y}[k]\mu_Q)\sqrt{\gamma_{x,y}},$$

$$\mathsf{t}_3(x, y) = (a_{m,y}[k] - 1 - a_{m,x}[k]\mu_Q - a_{m,y}[k]\mu_I)\sqrt{\gamma_{x,y}},$$

$$\mathsf{t}_4(x, y) = (a_{m,y}[k] + 1 - a_{m,x}[k]\mu_Q - a_{m,y}[k]\mu_I)\sqrt{\gamma_{x,y}}.$$

For ease of referencing, expressions for $\mathcal{P}_{C,x,y}(\beta\Delta)$ for other values of $x$ and $y$ are given in Table I. Notice that, each of the $\mathcal{P}_{C,x,y}(\beta\Delta)$ expressions in Table I can be expressed as linear combinations of $Q(a\sqrt{\beta\Delta}) \times Q(b\sqrt{\beta\Delta})$ for real values of $a$ and $b$. Let $\overline{\mathcal{P}}_{C,x,y} \triangleq E[\mathcal{P}_{C,x,y}(\beta\Delta)]$. To derive $\overline{\mathcal{P}}_{C,x,y}$, we must determine $E[Q(a\sqrt{\beta\Delta}) \times Q(b\sqrt{\beta\Delta})]$, averaged over $\beta$ and $\Delta$. To this end, we define the following functions:

$$\mathcal{H}_1(a, b, \Delta, t) \triangleq E[Q(a\sqrt{\beta\Delta})Q(b\sqrt{\beta\Delta})] \quad (23)$$
$$= \begin{cases} \mathcal{J}_1(|a|, |b|, \Delta, t) & \text{if } a \ge 0, b \ge 0 \\ \mathcal{K}_1(|a|, \Delta, t) - \mathcal{J}_1(|a|, |b|, \Delta, t) & \text{if } a \ge 0, b < 0 \\ \mathcal{K}_1(|b|, \Delta, t) - \mathcal{J}_1(|a|, |b|, \Delta, t) & \text{if } a < 0, b \ge 0 \\ 1 - \mathcal{K}_1(|a|, \Delta, t) - \mathcal{K}_1(|b|, \Delta, t) & \\ \quad + \mathcal{J}_1(|a|, |b|, \Delta, t) & \text{if } a < 0, b < 0 \end{cases},$$

$$\mathcal{J}_1(a, b, y, t) = E[Q(a\sqrt{y\beta})Q(b\sqrt{y\beta})],\ a, b, y \ge 0, \quad (24)$$

$$\mathcal{K}_1(a, y, t) = E[Q(a\sqrt{y\beta})] \quad \text{for } a, y \ge 0, \quad (25)$$
$$\mathcal{H}(a, b, t, B, \psi) = E[\mathcal{H}_1(a, b, \Delta, t)] \text{ for } a,\ b \text{ real}, \quad (26)$$
$$\mathcal{J}(a, b, t, B, \psi) = E[\mathcal{J}_1(a, b, \Delta, t)] \quad \text{for } a, b \ge 0, \quad (27)$$
$$\mathcal{K}(a, t, B, \psi) = E[\mathcal{K}_1(a, \Delta, t)] \quad \text{for } a \ge 0, \quad (28)$$

$$\mathcal{R}_1(a, y, t) \triangleq E[Q(a\sqrt{y\beta})] \quad (29)$$
$$= \begin{cases} \mathcal{K}_1(|a|, y, t) & \text{if } a, y \ge 0 \\ 1 - \mathcal{K}_1(|a|, y, t) & \text{if } a < 0, y \ge 0, \end{cases}$$

$$\mathcal{R}(a, t, B, \psi) = E[\mathcal{R}_1(a, \Delta, t)] \quad (30)$$
$$= \begin{cases} \mathcal{K}(|a|, t, B, \psi) & \text{if } a \ge 0 \\ 1 - \mathcal{K}(|a|, t, B, \psi) & \text{if } a < 0, \end{cases}$$

where, in Appendix, expressions for (24), (25), (27) and (28) are given in (36), (48), (41) and (51), respectively. Using (23) each row in Table I can be averaged over $\beta$ to arrive at $\tilde{\mathcal{P}}_{C,x,y}(\Delta)$. Upon further averaging $\tilde{\mathcal{P}}_{C,x,y}(\Delta)$, with the help of (26), over $\Delta$ of (11), we arrive at $\overline{\mathcal{P}}_{C,x,y}$ for FDDQ. For FDD and TDD schemes, note that $\overline{\mathcal{P}}_{C,x,y} = \tilde{\mathcal{P}}_{C,x,y}(1)$. For

$$\mathrm{P}_{S,PSK} = \frac{\mathrm{sgn}(\phi^-)}{2\pi}\,\overline{\mathcal{G}}\left(\pi - |\phi^-|, \mathfrak{z}^-, t, B, \psi\right) + \frac{\mathrm{sgn}(\phi^+)}{2\pi}\,\overline{\mathcal{G}}\left(\pi - |\phi^+|, \mathfrak{z}^+, t, B, \psi\right) \tag{17}$$

$$\tilde{\mathcal{P}}(k;\Delta) = \frac{\mathrm{sgn}(\theta_k^+)}{2\pi}\int_0^{\pi-|\theta_k^+|}\left(\frac{\sin^2\theta}{\sin^2\theta + \lambda\Delta\sin^2(\theta_k^+)}\right)^t d\theta - \frac{\mathrm{sgn}(\theta_k^-)}{2\pi}\int_0^{\pi-|\theta_k^-|}\left(\frac{\sin^2\theta}{\sin^2\theta + \lambda\Delta\sin^2(\theta_k^-)}\right)^t d\theta \tag{19}$$

$$\overline{\mathcal{P}}(k) = \frac{\mathrm{sgn}(\theta_k^+)}{2\pi}\,\overline{\mathcal{G}}\left(\pi - |\theta_k^+|, \ell_k^-, t, B, \psi\right) - \frac{\mathrm{sgn}(\theta_k^-)}{2\pi}\,\overline{\mathcal{G}}\left(\pi - |\theta_k^-|, \ell_k^+, t, B, \psi\right) \tag{20}$$

$$\begin{aligned}\mathcal{P}_{C,x,y}(\beta\Delta) &= \mathrm{Prob}(s_{m,x}[k] - d \le r_I[k] < s_{m,x}[k] + d|\beta\Delta) \times \mathrm{Prob}(s_{m,y}[k] - d \le r_Q[k] < s_{m,y}[k] + d|\beta\Delta)\\ &= \left\{Q(\mathrm{t}_1(x,y)\sqrt{\beta\Delta}) - Q(\mathrm{t}_2(x,y)\sqrt{\beta\Delta})\right\} \times \left\{Q(\mathrm{t}_3(x,y)\sqrt{\beta\Delta}) - Q(\mathrm{t}_4(x,y)\sqrt{\beta\Delta})\right\}\end{aligned} \tag{22}$$

convenience, $\overline{\mathcal{P}}_{C,x,y}$ are tabulated in Table III-B2 for FDDQ. Using them, the average SEP can be written as

$$\mathrm{P}_{S,QAM} = 1 - \frac{1}{M}\sum_{x=0}^{M_1-1}\sum_{y=0}^{M_2-1}\overline{\mathcal{P}}_{C,x,y}. \tag{31}$$

With $\kappa = 1$ and $\rho = 1$, it is easy to show that (31) coincides with the results presented in [13].

*2) Average Bit Error Probability:* To derive the average BEP, we follow the Gray code labeling approach of [22]. Let $k_1 = \log_2(M_1)$, $k_2 = \log_2(M_2)$, and the sets $\mathcal{X} = \{0, 1, \ldots, M_1 - 1\}$ and $\mathcal{Y} = \{0, 1, \ldots, M_2 - 1\}$. The vector $(a_{k_1-1}, a_{k_1-2}, \ldots, a_0)$ is the Gray code mapping for $s_{m,x}[k]$, and $(b_{k_2-1}, b_{k_2-2}, \ldots, b_0)$ is the Gray code mapping for $s_{m,y}[k]$. For $i = 0, \ldots, k_1 - 1$, let $X_1(i) = \{x : (x \bmod 2^{i+2}) = 2^i + l, l = 0, \ldots, 2^i - 1\} \cup \{x : (x \bmod 2^{i+2}) = 2^{i+1} + l, l = 0, \ldots, 2^i - 1\}$ and $X_0(i) = \{x : (x \bmod 2^{i+2}) = l, l = 0, \ldots, 2^i - 1\} \cup \{x : (x \bmod 2^{i+2}) = 3 \times 2^i + l, l = 0, \ldots, 2^i - 1\}$. In a similar manner, for $j = 0, \ldots, k_2 - 1$, we can define the sets $Y_1(j)$ and $Y_0(j)$. Using these sets, the decision statistic for each bit $a_i$, $i = 0, \ldots, k_1 - 1$, is given by the disjoint union of intervals on the $x$-axis in (32) (shown in the previous page), In (32), $\mathbf{1}_{\{x\}} = 1$ if '$x$' is true and $\mathbf{1}_{\{x\}} = 0$ if '$x$' is false. Following the approach presented in [23], it can be shown that the average BEP for bit $a_j$, $j = 0, \ldots, k_1 - 1$, with FDDQ is given by (33) (shown in the previous page). In (33), $\mathcal{R}(a, t, B, \psi)$ is defined in (30). In (33), the first triple summation captures the probability of transmitting $s_{m,x_0}[k] + js_{m,y}[k]$ and demodulating $s_{m,x_1}[k] + js_{m,y}[k]$, whereas the second triple summation captures the probability of transmitting $s_{m,x_1}[k] + js_{m,y}[k]$ and demodulating $s_{m,x_0}[k] + js_{m,y}[k]$. Similarly, the average BEP for $b_j$, $j = 0, \ldots, k_2 - 1$, with FDDQ denoted as $P_b(b_j)$, can be derived. For FDD and TDD schemes, the average BEP can be obtained by replacing $\mathcal{R}(a, t, B, \psi)$ by $\mathcal{R}_1(a, 1, t)$ of (29). Finally, the average BEP can be obtained as

$$\mathrm{P}_{\mathrm{b},QAM} = \frac{1}{\log_2(M)}\left\{\sum_{j=0}^{k_1-1} P_b(a_j) + \sum_{j=0}^{k_2-1} P_b(b_j)\right\}. \tag{34}$$

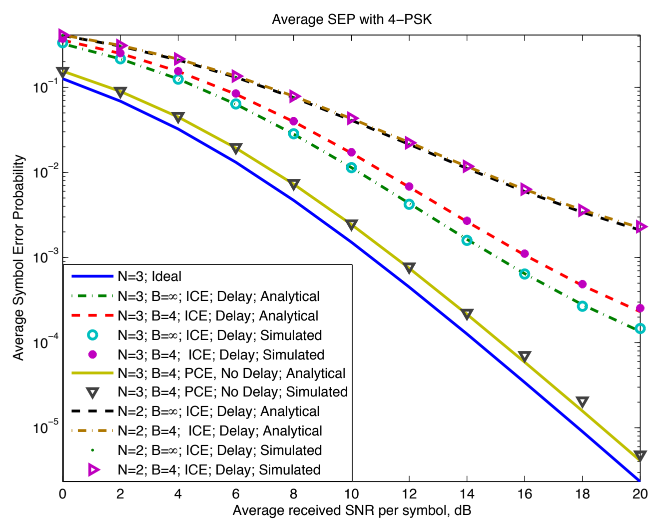


Fig. 2. Average SEP performance of QPSK modulation with imperfect channel estimation, feedback delay, and feedback channel quantization. Here, we assume $t = 2$ and 3 antennas, $B \in \{4, \infty\}$ feedback bits, $\rho_d = 0.99$ and average received SNR of the pilot channel $\gamma_p$ the same as the average operating data channel SNR. For comparison, SEP performance with channel quantization alone is also plotted. For all the cases, both analytical as well as simulation results are shown.

## IV. Results and Discussion

In this section, we present numerical (obtained through the analysis) and simulation results. In all the plots, we employ the Jakes model for, $\rho_d$, the time correlation. For an FDD system assuming MMSE estimation with a pilot SNR of $\gamma_p$, the estimation-error-only correlation, $\rho_e$, can be shown to be $\sqrt{\gamma_p/(1+\gamma_p)}$. Then, from (8), the combined correlation coefficient is $\rho = \rho_d\sqrt{\gamma_p/(1+\gamma_p)}$.

Fig. 3 shows the average BEP of Gray-coded QPSK with $t \in \{2, 3\}$ antennas, $B = 2$ bits, and $\gamma_p = 30$ dB. The simulation results in Fig. 3 match accurately with the numerical results, validating the presented analytical framework. Average SEP of

| $x$ | $y$ | $P_{C,x,y}(\beta\Delta) = \text{Prob}\left(s_{m,x}[k] + js_{m,y}[k] \text{ received successfully} \middle\vert \beta\Delta\right)$ |
|---|---|---|
| $\{1,\ldots,M_1-2\}$ | $\{1,\ldots,M_2-2\}$ | $\text{Prob}(s_{m,x}[k]-d \le r_I[k] < s_{m,x}[k]+d \vert \beta\Delta) \times \text{Prob}(s_{m,y}[k]-d \le r_Q[k] < s_{m,y}[k]+d \vert \beta\Delta) =$<br>$\left\{Q\left(\left[a_{m,x}[k]-1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) - Q\left(\left[a_{m,x}[k]+1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right)\right\}\times$<br>$\left\{Q\left(\left[a_{m,y}[k]-1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right) - Q\left(\left[a_{m,y}[k]+1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)\right\}$ |
| $\{1,\ldots,M_1-2\}$ | $\{0\}$ | $\text{Prob}(s_{m,x}[k]-d \le r_I[k] < s_{m,x}[k]+d \vert \beta\Delta) \times \text{Prob}(-\infty < r_Q[k] < s_{m,y}[k]+d \vert \beta\Delta) =$<br>$Q\left(-\left[a_{m,y}[k]+1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)\times$<br>$\left\{Q\left(\left[a_{m,x}[k]-1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) - Q\left(\left[a_{m,x}[k]+1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right)\right\}$ |
| $\{1,\ldots,M_1-2\}$ | $\{M_2-1\}$ | $\text{Prob}(s_{m,x}[k]-d \le r_I[k] < s_{m,x}[k]+d \vert \beta\Delta) \times \text{Prob}(s_{m,y}[k]-d \le r_Q[k] < \infty \vert \beta\Delta) =$<br>$Q\left(\left[a_{m,y}[k]-1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)\times$<br>$\left\{Q\left(\left[a_{m,x}[k]-1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) - Q\left(\left[a_{m,x}[k]+1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right)\right\}$ |
| $\{0\}$ | $\{1,\ldots,M_2-2\}$ | $\text{Prob}(-\infty < r_I[k] < s_{m,x}[k]+d \vert \beta\Delta) \times \text{Prob}(s_{m,y}[k]-d \le r_Q[k] < s_{m,y}[k]+d \vert \beta\Delta) =$<br>$Q\left(-\left[a_{m,x}[k]+1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right)\times$<br>$\left\{Q\left(\left[a_{m,y}[k]-1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right) - Q\left(\left[a_{m,y}[k]+1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)\right\}$ |
| $\{M_1-1\}$ | $\{1,\ldots,M_2-2\}$ | $\text{Prob}(s_{m,x}[k]-d \le r_I[k] < \infty \vert \beta\Delta) \times \text{Prob}(s_{m,y}[k]-d \le r_Q[k] < s_{m,y}[k]+d \vert \beta\Delta) =$<br>$Q\left(\left[a_{m,x}[k]-1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right)\times$<br>$\left\{Q\left(\left[a_{m,y}[k]-1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right) - Q\left(\left[a_{m,y}[k]+1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)\right\}$ |
| $\{0\}$ | $\{0\}$ | $\text{Prob}(-\infty < r_I[k] < s_{m,x}[k]+d \vert \beta\Delta) \times \text{Prob}(-\infty < r_Q[k] < s_{m,y}[k]+d \vert \beta\Delta) =$<br>$Q\left(-\left[a_{m,x}[k]+1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) Q\left(-\left[a_{m,y}[k]+1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)$ |
| $\{M_1-1\}$ | $\{0\}$ | $\text{Prob}(s_{m,x}[k]-d \le r_I[k] < \infty \vert \beta\Delta) \times \text{Prob}(-\infty < r_Q[k] < s_{m,y}[k]+d \vert \beta\Delta) =$<br>$Q\left(\left[a_{m,x}[k]-1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) Q\left(-\left[a_{m,y}[k]+1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)$ |
| $\{0\}$ | $\{M_2-1\}$ | $\text{Prob}(-\infty < r_I[k] < s_{m,x}[k]+d \vert \beta\Delta) \times \text{Prob}(s_{m,y}[k]-d \le r_Q[k] < \infty \vert \beta\Delta) =$<br>$Q\left(-\left[a_{m,x}[k]+1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) Q\left(\left[a_{m,y}[k]-1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)$ |
| $\{M_1-1\}$ | $\{M_2-1\}$ | $\text{Prob}(s_{m,x}[k]-d \le r_I[k] < \infty \vert \beta\Delta) \times \text{Prob}(s_{m,y}[k]-d \le r_Q[k] < \infty \vert \beta\Delta) =$<br>$Q\left(\left[a_{m,x}[k]-1-a_{m,x}[k]\mu_I+a_{m,y}[k]\mu_Q\right]\hat{\gamma}\right) Q\left(\left[a_{m,y}[k]-1-a_{m,x}[k]\mu_Q-a_{m,y}[k]\mu_I\right]\hat{\gamma}\right)$ |

TABLE I

FOR EACH $x \in \{0,1,\ldots,M_1-1\}$ AND $y \in \{0,1,\ldots,M_2-1\}$, CONDITIONED ON $\beta$ AND $\Delta$, THE PROBABILITY OF CORRECT RECEPTION OF THE SYMBOL $s_m[k] = s_{m,x}[k] + js_{m,y}[k]$ IS TABULATED ON THE THIRD COLUMN FOR AN $M_1 \times M_2$ RECTANGULAR QAM CONSTELLATION. IN THE ABOVE TABLE $\hat{\gamma} = \sqrt{\gamma_{x,y}\beta\Delta}$.

$$\widehat{a}_i = \begin{cases} 1 & \text{if } r_I[k] \in \cup_{x\in X_1(i)} \Big[ -\infty.\mathbf{1}_{\{x=0\}} + (s_{m,x}[k]-d), (s_{m,x}[k]+d) + \infty.\mathbf{1}_{\{x=M_1-1\}} \Big) \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

$$\begin{aligned} P_b(a_j) \;=\; & \frac{1}{M}\sum_{x_0\in X_0(j)}\sum_{x_1\in X_1(j)}\sum_{y\in\mathcal{Y}} \mathcal{R}\left(\left[-\infty.\mathbf{1}_{\{x_1=0\}} + a_{m,x_1}[k] - 1 - a_{m,x_0}[k]\mu_I + a_{m,y}[k]\mu_Q\right]\sqrt{\gamma_{x_0,y}}, t, B, \psi\right) \\ & -\mathcal{R}\left(\left[\infty.\mathbf{1}_{\{x_1=M_1-1\}} + a_{m,x_1}[k] + 1 - a_{m,x_0}[k]\mu_I + a_{m,y}[k]\mu_Q\right]\sqrt{\gamma_{x_0,y}}, t, B, \psi\right) + \\ & \frac{1}{M}\sum_{x_1\in X_1(j)}\sum_{x_0\in X_0(j)}\sum_{y\in\mathcal{Y}} \mathcal{R}\left(\left[-\infty.\mathbf{1}_{\{x_0=0\}} + a_{m,x_0}[k] - 1 - a_{m,x_1}[k]\mu_I + a_{m,y}[k]\mu_Q\right]\sqrt{\gamma_{x_1,y}}, t, B, \psi\right) - \\ & \mathcal{R}\left(\left[\infty.\mathbf{1}_{\{x_0=M_1-1\}} + a_{m,x_0}[k] + 1 - a_{m,x_1}[k]\mu_I + a_{m,y}[k]\mu_Q\right]\sqrt{\gamma_{x_1,y}}, t, B, \psi\right) \end{aligned} \tag{33}$$

QPSK with $t \in \{2,3\}$, and $B \in \{4,\infty\}$ is presented in Fig. 2. In Fig. 2 we set $\gamma_p$ equal to the operating SNR. As a result, the receiver does not suffer from any noticeable error floor. Fig. 2 shows that ICE and delay cause more degradation to the performance compared to quantization alone. For example, at an SNR of 20 dB, with $t=2$ and $B=4$ the SEP with ICE and delay is an order of magnitude worse than the SEP with PCE and no delay. Fig. 2 also shows that this performance gap increases when the number of antennas is increased by one. The reason for this is that the number of parameters to be fed back decreases as the number of antennas decreases, thus the impact of B is not much when there are fewer antennas.

The effects of pilot SNR and the feedback bits on the average BEP of 8-PSK is investigated in Fig. 4. In Fig. 4, we fix ‘t’ at 3, choose $B \in \{2,4,8\}$ and $\gamma_p \in \{15,30\}$ dB. From Fig. 4 we observe that, at high SNR, and for a given combination of $B$ and $\gamma_p$, increasing the pilot SNR is more beneficial to improving the average BEP than increasing $B$. This can be explained by the fact that ICE introduces error floor at high SNR, which can only be reduced by increasing the channel estimation quality. For example, in Fig. 4, with $(B,\gamma_p) = (2,15)$, increasing $B$ to 8 while keeping $\gamma_p$ at 15 dB improves the BEP by a factor of two, whereas even by keeping $B$ at 2 and increasing $\gamma_p$ to 30 dB improves the BEP by a factor of four *which emphasizes the importance of accurate channel estimation compared to the feedback quality improvement*. For a fixed value of $B=8$ bits, Fig. 5 plots the average SEP of 32-QAM, $M \in (4\times 8)$, by varying $t \in \{2,3,4\}$ and $\gamma_p \in \{10,30\}$ dB. From Fig. 5, we notice that, at high SNR and for a fixed $(t,\gamma_p)$, increasing the pilot SNR has a more direct effect in reducing the error floor than increasing the number of antennas. This is due to the fact that,

| $x$ | $y$ | $\overline{\mathcal{P}}_{C,x,y} = E\left[\mathcal{P}_{C,x,y}(\beta\Delta)\right]$ |
|---|---|---|
| $\{1,2,\ldots,M_1-2\}$ | $\{1,2,\ldots,M_2-2\}$ | $\mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_3(x,y),t,B,\psi\right) - \mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_4(x,y),t,B,\psi\right) -$ $\mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_3(x,y),t,B,\psi\right) + \mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_4(x,y),t,B,\psi\right)$ |
| $\{1,2,\ldots,M_1-2\}$ | $\{0\}$ | $\mathcal{R}\left(\mathsf{t}_1(x,y),t,B,\psi\right) - \mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_4(x,y),t,B,\psi\right) -$ $\mathcal{R}\left(\mathsf{t}_2(x,y),t,B,\psi\right) + \mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_4(x,y),t,B,\psi\right)$ |
| $\{1,2,\ldots,M_1-2\}$ | $\{M_2-1\}$ | $\mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_3(x,y),t,B,\psi\right) - \mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_3(x,y),t,B,\psi\right)$ |
| $\{0\}$ | $\{1,2,\ldots,M_2-2\}$ | $\mathcal{R}\left(\mathsf{t}_3(x,y),t,B,\psi\right) - \mathcal{R}\left(\mathsf{t}_4(x,y),t,B,\psi\right) -$ $\mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_3(x,y),t,B,\psi\right) + \mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_4(x,y),t,B,\psi\right)$ |
| $\{M_1-1\}$ | $\{1,2,\ldots,M_2-2\}$ | $\mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_3(x,y),t,B,\psi\right) - \mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_4(x,y),t,B,\psi\right)$ |
| $\{0\}$ | $\{0\}$ | $1 - \mathcal{R}\left(\mathsf{t}_4(x,y),t,B,\psi\right) -$ $\mathcal{R}\left(\mathsf{t}_2(x,y),t,B,\psi\right) + \mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_4(x,y),t,B,\psi\right)$ |
| $\{M_1-1\}$ | $\{0\}$ | $\mathcal{R}\left(\mathsf{t}_1(x,y),t,B,\psi\right) - \mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_4(x,y),t,B,\psi\right)$ |
| $\{0\}$ | $\{M_2-1\}$ | $\mathcal{R}\left(\mathsf{t}_3(x,y),t,B,\psi\right) - \mathcal{H}\left(\mathsf{t}_2(x,y),\mathsf{t}_3(x,y),t,B,\psi\right)$ |
| $\{M_1-1\}$ | $\{M_2-1\}$ | $\mathcal{H}\left(\mathsf{t}_1(x,y),\mathsf{t}_3(x,y),t,B,\psi\right)$ |

TABLE II

FOR EACH $x \in \{0,1,\ldots,M_1-1\}$ AND $y \in \{0,1,\ldots,M_2-1\}$ THE AVERAGE PROBABILITY OF CORRECT RECEPTION OF THE SYMBOL $s_m[k] = s_{m,x}[k] + js_{m,y}[k]$ IS TABULATED ON THE THIRD COLUMN FOR AN $M_1 \times M_2$ RECTANGULAR QAM CONSTELLATION WITH FDDQ SCHEME. THE FUNCTIONS $\mathsf{t}_1(x,y), \mathsf{t}_2(x,y), \mathsf{t}_3(x,y), \mathsf{t}_4(x,y)$ ARE DEFINED IN SECTION. (III-B1). THE FUNCTION $\mathcal{H}(a,b,t,B,\psi)$ IS DEFINED IN (26), WHEREAS THE FUNCTION $\mathcal{R}(a,t,B,\psi)$ IS DEFINED IN (30). EXPRESSIONS FOR $\overline{\mathcal{P}}_{C,x,y}$ FOR FDD AND TDD SCHEMES CAN EASILY BE OBTAINED BY REPLACING $\mathcal{H}(a,b,t,B,\psi)$ BY $\mathcal{H}_1(a,b,1,t)$ OF (23) AND $\mathcal{R}(a,t,B,\psi)$ BY $\mathcal{R}_1(a,1,t)$ OF (29).

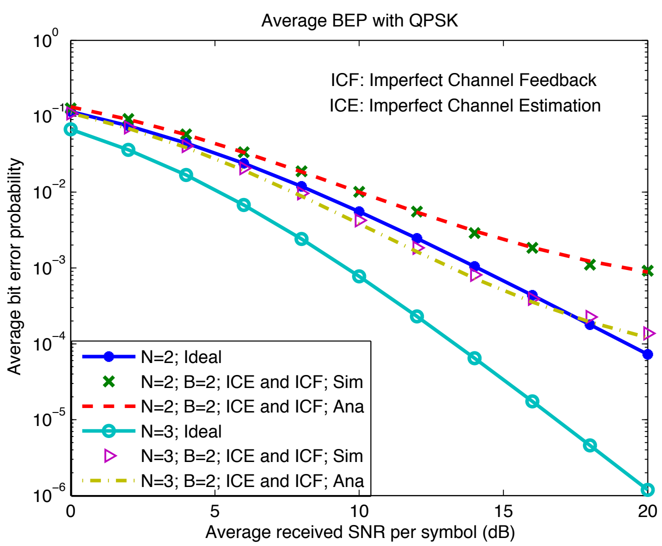


Fig. 3. Average BEP performance of QPSK modulation with imperfect channel estimation, feedback delay, and feedback channel quantization. Here, we assume $t = 2$ and 3 antennas, $B = 2$ feedback bits, $\rho_d = 0.99$ and average received SNR of the pilot channel $\gamma_p = 30$ dB. Both analytical as well as simulation results are shown.

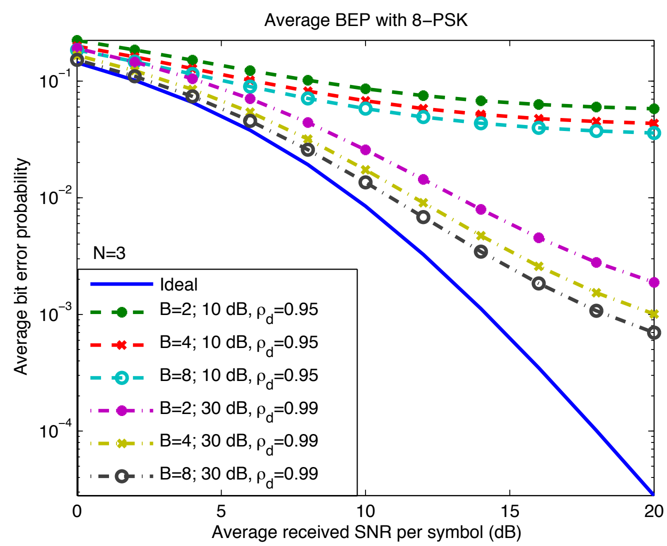


Fig. 4. Average BEP performance of Gray coded 8-PSK modulation as a function of the number of feedback bits $B \in \{2,4,8\}$ and the average pilot SNR $\gamma_p \in \{10,30\}$ dB. Here, we set $t = 3$ antennas and $\rho_d \in \{0.95, 0.99\}$.

at high SNR, error floor makes having an additional antenna less attractive from the diversity point-of-view.

Finally, in Fig. 6 we study the average SEP of 8-PSK with $t \in \{3,4\}$ and $B \in \{2,4,8\}$. Here, following the IEEE 802.16e WiMax standard [24], irrespective of the operating SNR, we set $\gamma_p$ 2.5 dB higher than the operating SNR. As expected, Fig. 6 shows that for a given number of antennas increasing $B$ monotonically improves the error performance. However, Fig. 6 also suggests that it is more beneficial to have a high feedback bit budget for a smaller number of antennas than to have more antennas with a smaller bit budget.

## V. CONCLUSION

In this paper, we have considered transmit beamforming for MISO systems on Rayleigh fading channels with imperfect channel feedback. We characterized the feedback imperfections in terms of noisy channel estimation, feedback delay, and finite-rate channel quantization. A general framework, valid for any two-dimensional constellation and FDD, FDDQ and TDD schemes, was presented to account for the feedback imperfections. We then derived the average SEP and BEP expressions for both $M$-PSK and rectangular $M$-QAM constellations with Gray code mapping. The simulations results showing an accurate agreement with analysis are also presented.

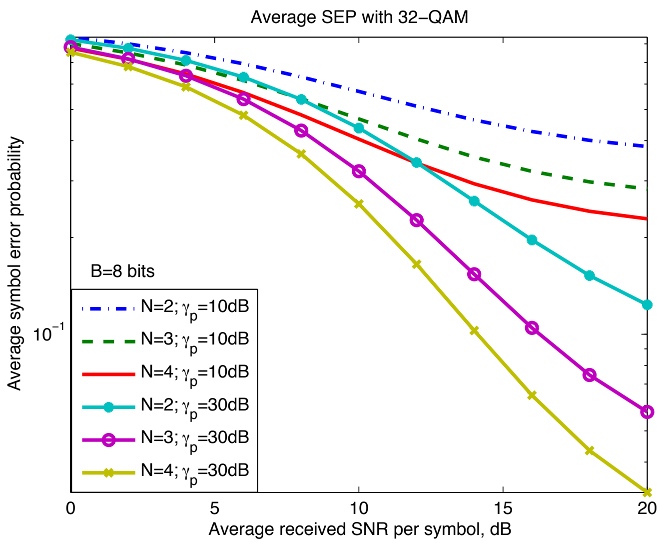


Fig. 5. Average SEP performance of 32-QAM modulation as a function of the number of antennas $t \in \{2, 3, 4\}$ and the average pilot SNR $\gamma_p \in \{10, 30\}$ dB, for a fixed feedback delay with $\rho_d = 0.99$. Here, we set $B = 8$ bits.

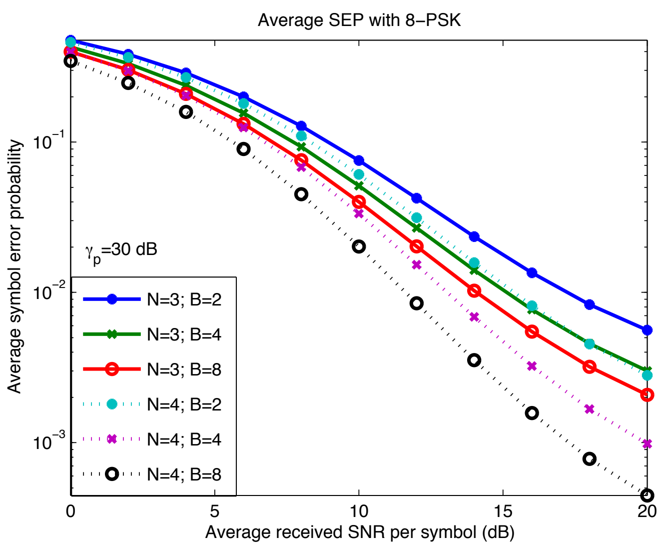


Fig. 6. Average SEP performance of 8-PSK modulation as a function of the number of antennas $t \in \{3, 4\}$ and the number of feedback bits $B \in \{2, 4, 8\}$. Here, we set $\gamma_p = 2.5$ dB higher than the operating SNR [24].

## APPENDIX

In this appendix, we derive expressions for (24), (25), (27) and (28). We begin with (27),

$$\mathcal{J}(a,b,t,B,\psi) \triangleq E\left[Q\left(a\sqrt{\beta\Delta}\right)Q\left(b\sqrt{\beta\Delta}\right)\right]$$
$$= \int_y f_\Delta(y)dy \int_{x=0}^{\infty} Q(a\sqrt{xy})Q(b\sqrt{xy})f_\beta(x)dx, \qquad (35)$$

where, in (35), $a, b \geq 0$, the pdf of $\Delta$ is given by (11). Let us first focus on the inner integral of (35), let

$$a_1 = a\sqrt{y} \quad \text{and} \quad b_1 = b\sqrt{y},$$

and using integration by-parts, we have

$$\mathcal{J}_1(a,b,y,t) \triangleq \int_{x=0}^{\infty} Q(a_1\sqrt{x})Q(b_1\sqrt{x})f_\beta(x)dx$$
$$= \mathcal{M}(a,b,y,t) + \mathcal{M}(b,a,y,t), \qquad (36)$$

where

$$\mathcal{M}(a,b,y,t) \triangleq \frac{b_1}{2\sqrt{2\pi}} \int_{x=0}^{\infty} F_\beta(x)Q(a_1\sqrt{x})e^{-\frac{b_1^2 x}{2}}x^{-\frac{1}{2}}dx. \qquad (37)$$

Upon using

$$Q(x) = (1/\pi)\int_{\theta=0}^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta,$$

for $x \geq 0$ [18], and

$$F_\beta(x) = 1 - e^{-x}\sum_{k=0}^{t-1}\frac{x^k}{\Gamma(k+1)} \quad x \geq 0,$$

$\mathcal{M}(a,b,y,t)$ of (37) can be simplified as (38) (shown in the next page). Upon using (36), (37) and (38) in (35), we have

$$\mathcal{J}(a,b,t,B,\psi) = \int_y f_\Delta(y)\mathcal{M}(a,b,y,t)dy + \int_y f_\Delta(y)\mathcal{M}(b,a,y,t)dy. \qquad (39)$$

In the absence of channel quantization (i.e., as $B \to \infty$, $\psi \to 0$ and $f_\Delta(y) = \delta(y-1)$), (39) becomes

$$\mathcal{J}(a,b,t,\infty,0) = \mathcal{M}(a,b,1,t) + \mathcal{M}(b,a,1,t), \qquad (40)$$

which is also equal to $\mathcal{J}_1(a,b,1,t)$ of (24). For finite $B$, using(11) for the pdf of $\Delta$, we have

$$\mathcal{J}(a,b,t,B,\psi) = 2^B(t-1)\left\{\mathcal{M}_1(a,b,\psi,t) + \mathcal{M}_1(b,a,\psi,t)\right\}, \qquad (41)$$

where $\mathcal{M}_1(a,b,\psi,t) \triangleq \int_{1-\psi}^{1}(1-y)^{t-2}\mathcal{M}(a,b,y,t)dy$. Using (38) $\mathcal{M}_1(a,b,\psi,t)$ can be simplified as (42) (shown in the next page). Owing to the fact that

$$\int_{1-\psi}^{1} f_\Delta(y)dy = 1,$$

the first double integral of (42) reduces to

$$\int_{1-\psi}^{1}(1-y)^{t-2}\int_{0}^{\pi/2}\sqrt{\frac{b^2\sin^2\theta}{b^2\sin^2\theta + a^2}}dy\,d\theta =$$
$$\frac{1}{2^B(t-1)}\frac{ab}{(a^2+b^2)}\,{}_2F_1\left(1,1;\frac{3}{2};\frac{b^2}{a^2+b^2}\right), \qquad (43)$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function, and we used [18, Eqn. (5.17)] to simplify (43). On the other hand, to simplify the inner integral of (42) let us consider the result [25]

$$\mathcal{M}(a,b,y,t) = \frac{b_1}{2\sqrt{2\pi}} \int\limits_{x=0}^{\infty} Q(a_1\sqrt{x}) e^{-\frac{b_1^2 x}{2}} x^{-\frac{1}{2}} dx - \frac{b_1}{2\sqrt{2\pi}} \sum_{j=0}^{t-1} \frac{1}{j!} \int\limits_{x=0}^{\infty} Q(a_1\sqrt{x}) e^{-(\frac{b_1^2}{2}+1)x} x^{j-\frac{1}{2}} dx$$

$$= \frac{1}{2\pi} \int\limits_{0}^{\pi/2} d\theta \sqrt{\frac{b_1^2 \sin^2\theta}{b_1^2 \sin^2\theta + a_1^2}} - \sum_{j=0}^{t-1} \frac{1}{j!} \frac{\Gamma\left(j+\frac{1}{2}\right) 2^{j+\frac{1}{2}}}{2\pi\sqrt{2\pi}} \int\limits_{0}^{\pi/2} d\theta b_1 \left(\frac{\sin^2\theta}{b_1^2\sin^2\theta + 2\sin^2\theta + a_1^2}\right)^{j+\frac{1}{2}}$$

$$= \frac{1}{2\pi} \int\limits_{0}^{\pi/2} \sqrt{\frac{b^2\sin^2\theta}{b^2\sin^2\theta + a^2}} - \sum_{j=0}^{t-1} \frac{1}{j!} \frac{b\Gamma\left(j+\frac{1}{2}\right) 2^{j+\frac{1}{2}}}{2\pi\sqrt{2\pi}} \int\limits_{0}^{\pi/2} d\theta \left(\frac{\sin^2\theta}{a^2 + b^2\sin^2\theta}\right)^{j+\frac{1}{2}} \sqrt{y} \left[y + \frac{2\sin^2\theta}{a^2+b^2\sin^2\theta}\right]^{-\left(j+\frac{1}{2}\right)} d\theta \quad (38)$$

---

$$\mathcal{M}_1(a,b,\psi,t) = \frac{1}{2\pi} \int\limits_{1-\psi}^{1} (1-y)^{t-2} \int\limits_{0}^{\pi/2} \sqrt{\frac{b^2\sin^2\theta}{b^2\sin^2\theta + a^2}} dy d\theta - \sum_{j=0}^{t-1} \frac{1}{j!} \frac{b\Gamma\left(j+\frac{1}{2}\right) 2^{j+\frac{1}{2}}}{2\pi\sqrt{2\pi}} \times$$
$$\int\limits_{0}^{\pi/2} d\theta \left(\frac{\sin^2\theta}{a^2\sin^2\theta + b^2}\right)^{j+\frac{1}{2}} \int\limits_{1-\psi}^{1} dy (1-y)^{t-2} \sqrt{y} \left[y + \frac{2\sin^2\theta}{a^2\sin^2\theta + b^2}\right]^{-\left(j+\frac{1}{2}\right)} \quad (42)$$

---

$$\mathcal{D}_1(\psi,\alpha,m,n) \triangleq \int\limits_{1-\psi}^{1} \sqrt{y}(1-y)^m (y+\alpha)^{-n} dy = \sum_{l=0}^{m} (-1)^l \binom{m}{l} \int\limits_{1-\psi}^{1} y^{l+\frac{1}{2}} (y+\alpha)^{-n} dy$$

$$= \sum_{l=0}^{m} \frac{(-1)^l \binom{m}{l} \alpha^{-n}}{(l+\frac{3}{2})} \left\{ {}_2F_1\left(l+\frac{3}{2}, n; l+\frac{5}{2}; \frac{-1}{\alpha}\right) - (1-\psi)^{l+\frac{3}{2}} \, {}_2F_1\left(l+\frac{3}{2}, n; l+\frac{5}{2}; \frac{\psi-1}{\alpha}\right) \right\} \quad (44)$$

---

for positive integer values of $m$, given in (44) (shown in the next page). Using (44), the inner integral of (42) simplifies to

$$\int\limits_{1-\psi}^{1} dy (1-y)^{t-2} \sqrt{y} \left[y + \frac{2\sin^2\theta}{a^2\sin^2\theta + b^2}\right]^{-\left(j+\frac{1}{2}\right)} = \mathcal{D}_1\left(\psi, \frac{2\sin^2\theta}{a^2\sin^2\theta+b^2}, t-2, j+\frac{1}{2}\right). \quad (45)$$

Using (43) and (45), (42) simplifies to (46) (shown in the next page). Finally, using (46) in (41), we arrive at a single-integral based formula for $\mathcal{J}(a,b,t,B,\psi)$. To simplify (25) and (28), now consider the following random variable

$$\mathcal{G}(\Delta,\phi,b,t) = \int\limits_{\theta=0}^{\phi} \left(\frac{\sin^2\theta}{\sin^2\theta + b^2\Delta}\right)^t d\theta. \quad (47)$$

Note that the function $\mathcal{K}_1(a,y,t)$ of (25) for $a, y \geq 0$ can be expressed in terms of (47) as

$$\mathcal{K}_1(a,y,t) = E[Q(a\sqrt{y\beta})] = \frac{1}{\pi} \int\limits_{0}^{\frac{\pi}{2}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{ya^2}{2}}\right)^t d\theta = \frac{1}{\pi}\mathcal{G}\left(y, \frac{\pi}{2}, \frac{a}{\sqrt{2}}, t\right). \quad (48)$$

The average of (47) over $\Delta$ can be performed as (49) (shown in the next page). The simplification in (49) is due to [25]. As a result, we have, for $a > 0$,

$$\mathcal{K}(a,t,B,\psi) = E\left[\mathcal{K}_1(a,\Delta,t)\right] = \frac{1}{\pi} E\left[\mathcal{G}\left(\Delta, \frac{\pi}{2}, \frac{a}{\sqrt{2}}, t\right)\right] = \frac{1}{\pi}\overline{\mathcal{G}}\left(\frac{\pi}{2}, \frac{a}{\sqrt{2}}, t, B, \psi\right) \quad (51)$$

## References


[1] T. K. Y. Lo, "Maximum ratio transmission," *IEEE Trans. Commun.,* vol. 47, no. 10, Oct. 1999, pp. 1458-1461.

[2] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, "Efficient use of side information in multiple-antenna data transmission over fading channels," *IEEE J. Sel. Areas in Comm.,* vol. 16, no. 8, Oct. 1998, pp. 1423-1436.

[3] A. J. Grant, "Performance analysis of transmit beamforming," *IEEE Transactions on Commun.,* vol. 53, Apr. 2005, pp. 738-744.

[4] E. Visotsky and U. Madhow, "Space-time transmit precoding with imperfect feedback," *IEEE Trans. Info. Theory,* vol. 47, no. 6, Sept. 2001, pp. 2632-2639.

[5] T. Yoo and A. J. Goldsmith, "Capacity and power allocation for fading MIMO channels with channel estimation error," *IEEE Trans. Info. Theory,* vol. 52, no. 5, May 2006, pp. 2203-2214.

[6] E. N. Onggosanusi, A. Gatherer, A. G. Dabak, and S. Hosur, "Performance analysis of closed-loop transmit diversity in the presence of feedback delay," *IEEE Trans. Commun.,* vol. 49, no. 11, Sept. 2001, pp. 1618-1630.

[7] Y. Chen, and C. Tellambura, "Performance analysis of maximum ratio transmission with imperfect channel estimation," *IEEE Comm. Letters,* vol. 9, no. 4, Apr. 2005, pp. 322-324.

$$\mathcal{M}_1(a,b,\psi,t) = \frac{1}{\pi 2^{B+1}(t-1)}\frac{ab}{(a^2+b^2)}\,{}_2F_1\left(1,1;\frac{3}{2};\frac{b^2}{a^2+b^2}\right) - \sum_{j=0}^{t-1}\frac{1}{j!}\frac{b\Gamma\left(j+\frac{1}{2}\right)2^{j+\frac{1}{2}}}{2\pi\sqrt{2\pi}} \times \int_0^{\pi/2} d\theta \left(\frac{\sin^2\theta}{a^2\sin^2\theta+b^2}\right)^{j+\frac{1}{2}} \times \mathcal{D}_1\left(\psi, \frac{2\sin^2\theta}{a^2\sin^2\theta+b^2}, t-2, j+\frac{1}{2}\right) \quad (46)$$

$$\overline{\mathcal{G}}(\phi,b,t,B,\psi) \triangleq E\left[\mathcal{G}(\Delta,\phi,b,t)\right] = 2^B(t-1)\int_{\theta=0}^{\phi}\left(\frac{\sin^2\theta}{b^2}\right)^t \int_{y=1-\psi}^{1}(1-y)^{t-2}\left(y+\frac{\sin^2\theta}{b^2}\right)^{-t} dy d\theta$$

$$= 2^B(t-1)\int_{\theta=0}^{\phi}\left(\frac{\sin^2\theta}{b^2}\right)^t \mathcal{D}_2\left(\psi,\frac{\sin^2\theta}{b^2},t-2,t\right)d\theta, \quad (49)$$

$$\mathcal{D}_2(\psi,\alpha,m,n) \triangleq \int_{1-\psi}^{1}(1-y)^m(y+\alpha)^{-n}dy = \frac{-\alpha^n}{1+m}\,{}_2F_1\left(1,n;2+m;-\frac{1}{\alpha}\right) - \frac{\alpha^{1-n}(1+\alpha)^m}{n-1}\,{}_2F_1\left(1-n,-m;2-n;\frac{\alpha}{\alpha+1}\right) + \frac{(1+\alpha)^m(1+\alpha-\psi)^{1-n}}{n-1}\,{}_2F_1\left(1-n,-m;2-n;\frac{1+\alpha-\psi}{\alpha+1}\right) \quad (50)$$


[8] N.-S. Kim and Y. H. Lee, "Effect of channel estimation errors and feedback delay on the performance of closed-loop transmit diversity system," in *Proc. IEEE SPAWC*, 2003, pp. 542-545.

[9] J. Hamalainen and R. Wichman, "The effect of feedback delay to the closed-loop transmit diversity in FDD WCDMA," in *Proc. IEEE PIMRC*, Sept. 2001, pp. 27-31.

[10] J. Hamalainen and R. Wichman, "Performance analysis of closed-loop transmit diversity in the presence of feedback errors," in *Proc. IEEE PIMRC*, Sept. 2002.

[11] D. J. Love, R. W. Heath Jr, and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Info. Theory,* vol. 49, no. 10, Oct. 2003, pp. 2735-2747.

[12] J. Zheng, E. R. Duni, and B. D. Rao, "Analysis of Multiple-Antenna Systems With Finite-Rate Feedback Using High-Resolution Quantization Theory," *IEEE Trans. Signal Processing,* vol. 55, no. 4, Apr. 2007, pp. 1461-1476.

[13] S. Zhou, Z. Wang, and G. Giannakis, "Quantifying the power loss when transmit beamforming relies on finite-rate feedback," *IEEE Trans. Wireless Commun.,* vol. 4, no. 4, July 2005, pp. 1948-1957.

[14] J. C. Roh and B. D. Rao, "Transmit beamforming in multiple-antenna systems with finite-rate feedback: A vq-vased approach," *IEEE Trans. Info. Theory,* vol. 52, no. 3, March 2006, pp. 1101-1112.

[15] C. R. Rao, *Linear Statistical Inference and Its Applications,* John Wiley & Sons, 1973.

[16] R. Mallik, D. Singh, and A. Kumari, "Analysis of Rake reception with multiple symbol weight estimation for antipodal signaling," *IEEE Transactions on Commun.,* vol. 51, pp. 1721-1729, Oct. 2003.

[17] R. Annavajjala, P. C. Cosman, and L. B. Milstein, "Performance analysis of linear modulation schemes with generalized diversity combining on Rayleigh fading channels with noisy channel estimates," *IEEE Trans. on Info. Theory*, vol. 53, pp. 4701-4727, Dec. 2007.

[18] M. K Simon and M.-S. Alouini, *Digital Communications Over Fading Channels: A Unified Approach to Performance Analysis,* Wiley Series, July 2000.

[19] K. Cho and Y. Yoon, "On the general BER expression of one- and two-dimensional amplitude modulation," *IEEE Trans. Commun.,* vol. 50, no. 7, July 2002, pp. 1074-1080.

[20] R. F. Pawula, S. O. Rice and J. H. Roberts, "Distribution of the phase angle between two vectors perturbed by Gaussian noise," *IEEE Trans. Commun.,* vol. COM-30, no. 8, pp. 1828-1841, August 1982.

[21] J. Lassing, E. Strom, E. Agrell and T. Ottosson, "Computation of the exact bit error rate of coherent M-ary PSK with Gray code bit mapping," *IEEE Trans. Commun.,* vol. 51, no. 10, November 2003, pp. 1758−1760.

[22] M. K. Simon and R. Annavajjala, "On the optimality of bit detection of certain digital modulations," *IEEE Trans. Commun.*, vol. 53, no. 2, Feb. 2005, pp. 299-307.

[23] R. Annavajjala, "Comments on "Exact error-rate analysis of diversity 16-QAM with channel estimation error," in *IEEE Trans. Commun.,*, vol. 54, no. 3, March 2006.

[24] *IEEE Standard for Local and Metropolitan Area Networks, Amendment 2: Physical and Medium Access Control Layers for Combined Fixed and Mobile Operation in Licensed Bands.* IEEE Std. 802.16-2005.

[25] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products,* Corrected and Enlarged Edition, Academic Press, Inc., 1980.