

Packet Error Probability of a Transmit Beamforming System with Imperfect Feedback

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Abstract—Average packet error probability (PEP) is an important error statistic for wireless communication system designers. In this paper we address the problem of analytically quantifying the effect of channel estimation errors, feedback delay and channel vector quantization on the PEP of transmit beamforming multiple input single output (MISO) systems in a spatially independent slow-fading wireless channel environment. We develop an accurate characterization of estimation errors as well as errors due to feedback delay, and tools relevant for deriving analytical expressions for the PEP. The modeling highlights the distinction between errors that arise due to channel estimation from those that arise due to feedback delay and represents an important departure from past work. Analytical expressions are derived for the PEP with BPSK signaling. The derived approximated closed-form analytical expression is complemented by simulations.

Index Terms: MISO systems, transmit beamforming, packet fading channel, channel estimation errors, feedback delay, channel quantization, average packet error probability

I. INTRODUCTION

Multiplicative fading is a major source of performance degradation in multipath wireless environment. Channel coding and interleaving can offer some protection from the negative effects of fading. However, in some wireless systems data has to be organized into small packets, which are confined to fixed time slots, with or without interleaving. One popular example of such a system is the slotted multiple access scheme. It is important for the system designers to know the impact of fading on the performance. An important metric for studying the performance of a non-interleaved wireless packet data transmission is the average packet error probability (PEP) [1]-[16]. Packet error probability is also increasingly becoming an important quality of service parameter for the wireless networking community since it determines how frequently the information packet has to be re-transmitted [17]-[21].

Extensive analytical results quantifying the impact of fading on average symbol and error probability (SEP/BEP) are available for various modulation schemes [22]. However, in slow fading situations, there is no mapping between the average SEP/BEP and the average PEP. Consequently knowing average SEP/BEP does not help in understanding the average PEP. Analysis of average PEP is a more complicated problem compared to the analysis of average SEP/BEP. Analytical quantification of packet error probability has received considerable attention in the literature [1]-[16]. Closed-form expressions for PEP have been derived

for the non-coherent FSK modulation. The non-coherent FSK's SEP, conditioned on the channel, is an exponential function and taking expectation of the higher powers of conditional SEP w.r.t. the fading random variable is analytically tractable. However, closed-form expressions are not available for coherent BPSK and other constellations. Conditional PEP (conditioned on a function of the wireless channel) for a scheme such as coherent BPSK results in integer powers of the Gaussian- Q function. This makes the analysis challenging because in order to derive the average PEP expression, one has to integrate the integer powers of the Gaussian- Q function w.r.t. the random variable that captures the fading environment, an analytically difficult exercise. We also note that, to the best of our knowledge, the effect of channel estimation errors on PEP has not been considered in the literature. In this paper we consider the problem of deriving analytical expressions for PEP of a multiple input single output (MISO) system with various forms of practical feedback imperfections. We later show that the derived average packet error probability expression, captures the analysis of other commonly interested performance metrics as special cases.

In a MISO system, if the channel state information (CSI) is available at the transmitter, one can achieve both the diversity and the array gains with transmit beamforming via maximal ratio transmission [23]. However, in frequency division duplexing MISO systems the CSI has to be fed back from the receiver to the transmitter, and practical feedback systems suffer from many forms of imperfections resulting in performance degradation. The first form of feedback imperfection considered is channel estimation error. Typically the channel is estimated at the receiver with the help of training symbols. Due to the presence of thermal noise, channel estimation errors are inevitable in any practical system. It is now a common practice to model the actual channel and its estimate as a jointly Gaussian random process, with an error term that is orthogonal to the channel estimate ([24], [34], [36], and [49]). The error term associated with a particular channel estimate is un-known to the receiver and hence it becomes part of noise when the performance analysis is carried out. If the channel under consideration is varying at symbol level, or if the performance criteria is average symbol/bit error probability, then the variance of the error term will be simply added (along with the symbol dependency) to the variance of the receiver noise resulting in an effective noise term with variance equaling the sum of variance of receiver noise and the variance of the estimation error term ([24], [36]-[38], and the references therein). In this paper, we also follow the standard model of joint Gaussianity between the channel and its estimate, but *adapt it to the packet fading model*. An important difference is that, in a packet fading model the error term is constant for the entire packet while each symbol experiences a different noise sample, requiring new analytical tools in order to study the performance.

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The second form of feedback imperfection we address is feedback delay. In any practical system, delay between constructing the beamforming vector at the receiver and using it at the transmitter is an un-avoidable form of feedback imperfection. Another well accepted formulation [25], [26], [34], and [37]-[43] is to treat the impact of feedback delay in a manner similar to estimation errors, i.e., actual channel and its delayed version are assumed to be jointly Gaussian with an un-known (to the receiver) error term that is orthogonal to the delayed version. Since the delay related error term is un-known to the receiver, similar to estimation related error term, during performance analysis, it becomes part of noise thus removing any conceptual distinction between the mismatch in beamforming due to feedback delay and estimation errors. Though much of the past work on feedback delay [25], [26], [34], and [37]-[43] effectively make the delay related error term part of receiver noise, alternate options were considered (primarily in the context of adaptive modulation) in [44]-[51]. However, it is important to note that much of the work treated estimation errors and feedback delay in a similar manner, i.e., either both the error terms are assumed to be known or un-known to the receiver. In this paper, based on feedback system considerations we feel it is appropriate to treat the errors due to feedback delay to be known at the receiver, while the errors due to estimation errors are un-known at the receiver. This modeling approach is adopted in this work and it shows that the impact of feedback delay on beamforming MISO system performance can be less severe and is also conceptually quite different from channel estimation errors. Such modeling of feedback delay has been considered by us in an earlier conference publication, albeit in a SEP context [33].

In frequency domain duplexing systems, the estimated channel has to be conveyed to the transmitter. In low-rate feedback systems, an effective way to send the channel information to the transmitter is by having a known codebook at both the receiver and the transmitter so that the receiver can just send the index number of the chosen codepoint. This leads to the third form of feedback imperfection considered in this paper, namely finite-rate quantization of the channel [27]- [32]. To summarize, the contributions of this paper are threefold: an accurate characterization of estimation errors in a packet fading context, a new modeling of feedback delay which shows improved performance for a beamforming MISO system and conceptually distinguishes it from estimation errors, and derivation of an analytical expression quantifying the impact of channel estimation errors, feedback delay and channel quantization on the average packet error probability. All these contributions further the understanding of feedback communication systems. As a side benefit, the analytical tools developed promise to be of general interest with broad applicability. The rest of this paper is organized as follows. A general framework for the modeling of imperfect feedback in the packet fading context is presented in Section II. An analytical expression for the average packet error probability with un-coded BPSK constellation is derived in Section III. Numerical and simulation results are presented in Section IV. We conclude this paper in Section V.

Notation: Small and upper case bold letters indicate vector and matrix respectively. $E(\cdot)$, $(\cdot)^T$, $(\cdot)^H$, $|\cdot|$, $(\bar{\cdot})$, and $\|\cdot\|$ denote expectation, transpose, Hermitian, absolute value, complex conjugate, and 2-norm respectively. $x \sim p(x)$ indicates that the random variable x is distributed as $p(x)$. $\mathbf{x} \sim \mathcal{NC}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

indicates a circularly symmetric complex Gaussian (CSCG) random variable \mathbf{x} with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$.

Important Variables: t -number of transmit antennas, ℓ -packet index, k -symbol index in the packet, N -number of symbols in a packet, $\tilde{\rho}_e$ - estimation related correlation co-efficient, ρ_e - magnitude of $\tilde{\rho}_e$, $\tilde{\rho}_d$ -delay related correlation co-efficient, ρ_d - magnitude of $\tilde{\rho}_d$, B -number of feedback bits, γ_b -SNR per bit, \mathbf{h}_ℓ -actual channel of ℓ^{th} packet, $\hat{\mathbf{h}}_\ell$ -estimated channel of ℓ^{th} packet.

II. SYSTEM MODEL

We consider a multiple input single output system with t antennas at the transmitter and one antenna at the receiver. Let \mathbf{h}_ℓ be the channel between the transmitter and the receiver for the ℓ^{th} packet. \mathbf{h}_ℓ is modeled as a spatially i.i.d frequency-flat Rayleigh fading channel which is constant for all the N un-coded symbols in packet ℓ . The vector valued channel $\mathbf{h}_\ell \sim \mathcal{NC}(\mathbf{0}, \mathbf{I})$. Furthermore, it is assumed that the channel varies from packet to packet but exhibits significant correlation. The transmitted k^{th} symbol in the packet ℓ is denoted by $s_\ell[k]$ and $E[|s_\ell[k]|^2] = E_s$. Let \mathbf{w}_ℓ be the unit norm beamforming vector (BV) at the transmitter for the packet ℓ . Then, the k^{th} received signal in the packet ℓ is given by

$$y_\ell[k] = \mathbf{h}_\ell^H \mathbf{w}_\ell s_\ell[k] + \eta_\ell[k], \quad k = 1, 2, \dots, N \quad (1)$$

where $\eta_\ell[k]$ is the thermal noise that effects the k^{th} symbol of ℓ^{th} packet, $\eta_\ell[k] \sim \mathcal{NC}(0, \sigma_n^2)$. We now discuss in detail the three forms of feedback imperfections and develop a general model that captures channel estimation errors, feedback delay, and finite-rate channel quantization for transmit beamforming MISO systems.

A. Channel Estimation Errors - Packet Fading Model

Let $\hat{\mathbf{h}}_\ell$ be the estimate of \mathbf{h}_ℓ . We assume that \mathbf{h}_ℓ and $\hat{\mathbf{h}}_\ell$ are jointly Gaussian, a reasonable assumption for many practical estimation techniques ([24], [36]-[38], and the references therein). The jointly Gaussian assumption allows us to relate them as follows:

$$\mathbf{h}_\ell = \frac{\tilde{\rho}_e}{\sqrt{\Lambda}} \hat{\mathbf{h}}_\ell + \sqrt{1 - \rho_e^2} \boldsymbol{\varepsilon}_{\ell e} \quad (2)$$

where $\boldsymbol{\varepsilon}_{\ell e} \sim \mathcal{NC}(\mathbf{0}, \mathbf{I})$, $\hat{\mathbf{h}}_\ell \sim \mathcal{NC}(\mathbf{0}, \Lambda \mathbf{I})$, \mathbf{I} denotes the identity matrix of size $t \times t$, $E[|\hat{h}_i|^2] = \Lambda$, $i = 1, \dots, t$, and $\tilde{\rho}_e = \rho_e e^{j\phi_{\rho_e}}$ is the complex correlation coefficient that determines the degree of accuracy in channel estimation. The closer ρ_e is to one, the more accurate is the channel estimate. $\tilde{\rho}_e$ can be assumed to be known at the receiver. Assuming instantaneous feedback and no channel quantization, the beamforming vector to be used at the transmitter is given by

$$\mathbf{w}_\ell = \frac{\hat{\mathbf{h}}_\ell}{\|\hat{\mathbf{h}}_\ell\|}. \quad (3)$$

The k^{th} received signal of the ℓ^{th} packet with the BV given in (3) and \mathbf{h}_ℓ given in (2) is

$$\begin{aligned} y_\ell[k] &= \mathbf{h}_\ell^H \mathbf{w}_\ell s_\ell[k] + \eta_\ell[k] \\ &= \left(\frac{\tilde{\rho}_e}{\sqrt{\Lambda}} \hat{\mathbf{h}}_\ell + \sqrt{1 - \rho_e^2} \boldsymbol{\varepsilon}_{\ell e} \right)^H \frac{\hat{\mathbf{h}}_\ell}{\|\hat{\mathbf{h}}_\ell\|} s_\ell[k] + \eta_\ell[k] \\ &= \frac{\tilde{\rho}_e}{\sqrt{\Lambda}} \|\hat{\mathbf{h}}_\ell\| s_\ell[k] + \left(\sqrt{1 - \rho_e^2} \boldsymbol{\varepsilon}_{\ell e} s_\ell[k] + \eta_\ell[k] \right) \end{aligned} \quad (4)$$

where

$$\tilde{\varepsilon}_{\ell e} = \frac{\varepsilon_{\ell e}^H \hat{\mathbf{h}}_{\ell}}{\|\hat{\mathbf{h}}_{\ell}\|} \sim \mathcal{NC}(0, 1).$$

In the above equation $\tilde{\varepsilon}_{\ell e}$ is *un-known* to the receiver and hence the receiver *can not compensate* for the phase rotation caused due to $\tilde{\varepsilon}_{\ell e}$. In our previous work ([33] and [34]), the performance criteria was average SEP/BEP and as a consequence the estimation error term simply got absorbed into the thermal noise and increased its variance (along with introducing a symbol dependency). In the packet fading model, the estimation error term $\tilde{\varepsilon}_{\ell e}$ is constant and impacts the entire packet (while each symbol experiences a different noise sample) and hence it can not be simply lumped into the additive noise term. As will be evident from the next section, not lumping the estimation error term into white noise also makes the analysis more complicated. Note that PEP analysis requires that we pay attention to the fact that $\tilde{\varepsilon}_{\ell e}$ is constant for the duration of the packet.

B. Feedback Delay with Imperfect Channel Estimation

In the above discussion, a simplistic assumption of feedback being available instantly was made at the transmitter. In reality there is a delay and to account for this it is assumed that the beamforming vector for the current packet ℓ is derived from the channel estimate of the previous packet ($\ell - 1$). Since there is a time lag between forming the BV at the receiver and its use at the transmitter, the BV \mathbf{w}_{ℓ} depends on $\hat{\mathbf{h}}_{\ell-1}$ as opposed to the current channel estimate $\hat{\mathbf{h}}_{\ell}$. Assuming that the channel estimate and its delayed version are jointly Gaussian, we can relate them as follows:

$$\hat{\mathbf{h}}_{\ell} = \tilde{\rho}_d \hat{\mathbf{h}}_{\ell-1} + \sqrt{(1 - \rho_d^2)\Lambda} \varepsilon_{\ell d} \quad (5)$$

where $\hat{\mathbf{h}}_{\ell} \sim \mathcal{NC}(\mathbf{0}, \Lambda \mathbf{I})$ and $\tilde{\rho}_d = \rho_d e^{j\phi_{\rho d}}$ is the complex correlation co-efficient between $\hat{\mathbf{h}}_{\ell}$ and $\hat{\mathbf{h}}_{\ell-1}$, $\varepsilon_{\ell d} \sim \mathcal{NC}(\mathbf{0}, \mathbf{I})$ is the error term due to delay and is assumed to be uncorrelated with $\hat{\mathbf{h}}_{\ell-1}$. The delay correlation co-efficient $\tilde{\rho}_d$ is assumed to be known to the receiver and measures the impact of delay. With the help of (2) and (5), the actual channel \mathbf{h}_{ℓ} in terms of delayed version of the estimated channel $\hat{\mathbf{h}}_{\ell-1}$ can be written as

$$\mathbf{h}_{\ell} = \tilde{\rho}_e \left\{ \frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \hat{\mathbf{h}}_{\ell-1} + \sqrt{1 - \rho_d^2} \varepsilon_{\ell d} \right\} + \sqrt{1 - \rho_e^2} \varepsilon_{\ell e}. \quad (6)$$

With the inclusion of estimation errors along with the delay, the beamforming vector (still un-quantized) is given by

$$\mathbf{w}_{\ell} = \frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|}. \quad (7)$$

The beamforming vector \mathbf{w}_{ℓ} indicates that the BV is formed with the help of channel estimate from the previous packet ($\ell - 1$) and is used to transmit packet ℓ . With this formulation, the k^{th}

received symbol of the packet ℓ can be written as

$$\begin{aligned} y_{\ell}[k] &= \mathbf{h}_{\ell}^H \mathbf{w}_{\ell} s_{\ell}[k] + \eta_{\ell}[k] \\ &= \left(\tilde{\rho}_e \left\{ \frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \hat{\mathbf{h}}_{\ell-1} + \sqrt{1 - \rho_d^2} \varepsilon_{\ell d} \right\} + \sqrt{1 - \rho_e^2} \varepsilon_{\ell e} \right)^H \\ &\quad \frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} s_{\ell}[k] + \eta_{\ell}[k] \\ &= \tilde{\rho}_e \left\{ \frac{\tilde{\rho}_d \|\hat{\mathbf{h}}_{\ell-1}\|}{\sqrt{\Lambda}} + \sqrt{1 - \rho_d^2} \varepsilon_{\ell d} \right\} s_{\ell}[k] + \\ &\quad \underbrace{\sqrt{1 - \rho_e^2} \varepsilon_{\ell e} s_{\ell}[k] + \eta_{\ell}[k]}_{\text{Un-known to the Receiver}} \end{aligned} \quad (8)$$

where

$$\begin{aligned} \varepsilon_{\ell d} &= \frac{\varepsilon_{\ell d}^H \hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \sim \mathcal{NC}(0, 1), \\ \varepsilon_{\ell e} &= \frac{\varepsilon_{\ell e}^H \hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \sim \mathcal{NC}(0, 1), \end{aligned}$$

$\tilde{\rho}_e$ and $\tilde{\rho}_d$ are complex conjugates of $\tilde{\rho}_e$ and $\tilde{\rho}_d$ respectively. For a particular packet both $\varepsilon_{\ell d}$ and $\varepsilon_{\ell e}$ are constant. The distributions of both $\varepsilon_{\ell d}$ and $\varepsilon_{\ell e}$ indicate their random nature over a number of packets.

As explained in the previous section, the estimation related error term $\varepsilon_{\ell e}$ is un-known to the receiver and hence the phase rotation caused by $\varepsilon_{\ell e}$ can not be compensated. The role of delay related error term $\varepsilon_{\ell d}$, which determines the penalty due to the feedback delay, will depend on the modeling assumptions. Notice that if there is no feedback delay, then $\tilde{\rho}_d = 1$ and the error term vanishes. If $\varepsilon_{\ell d}$ is also assumed to be un-known, then it can be treated in a manner similar to estimation error. In particular, if the performance metric is SEP/BEP, or if the channel is varying at a symbol level as opposed to packet level, then $\varepsilon_{\ell d}$ can be merged into the receiver noise greatly simplifying the analysis [25], [26], and [37]-[43]. A closer examination, as explained below, indicates that there is a distinction between estimation error related term and the delay related term and so lumping them together is a questionable simplification. We propose a model/approach to handle this error that we believe more accurately captures the impact of delay on feedback system performance. The model is based on the following system assumption: there is a pilot sequence before every packet of data and that the beamforming vector (7) is based on the channel estimate from the previous packet. Under this assumption, the receiver will be knowing both $\hat{\mathbf{h}}_{\ell}$ and $\hat{\mathbf{h}}_{\ell-1}$ and hence it knows the error term due to delay $\varepsilon_{\ell d}$, c.f. 5 (and subsequently $\varepsilon_{\ell d}$). As shown later, this simple change in the approach impacts the performance of the system considerably (more performance gain is seen at lower ρ_d values), and indicates more clearly the conceptual distinction between the imperfections resulting from delay and estimation. However this modeling also makes the problem of performance analysis more complicated. Note that even if the receiver knows $\varepsilon_{\ell d}$, it can not compensate for all the loss caused due to delay. Its performance lies in between a system with no feedback delay ($\rho_d = 1$), and a system where the receiver does not know $\varepsilon_{\ell d}$, ($0 < \rho_d < 1$) and is lumped into the receiver noise.

C. Quantization of Delayed Version of Channel Estimate

In the previous two sections we assumed that the channel state information is exactly conveyed to the transmitter. In practice,

the receiver estimates the channel, and quantizes it into one of $C = 2^B$ code words in the codebook which is known to both transmitter and receiver. The index, which is represented by B bits, of the code word corresponding to the channel estimate is fed back to the transmitter. We assume that the feedback channel is error free [27]-[32]. The beamforming vector formed by quantizing delayed version of the channel estimate is given by

$$\mathbf{w}_\ell = \mathcal{Q} \left[\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right] \quad (9)$$

where \mathcal{Q} is the vector quantization function and here we assume a vector quantization based codebook ([28] and [32]). Using (6) and (9), the k^{th} received signal of the packet ℓ is:

$$\begin{aligned} y_\ell[k] &= \mathbf{h}_\ell^H \mathbf{w}_\ell s_\ell[k] + \eta_\ell[k] \\ &= \left(\tilde{\rho}_e \left\{ \frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \hat{\mathbf{h}}_{\ell-1} + \sqrt{1 - \rho_d^2} \varepsilon_{\ell d} \right\} + \sqrt{1 - \rho_e^2} \varepsilon_{\ell e} \right)^H \\ &\quad \left(\mathcal{Q} \left[\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right] \right) s_\ell[k] + \eta_\ell[k] \\ &= \tilde{\rho}_e \left(\frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \vartheta_{\ell-1} \|\hat{\mathbf{h}}_{\ell-1}\| + \sqrt{1 - \rho_d^2} \hat{\varepsilon}_{\ell d} \right) s_\ell[k] + \\ &\quad \sqrt{1 - \rho_e^2} \hat{\varepsilon}_{\ell e} s_\ell[k] + \eta_\ell[k] \end{aligned} \quad (10)$$

where

$$\begin{aligned} \hat{\varepsilon}_{\ell d} &= \varepsilon_{\ell d}^H \left(\mathcal{Q} \left[\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right] \right), \\ \hat{\varepsilon}_{\ell d} &\sim \mathcal{NC}(0, 1), \\ \hat{\varepsilon}_{\ell e} &= \varepsilon_{\ell e}^H \left(\mathcal{Q} \left[\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right] \right), \\ \hat{\varepsilon}_{\ell e} &\sim \mathcal{NC}(0, 1), \\ \vartheta_{\ell-1} &= \left(\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right)^H \left(\mathcal{Q} \left[\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right] \right). \end{aligned} \quad (11)$$

We now take a closer look at the additional changes to the k^{th} symbol in the received packet ℓ because of use of a quantized beamforming vector. The received signal with and without quantized BV is given by (12) and (13) respectively

Un-Quantized BV (7)

$$\begin{aligned} \underbrace{y_\ell[k]}_{\text{Un-Quantized BV (7)}} &= \underbrace{\tilde{\rho}_e \left\{ \frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \|\hat{\mathbf{h}}_{\ell-1}\| + \sqrt{1 - \rho_d^2} \varepsilon_{\ell d} \right\}}_{\text{Known to the Receiver}} s_\ell[k] \\ &\quad + \underbrace{\sqrt{1 - \rho_e^2} \varepsilon_{\ell e} s_\ell[k] + \eta_\ell[k]}_{\text{Un-known to the Receiver}}, \end{aligned} \quad (12)$$

Quantized BV (9)

$$\begin{aligned} \underbrace{y_\ell[k]}_{\text{Quantized BV (9)}} &= \underbrace{\tilde{\rho}_e \left(\frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \vartheta_{\ell-1} \|\hat{\mathbf{h}}_{\ell-1}\| + \sqrt{1 - \rho_d^2} \hat{\varepsilon}_{\ell d} \right)}_{\tilde{\psi}_\ell = \psi_\ell e^{j\phi_{\psi_\ell}} \text{ (Known to the Receiver)}} \\ &\quad s_\ell[k] + \underbrace{\sqrt{1 - \rho_e^2} \hat{\varepsilon}_{\ell e} s_\ell[k] + \eta_\ell[k]}_{\text{Un-known to the Receiver}}. \end{aligned} \quad (13)$$

The above two equations mainly defer in three places. The effective error terms with un-quantized BV are $\varepsilon_{\ell d}$ and $\varepsilon_{\ell e}$ and the effective error terms with quantized BV are $\hat{\varepsilon}_{\ell d}$ and $\hat{\varepsilon}_{\ell e}$. All these effective error terms are different in an instantaneous sense, but all of them are CSCG random variables with same mean and variance. The main effect of quantization on the PEP comes from $\vartheta_{\ell-1}$. Note that $\vartheta_{\ell-1}$ is the inner product between the un-quantized and quantized BV and the statistical characterization

of $\tilde{\Delta} \triangleq |\vartheta_{\ell-1}|^2$ will be important for the performance analysis. Since finding the exact pdf of $\tilde{\Delta}$ is difficult, [28] and [32] upper bounded $\tilde{\Delta}$ by a r.v Δ , whose pdf is given by

$$p_\Delta(x) = 2^B (t-1)(1-x)^{t-2}, \quad 1-\omega < x < 1 \quad (14)$$

where $\omega = 2^{-B/(t-1)}$. In what follows, we use Δ in place of $\tilde{\Delta}$. Because of our modeling assumptions, the receiver knows $\tilde{\rho}_d$, $\tilde{\rho}_e$, $\hat{\mathbf{h}}_\ell$, $\hat{\mathbf{h}}_{\ell-1}$, and $\mathcal{Q} \left[\frac{\hat{\mathbf{h}}_{\ell-1}}{\|\hat{\mathbf{h}}_{\ell-1}\|} \right]$, so it knows $\tilde{\psi}_\ell$ which enables coherent detection of the transmitted symbol.

III. AVERAGE PACKET ERROR PROBABILITY

Since the feedback delay related error term and estimation related error term are constant for the entire packet, average packet error probability, a metric that requires averaging over the packet index ℓ thereby capturing the effect of imperfect feedback in transmit beamforming MISO systems, provides a meaningful way to study performance analysis. Also as pointed out in the introduction, for slotted multiple access schemes and as a quality of service parameter for the MAC layer of wireless networks, analytical understanding of PEP is important from system design point of view. Existing results primarily are for PEP with non-coherent FSK modulation (limited to single input single output systems and under ideal conditions), and so are not applicable to the present scenario of imperfect feedback with coherent BPSK modulation.

In this section we present the analysis of the average packet error probability of an un-coded packet of N BPSK symbols with imperfect feedback. We first begin with the decision statistic required for the detection of k^{th} symbol of ℓ^{th} packet and then focus on the PEP. The coherent detection of transmitted symbol is based on $r_\ell[k]$, obtained from $y_\ell[k]$ in (13).

$$r_\ell[k] = e^{-j\phi_{\psi_\ell}} y_\ell[k] \quad (15)$$

$$= \left(\psi_\ell + \sqrt{1 - \rho_e^2} \check{\varepsilon}_{\ell e} \right) s_\ell[k] + \check{\eta}_\ell[k],$$

$$\psi_\ell = \left| \tilde{\rho}_e \left(\frac{\tilde{\rho}_d}{\sqrt{\Lambda}} \vartheta_{\ell-1} \|\hat{\mathbf{h}}_{\ell-1}\| + \sqrt{1 - \rho_d^2} \hat{\varepsilon}_{\ell d} \right) \right|, \quad (16)$$

$$\check{\varepsilon}_{\ell e} = e^{-j\phi_{\psi_\ell}} \hat{\varepsilon}_{\ell e} \sim \mathcal{NC}(0, 1),$$

$$\check{\eta}_\ell[k] = e^{-j\phi_{\psi_\ell}} \eta_\ell[k] \sim \mathcal{NC}(0, \sigma_n^2).$$

Notice that in the above equation, ψ_ℓ and $\check{\varepsilon}_{\ell e}$ do not depend on $[k]$ indicating that these two terms are fixed for the entire packet, whereas the notation for the white noise is $\check{\eta}_\ell[k]$ indicating a different noise sample for each symbol. To highlight the differences with our previous work [34], we now briefly contrast the decision variable (DV) in (15) to that of the DV we used in [34]. The performance metric in [34] was average SEP/BEP and the DV in [34] was given as

$$r[k] = \kappa_o s[k] + \xi[k], \quad (17)$$

where κ_o is a complex constant number and ξ is conditionally (conditioned on both fading and quantization of channel) CSCG random variable. In [34] to derive analytical expression average SEP/BEP, we had to account for the fact that the transmitted symbol is scaled and rotated as well as the noise is symbol dependent. The DV (15) in this paper is more complicated. All the symbols of ℓ^{th} block are scaled by a known random variable ψ_ℓ . Note that knowledge of ψ_ℓ at the receiver is possible due to the modeling assumptions presented. Also, all the symbols of

ℓ^{th} block experience the same channel estimation related error term $\tilde{\epsilon}_{\ell e}$.

Since we are restricting our attention to the BPSK constellation, the receiver uses the real part of $r_\ell[k]$ to decode the transmitted symbol as

$$\begin{aligned}\tilde{r}_\ell[k] &= \text{Real}(r_\ell[k]) \\ &= \kappa s_\ell[k] + \hat{\eta}_\ell[k]\end{aligned}$$

where

$$\begin{aligned}\kappa &= \psi_\ell + \sqrt{(1 - \rho_e^2)/2} \hat{\epsilon}_{\ell er}, \\ \psi_\ell &> 0, \quad -\infty < \kappa < \infty,\end{aligned}\quad (18)$$

$\hat{\epsilon}_{\ell er}$ and $\hat{\eta}_\ell$ are both real random variables with $\hat{\epsilon}_{\ell er} = \text{Real}(\hat{\epsilon}_{\ell e}) \sim \mathcal{N}(0, 1)$ and $\hat{\eta}_\ell[k] = \text{Real}(\hat{\eta}_\ell[k]) \sim \mathcal{N}(0, \sigma_n^2/2)$. As pointed out earlier $\hat{\epsilon}_{\ell er}$ is a fixed constant for a particular packet, and viewed over a number of packets it is a statistical quantity. The derivation of an analytical expression for PEP is quite involved and here we outline the important steps in the derivation.

- 1) Derivation of pdf for κ : The signal scaling random variable in the decision statistic $\tilde{r}_\ell[k]$ is κ . So the pdf of κ is important for the analysis of PEP. The derivation of the pdf of κ is complicated by its dependency on the three forms of imperfection. Details of the derivation can be found in Section III-A.
- 2) Expectation of the Gaussian Q -function and its higher powers w.r.t. the random variable κ : Much of the analytical complexity in the performance analysis (PEP) revolves around the evaluation of expectations of the Gaussian Q -function and its higher powers w.r.t. the random variable κ in closed-form. For the first two powers of the Gaussian Q -function, we are able to evaluate the expectation using the exact form of the Gaussian Q -function. For higher powers (≥ 3), we make use of an approximation of the Gaussian Q -function and evaluate its expectation w.r.t. κ . Details are in Section III-B.2.

A. Derivation of pdf for κ

The signal scaling term κ in the decision statistic $\tilde{r}_\ell[k]$ is given in (18):

$$\kappa = \psi_\ell + \sqrt{(1 - \rho_e^2)/2} \hat{\epsilon}_{\ell er}, \quad \psi_\ell > 0, \quad -\infty < \kappa < \infty.$$

Conditioned on ψ_ℓ , the conditional pdf of κ is given by

$$p_\kappa(z|\psi_\ell = x) = \frac{1}{\sqrt{\pi(1 - \rho_e^2)}} e^{-\frac{(z-x)^2}{1-\rho_e^2}}.$$

and the pdf of κ can be obtained as

$$\begin{aligned}p_\kappa(z) &= \int_{-\infty}^{\infty} p_\kappa(z, x) dx \\ &= \int_{-\infty}^{\infty} p_\kappa(z|x) p_{\psi_\ell}(x) dx\end{aligned}\quad (19)$$

The evaluation of (19) requires the pdf of ψ_ℓ which is derived in Appendix-I and the final expression is given in (64). Note that if there are no estimation errors in the model, or if the performance criteria is average symbol/bit error probability [33], or if the performance criteria is average packet error probability with any constellation other than BPSK, then the pdf of ψ_ℓ

becomes central to the performance analysis. Here, not only is the pdf of ψ_ℓ required, the extra step discussed in (19) has to be carried out for the pdf of κ . Substituting the pdf of ψ_ℓ in (19), we obtain

$$p_\kappa(z) = \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{a2}(l, p, g, \mathcal{L}, z) + \mathcal{K}_{p_2} f_{b2}(l, p, g, \mathcal{L}, z) \} \quad (20)$$

where the variable and the corresponding defining equation are listed as pairs: n - (46), δ - (55), \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), and \mathcal{K}_{p_2} - (59), and

$$\begin{aligned}f_{b2}(l, p, g, \mathcal{L}, z) &= \frac{2 \mathcal{L}^{n-l-g} e^{-\frac{z^2}{1-\rho_e^2}}}{\rho_e^{2(n-l-g)} \Gamma(n-l-g) \sqrt{\pi(1-\rho_e^2)}} \\ &\int_0^\infty x^{2(n-l-g)-1} e^{-\frac{x^2[\rho_e^2 + \mathcal{L}(1-\rho_e^2)]}{\rho_e^2(1-\rho_e^2)} + \frac{2xz}{1-\rho_e^2}} dx.\end{aligned}$$

To evaluate the above integral we use the following identity [55]

$$\int_0^\infty x^{v-1} e^{-\beta x^2 - \gamma x} dx = (2\beta)^{-v/2} \Gamma(v) e^{\frac{\gamma^2}{8\beta}} D_{-v} \left(\frac{\gamma}{\sqrt{2\beta}} \right). \quad (21)$$

In the present context

$$\begin{aligned}\beta &= \frac{\rho_e^2 + \mathcal{L}(1 - \rho_e^2)}{\rho_e^2(1 - \rho_e^2)}, \\ \gamma &= \frac{-2z}{1 - \rho_e^2}, \\ v &= 2(n - l - g).\end{aligned}$$

$f_{b2}(l, p, g, \mathcal{L}, z)$ can now be written as

$$f_{b2}(l, p, g, \mathcal{L}, z) = \mathcal{H}_1 e^{-z^2 \mathcal{H}_2} D_{-2(n-l-g)}(-z \mathcal{H}_3), \quad (22)$$

$-\infty < z < \infty$

where

$$\mathcal{H}_1 = \frac{\mathcal{L}^{\frac{v}{2}} (1 - \rho_e^2)^{\frac{v}{2}} \Gamma(v)}{2^{(\frac{v}{2})-1} \Gamma(\frac{v}{2}) (\rho_e^2 + \mathcal{L}(1 - \rho_e^2))^{\frac{v}{2}} \sqrt{\pi(1 - \rho_e^2)}}, \quad (23)$$

$$\mathcal{H}_2 = \frac{\rho_e^2 + 2\mathcal{L}(1 - \rho_e^2)}{2(1 - \rho_e^2)(\rho_e^2 + \mathcal{L}(1 - \rho_e^2))}, \quad (24)$$

$$\mathcal{H}_3 = \frac{\sqrt{2} \rho_e}{\sqrt{(1 - \rho_e^2)(\rho_e^2 + \mathcal{L}(1 - \rho_e^2))}}, \quad (25)$$

and $D_{\tilde{p}}(\tilde{l})$ is the parabolic cylinder function [55]. Finally

$$f_{a2}(l, p, g, \mathcal{L}, z) = f_{b2}(l, p, g, 1, z), \quad (26)$$

which completes all the steps required to compute $p_\kappa(z)$ in (20). The analytical expression for $p_\kappa(z)$ is confirmed using simulations (histogram approach is used to get the simulated version of κ 's pdf) in Fig. 1 for number of transmit antennas $t \in \{2, 3\}$, delay correlation co-efficient $\rho_d \in \{0.98, 0.94\}$, and estimation error correlation co-efficient $\rho_e \in \{0.95, 0.91\}$. The number of feedback bits B which determines the quantization codebook size (2^B) is fixed at 4.

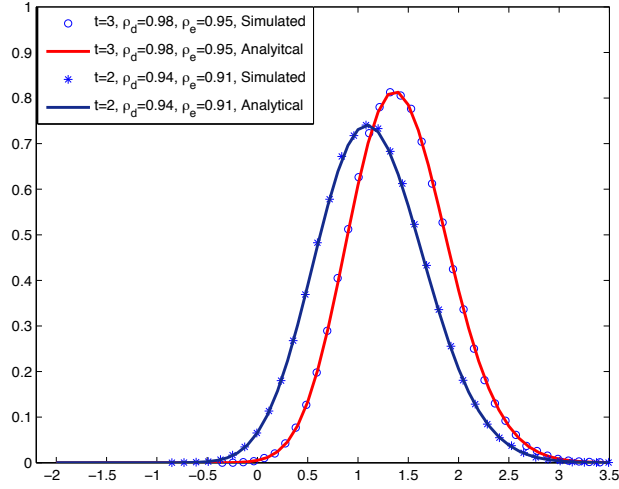


Fig. 1. Verification of the pdf for the signal scaling term κ defined in (18). Number of feedback bits $B=4$.

B. Analytical Expression for Packet Error Probability

Conditioned on κ , the packet error probability (the probability that at least one symbol in the packet is received incorrectly) is given by

$$\begin{aligned} P_{B,\ell}(\gamma_b, \rho_e, t, N) &= 1 - \{1 - p_{b,\ell}\}^N \\ &= 1 - \sum_{m=0}^N \binom{N}{m} (-1)^m (p_{b,\ell})^m \end{aligned}$$

where $p_{b,\ell}$ is the error probability of a symbol in the ℓ^{th} packet (calculated with the help of decision statistic $\tilde{r}_\ell[k]$)¹ Note that since κ is fixed for the entire packet (while the noise sample is different for each symbol), all the symbols have the same error probability. The average packet error probability is given by

$$\begin{aligned} \tilde{P}_B(\gamma_b, \rho_e, t, N) &= E_\ell [P_{B,\ell}(\gamma_b, \rho_e, t, N)] \\ &= 1 - \sum_{m=0}^N \binom{N}{m} (-1)^m E_\ell [(p_{b,\ell})^m]. \end{aligned} \quad (27)$$

Accounting for the fact that ' κ ' can be negative with a non-trivial probability, with BPSK constellation, $E_\ell [(p_{b,\ell})^m]$ can be written as

$$\begin{aligned} E_\ell [(p_{b,\ell})^m] &= A_p E_\ell \left[Q^m \left(\sqrt{2\gamma_b z_1} \right) \right] + \\ &\quad (1 - A_p) E_\ell \left\{ 1 - Q \left(\sqrt{2\gamma_b z_2} \right) \right\}^m \\ &= A_p E_{z_1} \left[Q^m \left(\sqrt{2\gamma_b z_1} \right) \right] + (1 - A_p) \\ &\quad \sum_{w=0}^m (-1)^w E_{z_2} \left\{ Q^w \left(\sqrt{2\gamma_b z_2} \right) \right\} \end{aligned} \quad (28)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{u=x}^{\infty} \exp\left(-\frac{u^2}{2}\right) du$$

¹Similar to [8], the above equation can be easily modified to the scenario of a block channel coded system that can correct an arbitrary number of errors.

is the standard Gaussian tail function [22] and A_p is the area under the positive side of pdf $p_\kappa(z)$. i.e. $A_p = \int_0^\infty p_\kappa(z) dz$. In (28), γ_b is the SNR of a symbol in the packet and

$$\begin{aligned} z_1 &= \kappa^2, & 0 < \kappa < \infty, \\ z_2 &= \kappa^2, & -\infty < \kappa < 0. \end{aligned}$$

Using transformation of random variables, the pdfs of z_1 and z_2 , needed to evaluate (28), can be shown to be given by (67) and (68) respectively. For readability purpose the pdfs of z_1 (67) and z_2 (68) are given at the end of Appendix-I. To evaluate (28), we need to find an expression for A_p and evaluate the expectations of the Gaussian Q -function and its higher powers w.r.t. the random variable κ . These steps are described below.

1) *Evaluation of A_p* : A closed-form expression for A_p can be evaluated as

$$\begin{aligned} A_p &= \int_0^\infty p_\kappa(z) dz \\ &= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \left\{ \mathcal{K}_{p_1} f_{ap1}(l, p, g, \mathcal{L}, z) + \right. \\ &\quad \left. \mathcal{K}_{p_2} f_{ap2}(l, p, g, \mathcal{L}, z) \right\}, \end{aligned} \quad (29)$$

$$f_{ap2}(l, p, g, \mathcal{L}, z) = 1 - \mathcal{H}_1 \int_0^\infty e^{-z^2 \mathcal{H}_2} D_{-\nu}(\mathcal{H}_3 z) dz.$$

where $\nu = 2(n-l-g)$. Let $z = \sqrt{x}$, $dz = \frac{1}{2}x^{-\frac{1}{2}} dx$,

$$\begin{aligned} f_{ap2}(l, p, g, \mathcal{L}, z) &= 1 - \frac{\mathcal{H}_1 2^{-(n-l-g+1)} \sqrt{\pi}}{\Gamma(n-l-g+1)} \\ &\quad {}_2F_1\left(v/2, \frac{1}{2}; v/2+1; \frac{4\mathcal{H}_2 - \mathcal{H}_3^2}{4\mathcal{H}_2 + \mathcal{H}_3^2}\right) \end{aligned} \quad (30)$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric function. To evaluate the above integral we used [55]

$$\begin{aligned} \int_0^\infty x^{-\frac{\beta}{2}+1} e^{-xc} D_{-\nu}\left(2(kx)^{\frac{1}{2}}\right) dx &= \frac{2^{1-\beta-\frac{\nu}{2}} \sqrt{\pi} \Gamma(\beta)}{\Gamma\left(\frac{\nu+\beta+1}{2}\right)} \\ &\quad (c+k)^{-\frac{\beta}{2}} {}_2F_1\left(\frac{\nu}{2}, \frac{\beta}{2}; \frac{\nu+\beta+1}{2}; \frac{c-k}{c+k}\right), \end{aligned} \quad (31)$$

$$\left[\text{Re}(c+k) > 0, \quad \text{Re}\left(\frac{c}{k}\right) > 0 \right].$$

In (29) $f_{ap1}(l, p, g, \mathcal{L}, z) = f_{ap2}(l, p, g, 1, z)$.

2) *Evaluating the expectations of the Gaussian Q -functions in (28)*: In this subsection we evaluate the expectations of the Gaussian Q -function and its higher powers w.r.t. the random variable κ . As explained in the previous subsection, the random variable κ is now split into two random variables z_1 (capturing the positive part of κ) and z_2 (capturing the negative part of κ) in closed-form. For $m \in \{1, 2\}$ (m being the power of the Gaussian Q -function), we evaluate the expectation using the exact form of the Gaussian Q -function. For higher powers ($m \geq 3$), we make use of an approximation of the Gaussian Q -function and evaluate its expectation w.r.t. z_1 and z_2 . We start with $A_p E_{z_1} [Q^m(\sqrt{2\gamma_b z_1})]$, $m = 1$ and 2, required to compute (28):

$$\begin{aligned} A_p E_{z_1} \left[Q^m \left(\sqrt{2\gamma_b z_1} \right) \right] &= \int_{z_1=0}^{\infty} Q^m \left(\sqrt{2\gamma_b z_1} \right) \\ &\quad \left[\hat{p}_1(z_1) + \hat{p}_2(z_1) \right] dz_1. \end{aligned} \quad (32)$$

For $m = 1$, $A_p E_{z_1} [Q^m(\sqrt{2\gamma_b z_1})]$ can be written as

$$\begin{aligned} A_p E_{z_1} [Q(\sqrt{2\gamma_b z_1})] &= \int_{z_1=0}^{\infty} Q(\sqrt{2\gamma_b z_1}) \hat{p}_1(z_1) dz_1 \\ &+ \int_{z_1=0}^{\infty} Q(\sqrt{2\gamma_b z_1}) \hat{p}_2(z_1) dz_1 \\ &= \mathcal{G}_1\left(\frac{\pi}{2}\right) + \mathcal{G}_2\left(\frac{\pi}{2}\right) \end{aligned} \quad (33)$$

where $\mathcal{G}_1(\varphi)$ and $\mathcal{G}_2(\varphi)$ are derived in Appendix-II. For $m = 1$ and 2, in Appendix-II we exploit the fact that the first and second powers of the Gaussian Q -function are parameterized by $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{4}$ respectively in the function $\tilde{Q}(x)$ given below [22], i.e., $\tilde{Q}(\frac{\pi}{2}) = Q(x)$ and $\tilde{Q}(\frac{\pi}{4}) = Q^2(x)$

$$\tilde{Q}(x) = \frac{1}{\pi} \int_{\theta=0}^{\varphi} e^{-\frac{x^2}{2\sin^2\theta}} d\theta, \quad x \geq 0. \quad (34)$$

Following the above steps and Appendix-II, it can be shown that

$$\begin{aligned} A_p E_{z_1} \left\{ Q^2(\sqrt{2\gamma_b z_1}) \right\} &= \mathcal{G}_1\left(\frac{\pi}{4}\right) + \mathcal{G}_2\left(\frac{\pi}{4}\right), \\ (1 - A_p) E_{z_2} \left\{ Q(\sqrt{2\gamma_b z_2}) \right\} &= \mathcal{G}_1\left(\frac{\pi}{2}\right) - \mathcal{G}_2\left(\frac{\pi}{2}\right), \\ (1 - A_p) E_{z_2} \left\{ Q^2(\sqrt{2\gamma_b z_2}) \right\} &= \mathcal{G}_1\left(\frac{\pi}{4}\right) - \mathcal{G}_2\left(\frac{\pi}{4}\right). \end{aligned}$$

For $m \geq 3$, exact expressions for $A_p E_{z_1} [Q^m(\sqrt{2\gamma_b z_1})]$ and $(1 - A_p) E_{z_2} [Q^m(\sqrt{2\gamma_b z_2})]$ are difficult to derive. With the help of results in [53], the following series representation can be given for an analytically tractable form for $Q^m(x)$. We begin with the approximated form for $Q(x)$

$$\begin{aligned} Q(x) &\approx e^{-\frac{x^2}{2}} \sum_{\tilde{m}=0}^{m_a} c_{\tilde{m}} x^{\tilde{m}}, \\ c_{\tilde{m}} &= \frac{(-1)^{\tilde{m}+2} (A)^{\tilde{m}+1}}{B\sqrt{\pi} (\sqrt{2})^{\tilde{m}+2} (\tilde{m} + 1)!}. \end{aligned}$$

where $A = 1.98$, $B = 1.135$, and $m_a = 10$ [53]. $Q^m(x)$ can be written as

$$Q^m(x) \approx e^{-\frac{m x^2}{2}} \sum_{k_1=0}^{m m_a} c_{k_1} x^{k_1}. \quad (35)$$

The co-efficients c_{k_1} can be calculated in an easy manner using the fourier transform properties in a programming language like MATLAB. The expectations of the Gaussian- Q approximation with respect to z_1 and z_2 are carried out in Appendix-III and the final expressions are given by (96) and (101).

C. Special Cases of PEP Expression in (27)

In this subsection we briefly discuss a few special cases of the PEP expression given in (27).

- 1) With block length $N = 1$ and assuming that the delay related error term known at the receiver, the PEP expression (27) coincides with the analytical expression for average SEP in [33] (for BPSK constellation). In [33] we derived closed-form average SEP expressions for $M_1 \times M_2$ -QAM constellation assuming that the delay related error term is known at the receiver. The average SEP

expression for BPSK, based on (27) (i.e., by substituting $N = 1$ in (27) and after some simplification) is given by

$$P_{SEP} = 2\mathcal{G}_1\left(\frac{\pi}{2}\right) + (1 - A_p),$$

$\mathcal{G}_1(\varphi)$ is defined in Appendix-II.

- 2) PEP expression in (27) can also be applied to systems where the delay related error term is not known at the receiver. In (27) with $\rho_d = 1$ and changing the value of ρ_e such that it captures both estimation related correlation co-efficient and delay related correlation co-efficient² PEP in (27) gives the performance of a system where the delay related error term is not known at the receiver.
- 3) By making the block length $N = 1$, and assuming that the delay related error term is not known at the receiver, the PEP expression (27) coincides with the results presented in [34] (for BPSK constellation).
- 4) By making $\rho_e = 1$, $\rho_d = 1$, and assuming perfect quantization, with $t = 1$ the results in an approximated analytical expression for PEP with BPSK. Except for a few modulation schemes (such as non-coherent FSK³), PEP is generally studied with the help of computer simulations, and to the best of our knowledge this approximated analytical form (27) is the first available in the literature.
- 5) With arbitrary t and perfect feedback ($\rho_e = 1$, $\rho_d = 1$, and $B = \infty$), (27) gives the average packet error probability with t degrees of diversity.
- 6) By making appropriate changes to the PEP expression in (27), one can obtain an analytical understanding into the effects of fading, i.e., we can study the effects of channel estimation errors alone, or delay alone, or channel quantization alone, or other possible combinations of feedback imperfection. We believe the general framework (and the closed-form pdfs of random variables) to derive PEP presented in this paper can be leveraged to analytically study the average packet error probability in other system settings as well.

IV. SIMULATION RESULTS

In this section we present a sample simulation to verify the accuracy of the derived analytical expression for the average packet error probability and also show the effectiveness of the modeling of feedback delay. Fig. 2 shows the accuracy of derived analytical expression (in the form of an infinite series) for average packet error probability of transmit beamforming with imperfect feedback of a packet of $N \in \{30, 50\}$ un-coded BPSK symbols with $t \in \{2, 3\}$ transmit antennas, the estimation error correlation co-efficient $\rho_e \in \{0.98, 0.95\}$, the feedback delay related correlation co-efficient $\rho_d \in \{0.96, 0.93\}$, and number of feedback bits $B \in \{4, 5\}$. As pointed out earlier, the first two powers of Q -function are evaluated exactly. For higher powers ($m \geq 3$ in (28)), we calculate the expectation w.r.t. to the tractable approximation of the Q -function given in (35). According to the popular Clark's model [54], the correlation between channel samples with a lag of τ is given by $R(\tau) =$

²As explained in [34], when both estimation and delay related noise terms are absorbed into the receiver noise, then the effective correlation co-efficient becomes the product of delay only correlation co-efficient and estimation only correlation co-efficient.

³Or other modulation schemes where the conditional BEP/SEP expressions result in an exponential function.

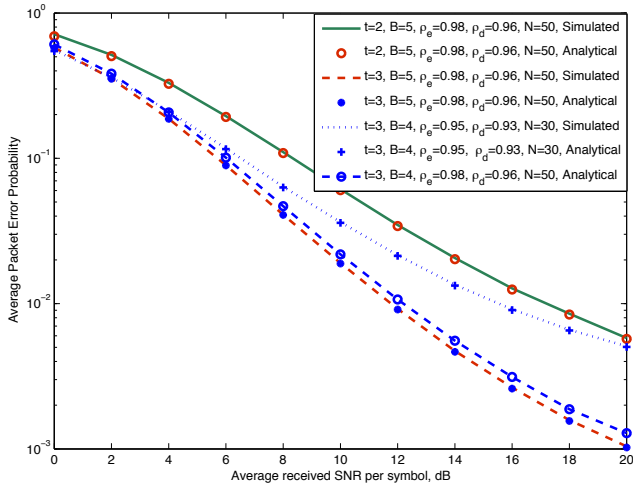


Fig. 2. Effect of imperfections in feedback (estimation, delay and quantization) on the average packet error probability.

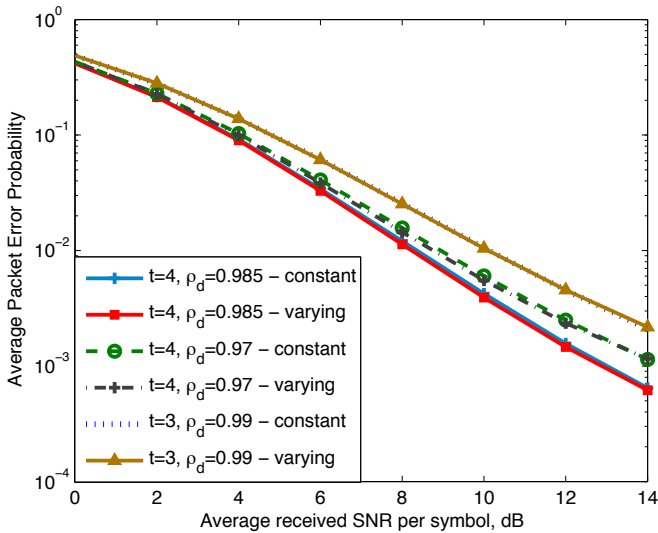


Fig. 3. Verifying the constant channel assumption.

$J_0(2\pi f_m \tau)$ where $J_k(\cdot)$ is the k^{th} order Bessel function of the first kind, $f_m = v/\lambda$, v is the velocity of mobile, and λ is the carrier wavelength. $\rho_d = 0.96$ correspond to a Doppler frequency of 80 rad, and a delay of 5 milliseconds. In Fig. 2, by fixing $N = 50$, $\rho_d = 0.98$, $\rho_e = 0.96$, and $B = 5$, improvement in performance can be seen as the number of antennas are increased from $t = 2$ to $t = 3$. Similarly, by fixing $t = 3$, $N = 50$, $\rho_d = 0.98$, and $\rho_e = 0.96$, an improvement in performance is noticed as the number of feedback bits are increased from $B = 4$ to $B = 5$. In Fig. 3 we verify the constant channel (for the entire packet) assumption. In Fig. 3 curves with legend ‘constant’ implies that the channel is assumed to be constant and ‘varying’ implies that the channel is modeled to be varying from symbol-to-symbol such that the effective correlation coefficient is ρ_d . Simulation parameters for Fig. 3: $B = 4$, $N = 30$, and $\rho_e = 0.98$. Both the curves for all three different sets of parameters match quite well showing that the assumption of constant channel is well justified.

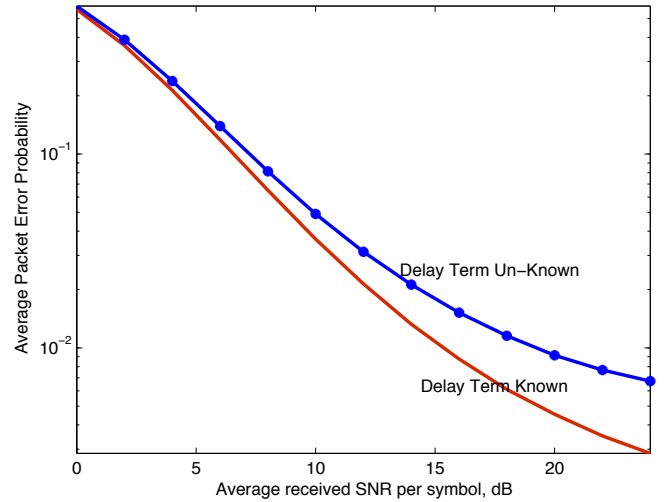


Fig. 4. Impact of delay related error term.

As discussed in the modeling of imperfect feedback, one of the contributions of this paper is the modeling aspect of feedback delay. The impact of the knowledge of delay related error term (at the receiver) on the performance can be seen in Fig.4. With all system parameters being the same, the solid curve shows the performance of the system with delay related error term assumed known to the receiver as in this paper. The dotted curve shows the performance of the system when the receiver is assumed to not know the delay related error term as has been done in the past. Apart from being conceptually distinct from estimation error related error term, the delay related error term as modeled in the paper shows improved system performance. The past modeling [34] (and references therein) simplifies the analysis but can lead to erroneous conclusions on system performance. The simulation parameters for Fig.4 are: Number of transmit antennas $t = 3$, packet size $N = 30$, estimation related correlation co-efficient $\rho_e = 0.97$, delay related correlation co-efficient $\rho_d = 0.9$, and number of feedback bits $B = 4$.

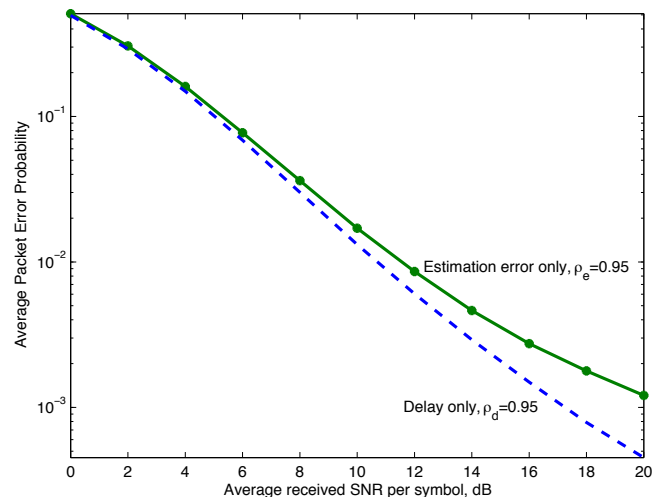


Fig. 5. Contrast between delay only system and estimation error only system.

Fig.5 contrasts the effect due to feedback delay alone and

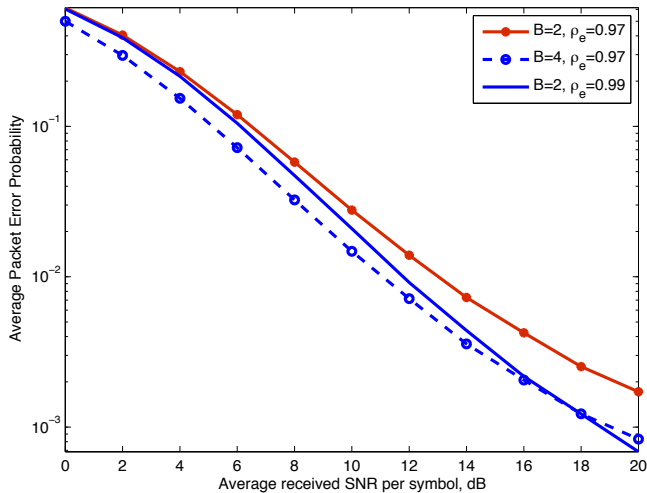


Fig. 6. Trade-off between channel estimation errors and channel quantization

due to estimation errors alone. The solid dotted curve shows PEP due to estimation errors alone with $\rho_e = 0.95$ and the dotted curve shows the PEP due to feedback delay alone with $\rho_d = 0.95$. Clearly the performance due to estimation errors alone is worse than performance due to delay alone. In this figure the feedback delay error term is known at the receiver thus it is able to perform better. If the feedback delay is not modeled in the way explained in this paper, then both curves would be same. The important message from this figure is: if a trade-off is possible it is better to put more resources into reducing estimation errors as opposed to trying to reduce the feedback delay. In Fig.5- number of feedback bits $B = 4$ and the block length $N = 30$. In Fig.6 we consider a possible trade-off between the number of feedback bits and the channel estimation quality. Quality of channel estimation depends on number of pilots and the pilot SNR [34]. If the forward link budget is constrained then increasing the number of feedback bits, which in turn improves the quality of channel quantization, can help in achieving a performance which is equivalent to increasing the pilot SNR. Similarly if the feedback link is constrained then pilot SNR can be increased to achieve an improvement in performance. Fig.6 illustrates this observation. In Fig.6- number of transmit antennas $t = 3$, block size $N = 30$, and delay related correlation coefficient $\rho_d = 0.97$.

V. CONCLUSION

We considered the problem of analyzing the average packet error probability, an important quality of service parameter, of closed loop MISO systems with imperfect feedback. The feedback imperfections include channel estimation errors, feedback delay and finite-rate channel quantization. Modeling of channel estimation errors in the packet fading context makes use of the fact that the channel estimate and the related error term are constant for the entire packet. The modeling approach distinguishes between errors due to channel estimation (un-known to the receiver) from those due to feedback delay (known to the receiver). Knowledge of delay related error term at the receiver helps in improved performance compared to a system without the knowledge of delay error term. An approximated analytical

expression for the PEP of an un coded (or a simple block channel coded) packet of BPSK symbols is derived. The general expressions derived are quite complex and not easily amenable to interpretation. Nevertheless, the steps taken in this paper are necessary and hopefully will prove to be useful for future work that has greater interpretability. Special cases of interest are discussed. The derived closed-form analytical expression is validated by simulations. Simulation results have also been used to contrast the relative effects of the three forms of feedback imperfections on the average packet error probability.

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APPENDIX-I

In this appendix we derive the pdf of the random variable ψ_ℓ

$$\psi_\ell = \left| \bar{\rho}_e \left(\bar{\rho}_d \vartheta_{\ell-1} \frac{\|\hat{\mathbf{h}}_{\ell-1}\|}{\sqrt{\Lambda}} + \sqrt{1 - \rho_d^2} \hat{\varepsilon}_{\ell d} \right) \right|.$$

ψ_ℓ has the parameters associated with all three forms of feedback imperfection. $\bar{\rho}_e$ determines the estimation quality, $\bar{\rho}_d$ determines the effect due to delay, $\sqrt{1 - \rho_d^2} \hat{\varepsilon}_{\ell d}$ is the error term due to feedback delay that is known to the receiver (because of the modeling approach presented in section II-B), and $\vartheta_{\ell-1}$ is the inner product between the estimated and delayed channel direction and its quantized version. Because of the analytical complexity involved in deriving p_{ψ_ℓ} , the pdf of ψ_ℓ , we begin with first writing ψ_ℓ as

$$\begin{aligned} \psi_\ell &= \rho_e \varphi, \\ \varphi &= \left| \bar{\rho}_d \vartheta_{\ell-1} \frac{\|\hat{\mathbf{h}}_{\ell-1}\|}{\sqrt{\Lambda}} + \sqrt{1 - \rho_d^2} \hat{\varepsilon}_{\ell d} \right|. \end{aligned} \quad (36)$$

pdf of φ^2 :

We first derive $p_{\varphi^2}(z)$, and then use a simple transformation to get the pdf of ψ_ℓ . In φ ,

$$\frac{\|\hat{\mathbf{h}}_{\ell-1}\|}{\sqrt{\Lambda}} \sim \frac{2x^{2t-1}e^{-x^2}}{\Gamma(t)}, \quad x \geq 0, \quad (37)$$

$$\hat{\varepsilon}_{\ell d} \sim \mathcal{NC}(0, 1), \quad (38)$$

and the pdf of $\Delta = |\vartheta_{\ell-1}|^2$ (later in the derivation we will be needing the pdf of Δ not $\vartheta_{\ell-1}$) is

$$\begin{aligned} p_\Delta(x) &= 2^B(t-1)(1-x)^{t-2}, \quad 1-\omega < x < 1, \\ \omega &= 2^{-\frac{B}{t-1}}. \end{aligned} \quad (39)$$

Note that φ has three random variables ($\vartheta_{\ell-1}$, $\|\hat{\mathbf{h}}_{\ell-1}\|$, and $\hat{\varepsilon}_{\ell d}$), and all three are independent of each other. To begin with assume that ϑ is a constant (we will relax this assumption at a later stage). If X_1 and X_2 are statistically independent Gaussian random variables, each one with same variance σ^2 and with the non-centrality parameter $s^2 = m_1^2 + m_2^2$ (m_1 and m_2 are the means of X_1 and X_2 respectively), then the non-central chi-squared distribution, $Z = X_1^2 + X_2^2$ is given by

$$p(z|s) = \frac{1}{2\sigma^2} e^{-\frac{(s^2+z)}{2\sigma^2}} \mathbf{I}_0\left(\frac{\sqrt{z}s}{\sigma^2}\right) \quad (40)$$

where $I_0(x)$ is the modified bessel function of 0^{th} order [55]. where
Let

$$\sigma = \sqrt{\frac{1 - \rho_d^2}{2}}, \quad (41)$$

$$s^2 \triangleq y = \frac{\rho_d^2 \Delta \|\hat{\mathbf{h}}_{\ell-1}\|^2}{\Lambda}. \quad (42)$$

After a simple transformation, the conditional pdf of y is given by

$$p(y|\Delta) = \frac{e^{-y/(\rho_d^2 \Delta)} y^{t-1}}{(\rho_d^2 \Delta)^t \Gamma(t)}, \quad y \geq 0. \quad (43)$$

With σ defined in (41) and y defined in (42), (40) can be applied to the present context to get $p_{\varphi^2}(z|y, \Delta)$, the conditional pdf of φ^2

$$p_{\varphi^2}(z|y, \Delta) = \frac{1}{2\sigma^2} e^{-\frac{(y+z)}{2\sigma^2}} I_0\left(\frac{\sqrt{zy}}{\sigma^2}\right).$$

The conditional pdf of φ^2 is given by

$$\begin{aligned} p_{\varphi^2}(z|\Delta) &= \int_{-\infty}^{\infty} p_{\varphi^2}(z|y, \Delta) p(y|\Delta) dy \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2\sigma^2} e^{-\frac{(y+z)}{2\sigma^2}} \sum_{\tilde{k}=0}^{\infty} \frac{(zy)^{\tilde{k}}}{4^{\tilde{k}} \sigma^{4\tilde{k}} (\tilde{k}!)^2} \right. \\ &\quad \left. \left(\frac{e^{-y/(\rho_d^2 \Delta)} y^{t-1}}{(\rho_d^2 \Delta)^t \Gamma(t)} \right) dy \right. \\ &= \lambda e^{-\frac{z}{2\sigma^2}} \sum_{\tilde{k}=0}^{\infty} \mathcal{L}(\tilde{k}) z^{\tilde{k}}, \quad z > 0 \end{aligned} \quad (44)$$

where

$$\begin{aligned} \lambda &= \frac{(1 - \rho_d^2)^{t-1}}{\Gamma(t) [1 - \rho_d^2 (1 - \Delta)]^t}, \\ \mathcal{L}(\tilde{k}) &= \frac{\Gamma(\tilde{k} + t) \rho_d^{2\tilde{k}} \Delta^{\tilde{k}}}{(1 - \rho_d^2)^{\tilde{k}} [1 - \rho_d^2 (1 - \Delta)]^{\tilde{k}} (\tilde{k}!)^2}. \end{aligned}$$

In the above derivation we used the infinite series representation for $I_0(x)$:

$$I_\alpha(x) = \sum_{\bar{k}=0}^{\infty} \frac{(x/2)^{\alpha+2\bar{k}}}{\bar{k}! \Gamma(\alpha + \bar{k} + 1)},$$

and the following identity [55]

$$\int_0^{\infty} x^n e^{-\omega x} dx = n! \omega^{-(n+1)}. \quad (45)$$

We can further simplify $p_{\varphi^2}(z|\Delta)$ given in (44) by writing the infinte series as

$$\begin{aligned} \sum_{\tilde{k}=0}^{\infty} \mathcal{L}(\tilde{k}) z^{\tilde{k}} &= \sum_{\tilde{k}=0}^{\infty} (\tilde{k} + t - 1) \cdots (\tilde{k} + 1) \frac{\beta^{\tilde{k}}}{\tilde{k}!} \\ &= \frac{d^n}{d\beta^n} \left(\sum_{\tilde{k}=0}^{\infty} \frac{\beta^{\tilde{k}+n}}{\tilde{k}!} \right) = \frac{d^n}{d\beta^n} (\beta^n e^\beta) \\ &= e^\beta \left(\sum_{l=0}^n n_{c_l} n_{p_l} \beta^{n-l} \right) \end{aligned}$$

$$\begin{aligned} n &= t - 1, \\ n_{c_l} &= \frac{n!}{l!(n-l)!}, \\ n_{p_l} &= \frac{n!}{(n-l)!}, \\ \beta &= \frac{\rho_d^2 \Delta z}{(1 - \rho_d^2) [1 - \rho_d^2 (1 - \Delta)]}. \end{aligned} \quad (46)$$

After some simplification $p_{\varphi^2}(z|\Delta)$ can be written as

$$\begin{aligned} p_{\varphi^2}(z|\Delta) &= \frac{(1 - \rho_d^2)^n}{\Gamma(t) [1 - \rho_d^2 (1 - \Delta)]^{n+1}} e^{-\frac{z}{2\sigma^2} + \beta} \\ &\quad \left(\sum_{l=0}^n n_{c_l} n_{p_l} \beta^{n-l} \right) \\ &= \frac{(1 - \rho_d^2)^n e^{-z/[1 - \rho_d^2 (1 - \Delta)]}}{\Gamma(t) [1 - \rho_d^2 (1 - \Delta)]^{n+1}} \\ &\quad \sum_{l=0}^n n_{c_l} \{n_{p_l} \Gamma(l+1)\} \frac{z^{n-l}}{\Gamma(l+1)} \left(\frac{\rho_d^2 \Delta}{1 - \rho_d^2} \right)^{n-l} \\ &= \frac{(1 - \rho_d^2)^n}{[1 - \rho_d^2 (1 - \Delta)]^n} \sum_{l=0}^n n_{c_l} f_l(z) \left(\frac{\rho_d^2 \Delta}{1 - \rho_d^2} \right)^{n-l} \end{aligned} \quad (47)$$

where

$$f_l(z) = \frac{e^{-z/[1 - \rho_d^2 (1 - \Delta)]} z^{n-l}}{[1 - \rho_d^2 (1 - \Delta)]^{n-l+1} \Gamma(n-l+1)}, \quad 0 \leq l \leq n.$$

For sanity check the area under the pdf $p_{\varphi^2}(z|\Delta)$ can be verified to be one.

$$\begin{aligned} \int_0^{\infty} p_{\varphi^2}(z|\Delta) dz &= \int_0^{\infty} \frac{(1 - \rho_d^2)^n}{[1 - \rho_d^2 (1 - \Delta)]^n} \\ &\quad \sum_{l=0}^n n_{c_l} f_l(z) \left(\frac{\rho_d^2 \Delta}{1 - \rho_d^2} \right)^{n-l} \\ &= \frac{(1 - \rho_d^2)^n}{[1 - \rho_d^2 (1 - \Delta)]^n} \sum_{l=0}^n n_{c_l} \left(\frac{\rho_d^2 \Delta}{1 - \rho_d^2} \right)^{n-l} \\ &= \frac{(1 - \rho_d^2)^n}{[1 - \rho_d^2 (1 - \Delta)]^n} \left(1 + \frac{\rho_d^2 \Delta}{1 - \rho_d^2} \right)^n = 1. \end{aligned}$$

We now look at the case where Δ is random. The pdf for Δ is given in (14). We now integrate out the randomness due to Δ to get $p_{\varphi^2}(z)$, the pdf of φ^2 .

$$\begin{aligned} p_{\varphi^2}(z) &= \int_{-\infty}^{\infty} p_{\varphi^2}(z|\Delta) p_\Delta(\Delta) d\Delta \\ &= \int_{1-\omega}^1 \left(\frac{(1 - \rho_d^2)^n}{[1 - \rho_d^2 (1 - \Delta)]^n} \sum_{l=0}^n n_{c_l} f_l(z) \right. \\ &\quad \left. \left\{ \frac{\rho_d^2 \Delta}{1 - \rho_d^2} \right\}^{n-l} \right) (2^B n (1 - \Delta)^{n-1}) d\Delta \\ &= 2^B n (1 - \rho_d^2)^n \sum_{l=0}^n \frac{n_{c_l} z^{n-l}}{\Gamma(n-l+1)} \left\{ \frac{\rho_d^2}{1 - \rho_d^2} \right\}^{n-l} \\ &\quad \underbrace{\int_{1-\omega}^1 \frac{e^{-\frac{z}{1 - \rho_d^2 (1 - \Delta)}} (\Delta)^{n-l} (1 - \Delta)^{n-1}}{[1 - \rho_d^2 (1 - \Delta)]^{2n-l+1}} d\Delta}_{J(z)}. \end{aligned} \quad (48)$$

In order to evaluate $J(z)$, let $\frac{1}{[1-\rho_d^2(1-\Delta)]} = \alpha$ which implies $\Delta = \frac{\alpha\rho_d^2 - \alpha + 1}{\alpha\rho_d^2}$ and $d\Delta = -\frac{d\alpha}{\alpha^2\rho_d^2}$.

$$\begin{aligned} J(z) &= \int_{1-\omega}^1 \frac{e^{-z/[1-\rho_d^2(1-\Delta)]} \Delta^{n-l} (1-\Delta)^{n-1}}{[1-\rho_d^2(1-\Delta)]^{n-l+1}} d\Delta \\ &= \frac{(-1)^n (\rho_d^2 - 1)^{n-l}}{(\rho_d^2)^{2n-l}} \int_{\frac{1}{1-\rho_d^2\omega}}^1 e^{-z\alpha} \left(\frac{1}{\rho_d^2 - 1} + \alpha \right)^{n-l} \\ &\quad (1-\alpha)^{n-1} d\alpha \\ &= \frac{(-1)^n (\rho_d^2 - 1)^{n-l}}{(\rho_d^2)^{2n-l}} \int_{\frac{1}{1-\rho_d^2\omega}}^1 e^{-z\alpha} \sum_{p=0}^{2n-l-1} c_p \alpha^p d\alpha. \end{aligned} \quad (49)$$

The co-efficients c_p are calculated as follows. Let $F(x)$ be a function defined as follows:

$$\begin{aligned} F(x) &= (d+x)^s (1-x)^r = \sum_{h=0}^{s+r} c_h x^h, \\ c_h &= \frac{1}{h!} \left. \frac{d^h (F(x))}{dx^h} \right|_{x=0} \\ &= \frac{1}{h!} \sum_{q=0}^h \binom{h}{q} L_r L_s (-1)^{h-q} d^{s-q}, \quad (50) \\ L_r &= \frac{r!}{(r-h+q)!} \quad \text{if } r \geq (h-q) \quad \text{else } L_r = 0, \\ L_s &= \frac{s!}{(s-q)!} \quad \text{if } s \geq q \quad \text{else } L_s = 0. \end{aligned}$$

In the present context,

$$x = \alpha, \quad s = n-l, \quad r = n-1, \quad \text{and } d = \frac{1}{\rho_d^2 - 1}.$$

With the help of the above function, $J(z)$ can now be expressed as

$$J(z) = \frac{(-1)^n (\rho_d^2 - 1)^{n-l}}{(\rho_d^2)^{2n-l}} \sum_{p=0}^{2n-l-1} c_p I_p \quad (51)$$

where c_p is calculated with the help of (50) and I_p can be evaluated as

$$\begin{aligned} I_p &= \int_{\frac{1}{1-\rho_d^2\omega}}^1 e^{-z\alpha} \alpha^p d\alpha \\ &= \left[-e^{-z\alpha} \left(\frac{\alpha^p}{z} + \sum_{g=1}^p \frac{p(p-1)\dots(p-g+1)}{z^{g+1}} \alpha^{p-g} \right) \right]_{\mathcal{L}}^1 \\ &= \sum_{g=0}^p \frac{p_{p_g} [e^{-z\mathcal{L}} \mathcal{L}^{p-g} - e^{-z}]}{z^{g+1}} \end{aligned}$$

where

$$\mathcal{L} = \frac{1}{1-\rho_d^2\omega} \quad (52)$$

where ω is defined in (39). Substituting (51) in (48), the final closed for pdf $p_{\varphi^2}(z)$ is given by

$$\begin{aligned} p_{\varphi^2}(z) &= \frac{2^B n (1-\rho_d^2)^n}{\rho_d^{2n}} \sum_{l=0}^n \frac{(-1)^l n_{c_l} z^{n-l}}{\Gamma(n-l+1)} \\ &\quad \sum_{p=0}^{2n-l-1} c_p \left(\sum_{g=0}^p \frac{p_{p_g} [e^{-z\mathcal{L}} \mathcal{L}^{p-g} - e^{-z}]}{z^{g+1}} \right) \\ &= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} \tilde{f}_1(l, p, g, \mathcal{L}, z) + \\ &\quad \mathcal{K}_{p_2} \tilde{f}_2(l, p, g, \mathcal{L}, z) \} + \mathcal{R}_{n,z} \end{aligned} \quad (53)$$

where

$$\mathcal{R}_{n,z} = \sum_{l=0}^n \sum_{p=n-l}^{2n-l-1} \sum_{g=n-l}^p \mathcal{K}_{p_3} (-1)^l n_{c_l} c_p p_{p_g} \frac{[e^{-z\mathcal{L}} \mathcal{L}^{p-g} - e^{-z}]}{\Gamma(n-l+1) z^{g-n+l+1}}, \quad (54)$$

$$\delta = \begin{cases} p & \text{if } p \leq n-l-1 \\ n-l-1 & \text{if } p \geq n-l \end{cases}, \quad (55)$$

$$\tilde{f}_2(l, p, g, \mathcal{L}, z) = \frac{\mathcal{L}^{n-l-g} e^{-\mathcal{L}z} z^{n-l-g-1}}{\Gamma(n-l-g)}, \quad (56)$$

$$\tilde{f}_1(l, p, g, \mathcal{L}, z) = \tilde{f}_2(l, p, g, 1, z), \quad (57)$$

$$\mathcal{K}_{p_1} = c_p 2^B n (1-\rho_d^2)^n (-1)^{l+1} n_{c_l} \frac{\Gamma(n-l-g) p_{p_g}}{\rho_d^{2n} \Gamma(n-l+1)}, \quad (58)$$

$$\mathcal{K}_{p_2} = -\mathcal{K}_{p_1} \mathcal{L}^{p-n+l}, \quad (59)$$

$$\mathcal{K}_{p_3} = \frac{2^B n (1-\rho_d^2)^n}{\rho_d^{2n}}, \quad (60)$$

and c_p 's can be calculated with the help of (50). \mathcal{L} is defined in (52).

Simplification of $p_{\varphi^2}(z)$:

Note that the form of $p_{\varphi^2}(z)$ given in (53) is separated into two parts, terms with powers of z in the numerator and terms with powers of z in the denominator. $\mathcal{R}_{n,z}$ defined in (54) captures the terms with powers of z in the denominator. The negative exponent of z in $\mathcal{R}_{n,z}$ will make the performance analysis intractable. We now take a closer look and analytically prove that $\mathcal{R}_{n,z} = 0$.

$$\mathcal{R}_{n,z} = \mathcal{K}_{p_3} \sum_{l=0}^n \sum_{p=n-l}^{2n-l-1} \sum_{g=n-l}^p (-1)^l n_{c_l} c_p p_{p_g} \frac{[e^{-z\mathcal{L}} \mathcal{L}^{p-g} - e^{-z}]}{\Gamma(n-l+1) z^{g-n+l+1}}.$$

Let $g-n+l = e$ and $p-n+l = f$.

$$\mathcal{R}_{n,z} = \mathcal{K}_{p_3} \sum_{l=0}^n \sum_{f=0}^{n-1} \sum_{e=0}^f \frac{(-1)^l n_{c_l} c_{f+n-l} (n-l+f)!}{(n-l)! (f-e)!} \left(\frac{e^{-z\mathcal{L}} \mathcal{L}^{f-e} - e^{-z}}{z^{e+1}} \right). \quad (61)$$

We now look at the co-efficient c_{n-l+f} , which are evaluated with the help of (50)

$$c_{n-l+f} = \frac{1}{(n-l+f)!} \sum_{q=0}^{n-l+f} \binom{n-l+f}{q} L_r L_s (-1)^{n-l+f-q} d^{s-q}$$

$$L_r \neq 0 \quad \text{if } n-1 \geq (n-l+f-q) \text{ else } L_r = 0$$

$$L_s \neq 0 \quad \text{if } n-l \geq q \text{ else } L_s = 0$$

The two in-equalities in L_r and L_s in the above equations are satisfied only if $q = n-l$, then c_{n-l+f} is

$$c_{n-l+f} = \frac{(n-1)!(-1)^f}{f!(n-f-1)!}. \quad (62)$$

The important point in the above equation is that c_{n-l+f} is independent of l . We can now write $\mathcal{R}_{n,z}$ in (61) as

$$\mathcal{R}_{n,z} = \mathcal{K}_{p_3} \sum_{l=0}^n (-1)^l n_{c_l} \sum_{f=0}^{n-1} \frac{(n-l+f)!}{(n-l)!} \mathcal{T}(f)$$

$$= \mathcal{K}_{p_3} \sum_{l=0}^n (-1)^l n_{c_l} \mathcal{W}(f)$$

where

$$\mathcal{T}(f) = \sum_{e=0}^f \frac{c_{f+n-l} [e^{-zL} L^{f-e} - e^{-z}]}{(f-e)! z^{e+1}},$$

$$\mathcal{W}(f) = \sum_{f=0}^{n-1} \frac{(n-l+f)!}{(n-l)!} \mathcal{T}(f)$$

$$= \sum_{f=0}^{n-1} m_f l^f$$

where m_f 's are constants that are independent of index l , as shown in the next step we do not need to calculate them explicitly. $\mathcal{R}_{n,z}$ can now be written as

$$\mathcal{R}_{n,z} = \mathcal{K}_{p_3} \sum_{f=0}^{n-1} m_f \underbrace{\sum_{l=0}^n (-1)^l n_{c_l} l^f}_{C_f}.$$

By using the following property of binomial co-efficients

$$\sum_{k=0}^n (-1)^k n_{c_k} l^{b-1} = 0, \quad n \geq b \geq 1,$$

it is clear that

$$C_f = 0, \quad n-1 \geq f \geq 0,$$

and subsequently $\mathcal{R}_{n,z} = 0$. This result is important because the presence of negative exponents of z in the pdf make the performance analysis difficult. The final simplified form of pdf $p_{\varphi^2}(z)$ can now be written as

$$p_{\varphi^2}(z) = \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_1(l, p, g, \mathcal{L}, z) + \mathcal{K}_{p_2} f_2(l, p, g, \mathcal{L}, z) \} \quad (63)$$

where (variable - definition): n - (46), δ - (55), \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), \mathcal{K}_{p_2} - (59), $f_1(, , , ,)$ - (57), and $f_2(, , , ,)$ - (56).

pdf of ψ_ℓ :

Note that the random variable that is of interest to us ψ_ℓ . From earlier discussion ψ_ℓ is related to φ as $\psi_\ell = \rho_e \varphi$. (63) is the pdf of φ^2 . Using transformation of random variables, p_{ψ_ℓ} , the pdf of ψ_ℓ can be shown to be given by

$$p_{\psi_\ell}(x) = \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{a1}(l, p, g, \mathcal{L}, x) + \mathcal{K}_{p_2} f_{a2}(l, p, g, \mathcal{L}, x) \}, \quad x > 0, \quad (64)$$

$$f_{a2}(l, p, g, \mathcal{L}, x) = \frac{2 \mathcal{L}^{n-l-g} e^{-\frac{\mathcal{L} x^2}{\rho_e^2}} x^{2(n-l-g)-1}}{\rho_e^{2(n-l-g)} \Gamma(n-l-g)}. \quad (65)$$

$$f_{a1}(l, p, g, \mathcal{L}, x) = f_{a2}(l, p, g, 1, x) \quad (66)$$

where (variable - definition): n - (46), δ - (55), \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), and \mathcal{K}_{p_2} - (59).

pdfs of z_1 and z_2 :

z_1 and z_2 are the random variables defined in section.III-B, here we present the expressions for the pdfs of z_1 and z_2 .

$$p_{z_1}(z_1) = \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{s1}(l, p, g, \mathcal{L}, z_1) + \mathcal{K}_{p_2} f_{s2}(l, p, g, \mathcal{L}, z_1) \}, \quad (67)$$

$$f_{s2}(l, p, g, \mathcal{L}, z_1) = \frac{\mathcal{H}_1}{2 A_p} z_1^{-\frac{1}{2}} e^{-z_1} \mathcal{H}_2 D_{-2(n-l-g)}(-\sqrt{z_1} \mathcal{H}_3),$$

$$f_{s1}(l, p, g, \mathcal{L}, z_1) = f_{s2}(l, p, g, 1, z_1).$$

$$p_{z_2}(z_2) = \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{s3}(l, p, g, \mathcal{L}, z_2) + \mathcal{K}_{p_2} f_{s4}(l, p, g, \mathcal{L}, z_2) \}, \quad (68)$$

$$f_{s4}(l, p, g, \mathcal{L}, z_1) = \frac{\mathcal{H}_1}{2(1-A_p)} z_2^{-\frac{1}{2}} e^{-z_2} \mathcal{H}_2 D_{-2(n-l-g)}(\sqrt{z_2} \mathcal{H}_3),$$

$$f_{s3}(l, p, g, \mathcal{L}, z_1) = f_{s4}(l, p, g, 1, z_1).$$

where the variables and the corresponding defining equation are listed as pairs: \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), \mathcal{K}_{p_2} - (59), \mathcal{H}_1 - (23), \mathcal{H}_2 - (24), and \mathcal{H}_3 - (25), and $D_{\tilde{p}}(\tilde{l})$ is the parabolic cylinder function. Parabolic cylinder function has many representations, for analytical simplicity we choose to work with the following representation [55]:

$$D_{\tilde{p}}(\tilde{l}) = 2^{\frac{\tilde{p}}{2}} e^{-\frac{\tilde{l}^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\tilde{p}}{2}\right)} {}_1F_1\left(-\frac{\tilde{p}}{2}, \frac{1}{2}; \frac{\tilde{l}^2}{2}\right) - \frac{\sqrt{2\pi}\tilde{l}}{\Gamma\left(\frac{-\tilde{p}}{2}\right)} {}_1F_1\left(\frac{1-\tilde{p}}{2}, \frac{3}{2}; \frac{\tilde{l}^2}{2}\right) \right\} \quad (69)$$

where ${}_1F_1(\cdot, \cdot; \cdot)$ is the confluent hypergeometric function of the first kind. Using the representation in (69) for the parabolic

cylinder function, (67) and (68) can be written as

$$\begin{aligned} p_{z_1}(z_1) &= \frac{\hat{p}_1(z_1) + \hat{p}_2(z_1)}{A_p}, \quad z_1 > 0, \\ p_{z_2}(z_2) &= \frac{\hat{p}_1(z_2) - \hat{p}_2(z_2)}{(1 - A_p)}, \quad z_2 > 0, \\ \hat{p}_1(x) &= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{s5}(l, p, g, \mathcal{L}, x) + \\ &\quad \mathcal{K}_{p_2} f_{s6}(l, p, g, \mathcal{L}, x) \}, \quad x > 0, \end{aligned} \quad (70)$$

$$f_{s6}(l, p, g, \mathcal{L}, x) = \mathcal{R}_1 x^{-\frac{1}{2}} e^{-x \mathcal{R}_h} {}_1F_1 \left(\mathcal{R}_n, \frac{1}{2}; x \mathcal{R}_c \right), \quad (71)$$

$$\mathcal{R}_1 = \frac{\mathcal{H}_1 \sqrt{\pi}}{2^{n-l-g+1} \Gamma \left(\frac{1+2(n-l-g)}{2} \right)}, \quad (72)$$

$$\mathcal{R}_h = \frac{(4\mathcal{H}_2 + \mathcal{H}_3^2)}{4}, \quad (73)$$

$$\mathcal{R}_n = n - l - g, \quad (74)$$

$$\mathcal{R}_c = \frac{\mathcal{H}_3^2}{2}, \quad (75)$$

$$f_{s5}(l, p, g, \mathcal{L}, x) = f_{s6}(l, p, g, 1, x). \quad (76)$$

$$\begin{aligned} \hat{p}_2(x) &= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{s7}(l, p, g, \mathcal{L}, x) + \\ &\quad \mathcal{K}_{p_2} f_{s8}(l, p, g, \mathcal{L}, x) \}, \quad x > 0, \end{aligned} \quad (77)$$

$$f_{s8}(l, p, g, \mathcal{L}, x) = \mathcal{R}_2 e^{-x \mathcal{R}_h} {}_1F_1 \left(\mathcal{R}_{n2}, \frac{3}{2}; x \mathcal{R}_c \right), \quad (78)$$

$$\mathcal{R}_2 = \frac{\mathcal{H}_1 \mathcal{H}_3 \sqrt{2\pi}}{2^{n-l-g+1} \Gamma(n-l-g)}, \quad (79)$$

$$\mathcal{R}_{n2} = n - l - g + \frac{1}{2}, \quad (80)$$

$$f_{s7}(l, p, g, \mathcal{L}, x) = f_{s8}(l, p, g, 1, x). \quad (81)$$

APPENDIX-II

In this appendix we derive the analytical expressions for $\mathcal{G}_1(\varphi)$ and $\mathcal{G}_2(\varphi)$, which are used in the evaluation of $A_p E[Q^m(\sqrt{2}\gamma_b z_1)]$ and $(1 - A_p) E[Q^m(\sqrt{2}\gamma_b z_2)]$, where $m \in \{1, 2\}$.

Derivation of $\mathcal{G}_1(\varphi)$:

$$\mathcal{G}_1(\varphi) = \int_{\tilde{y}=0}^{\infty} \tilde{Q}(\sqrt{2\gamma_b \tilde{y}}) \hat{p}_1(\tilde{y}) d\tilde{y}, \quad (82)$$

$$= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{c1}(l, p, g, \mathcal{L}, \varphi) + \\ \mathcal{K}_{p_2} f_{c2}(l, p, g, \mathcal{L}, \varphi) \} \quad (83)$$

where (variable - definition): n - (46), δ - (55), \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), \mathcal{K}_{p_2} - (59), $\tilde{Q}(x)$ - (34), and $\hat{p}_1(x)$ - (70).

$$\begin{aligned} f_{c2}(l, p, g, \mathcal{L}, \varphi) &= \int_{\tilde{y}=0}^{\infty} \tilde{Q}(\sqrt{2\gamma_b \tilde{y}}) f_{s6}(l, p, g, \mathcal{L}, \tilde{y}) d\tilde{y} \\ &= \mathcal{R}_1 \int_{\tilde{y}=0}^{\infty} \tilde{Q}(\sqrt{2\gamma_b \tilde{y}}) \tilde{y}^{-\frac{1}{2}} e^{-\tilde{y} \mathcal{R}_h} \\ &\quad {}_1F_1 \left(\mathcal{R}_n, \frac{1}{2}; \tilde{y} \mathcal{R}_c \right) d\tilde{y} \\ &= \frac{\mathcal{R}_1}{\pi} \int_{\theta=0}^{\varphi} d\theta \int_{\tilde{y}=0}^{\infty} \tilde{y}^{-\frac{1}{2}} e^{-\tilde{y} \left(\frac{\gamma_b}{\sin^2 \theta} + \mathcal{R}_h \right)} \\ &\quad {}_1F_1 \left(\mathcal{R}_n, \frac{1}{2}; \tilde{y} \mathcal{R}_c \right) d\tilde{y} \end{aligned} \quad (84)$$

where (variable - definition): $f_{s6}(\dots)$ - (71), \mathcal{R}_1 - (72), \mathcal{R}_h - (73), \mathcal{R}_n - (74), and \mathcal{R}_c - (75). With $\tilde{y} \mathcal{R}_c = x$, (84) becomes

$$f_{c2}(l, p, g, \mathcal{L}, \varphi) = \frac{\mathcal{R}_1}{\pi \mathcal{R}_c^{\frac{3}{2}}} \int_{\theta=0}^{\varphi} d\theta \int_{x=0}^{\infty} x^{-\frac{1}{2}} e^{-x S} \\ {}_1F_1 \left(\mathcal{R}_n, \frac{1}{2}; x \right) dx \quad (85)$$

where

$$S = \frac{\gamma_b}{\sin^2 \theta \mathcal{R}_c} + \frac{\mathcal{R}_h}{\mathcal{R}_c}. \quad (86)$$

Notice that $S > 1$ for $\rho_e < 1$. To evaluate the above equation, we use the identity given below [55]:

$$\begin{aligned} \int_{r=0}^{\infty} r^{c-1} e^{-r d} {}_1F_1(a, c; r) dr &= \Gamma(c) d^{-c} \left(\frac{d-1}{d} \right)^{-a} \\ &= \Gamma(c) d^{-c} (1 - d^{-1})^a, \quad \text{Re } c > 0, \text{ Re } d > 1. \end{aligned} \quad (87)$$

In the present context, in (87),

$$c = \frac{1}{2}, \quad d = S, \quad \text{and} \quad q = -\mathcal{R}_n.$$

With the help of above identity, (85) becomes

$$f_{c2}(l, p, g, \mathcal{L}, \varphi) = \mathcal{M} \int_{\theta=0}^{\varphi} d^{-c} (1 - d^{-1})^q d\theta$$

where

$$d^{-1} = \mathcal{R}_d \left(\frac{\sin^2 \theta}{\sin^2 \theta + c_1} \right),$$

$$\mathcal{M} = \frac{\mathcal{R}_1 \Gamma \left(\frac{1}{2} \right)}{\pi \mathcal{R}_c^{\frac{3}{2}}},$$

$$\mathcal{R}_d = \frac{\mathcal{R}_c}{\mathcal{R}_h},$$

and

$$c_1 = \frac{\gamma_b}{\mathcal{R}_h}. \quad (88)$$

By using the generalized binomial series expansion,

$$\begin{aligned} (1-x)^q &= \sum_{\tilde{n}=0}^{\infty} \frac{(-1)^{\tilde{n}} \bar{p}_{k, \tilde{n}}}{\tilde{n}!} (x)^{\tilde{n}}, \\ \bar{p}_{k, \tilde{n}} &= q(q-1) \cdots (q-\tilde{n}+1). \end{aligned}$$

$f_{c2}(l, p, g, \mathcal{L}, \varphi)$ can now be written in a series of steps as ⁴

$$\begin{aligned}
f_{c2}(l, p, g, \mathcal{L}, \varphi) &= \mathcal{M} \int_{\theta=0}^{\varphi} \sum_{\tilde{n}=0}^{\infty} \frac{\mathcal{R}_d^{\tilde{n}+\frac{1}{2}} (-1)^{\tilde{n}} \bar{p}_{k, \tilde{n}} \tilde{d}^{\tilde{n}+c}}{\tilde{n}!} d\theta \\
&= \mathcal{M} \int_{\theta=0}^{\varphi} \sum_{\tilde{n}=0}^{\infty} \frac{\mathcal{R}_d^{\tilde{n}+\frac{1}{2}} (-1)^{\tilde{n}} \bar{p}_{k, \tilde{n}}}{\tilde{n}!} \\
&\quad \left(1 - (1 - \tilde{d})\right)^{\tilde{n}+c} d\theta \\
&= \mathcal{M} \int_{\theta=0}^{\varphi} \sum_{\tilde{n}=0}^{\infty} \frac{\mathcal{R}_d^{\tilde{n}+\frac{1}{2}} (-1)^{\tilde{n}} \bar{p}_{k, \tilde{n}}}{\tilde{n}!} \\
&\quad \sum_{\tilde{n}=0}^{\infty} \frac{(-1)^{\tilde{n}} \bar{b}_{k, \tilde{n}}}{\tilde{n}!} (1 - \tilde{d})^{\tilde{n}} d\theta \\
&= \mathcal{M} \sum_{\tilde{n}=0}^{\infty} \frac{\mathcal{R}_d^{\tilde{n}+\frac{1}{2}} (-1)^{\tilde{n}} \bar{p}_{k, \tilde{n}}}{\tilde{n}!} \\
&\quad \sum_{\tilde{n}=0}^{\infty} \frac{(-1)^{\tilde{n}} \bar{b}_{k, \tilde{n}}}{\tilde{n}!} \sum_{\tilde{n}_1=0}^{\tilde{n}} (-1)^{\tilde{n}_1} \binom{\tilde{n}}{\tilde{n}_1} \\
&\quad \mathcal{D}(\varphi, c_1, \tilde{n}_1) \quad (89)
\end{aligned}$$

where

$$\begin{aligned}
\bar{b}_{k, \tilde{n}} &= (\tilde{n} + c) (\tilde{n} + c - 1) \cdots (\tilde{n} + c - \tilde{n} + 1), \\
\tilde{d} &= \frac{\sin^2 \theta}{\sin^2 \theta + c_1},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\varphi, c_1, \tilde{n}_1) &= \int_{\theta=0}^{\varphi} \tilde{d} d\theta = \int_{\theta=0}^{\varphi} \left(\frac{\sin^2 \theta}{\sin^2 \theta + c_1} \right)^{\tilde{n}_1} d\theta \quad (90) \\
&= \pi \left\{ \frac{\varphi}{\pi} - \frac{T}{\pi} \sqrt{\frac{c_1}{1+c_1}} \sum_{k=0}^{\tilde{n}_1-1} \binom{2k}{k} \right. \\
&\quad \left. \frac{1}{[4(1+c_1)]^k} - \frac{2}{\pi} \sqrt{\frac{c_1}{1+c_1}} \sum_{k=0}^{\tilde{n}_1-1} \right. \\
&\quad \left. \sum_{j=0}^{k-1} \binom{2k}{j} \frac{(-1)^{j+k}}{[4(1+c_1)]^k} \frac{\sin[(2k-2j)T]}{2k-2j} \right\}, \\
&\quad 0 \leq \varphi \leq 2\pi \quad (91)
\end{aligned}$$

where

$$\begin{aligned}
T &= \frac{1}{2} \tan^{-1} \left(\frac{2\sqrt{c_1(1+c_1)} \sin 2\varphi}{(1+2c_1) \cos 2\varphi - 1} \right) + \frac{\pi}{2} \\
&\quad \left[1 - 2\sqrt{c_1(1+c_1)} \sin 2\varphi \left(\frac{(1+2c_1) \cos 2\varphi}{2} \right) \right].
\end{aligned}$$

Equation (89) gives the final expression for $f_{c2}(l, p, g, \mathcal{L}, \varphi)$. To complete the calculation of $\mathcal{G}_1(\varphi)$ in (83) we still need $f_{c1}(l, p, g, \mathcal{L}, \varphi)$ which is given by

$$f_{c1}(l, p, g, \mathcal{L}, \varphi) = f_{c2}(l, p, g, 1, \varphi).$$

Derivation of $\mathcal{G}_2(\varphi)$:

$$\begin{aligned}
\mathcal{G}_2(\varphi) &= \int_{\tilde{y}=0}^{\infty} \tilde{Q} \left(\sqrt{2\gamma_b \tilde{y}} \right) \hat{p}_2(\tilde{y}) d\tilde{y}, \quad (92) \\
&= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{d1}(l, p, g, \mathcal{L}, \varphi) + \\
&\quad \mathcal{K}_{p_2} f_{d2}(l, p, g, \mathcal{L}, \varphi) \} \quad (93)
\end{aligned}$$

where (variable - definition): n - (46), δ - (55), \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), \mathcal{K}_{p_2} - (59), $\tilde{Q}(x)$ - (34), and $\hat{p}_2(x)$ - (77), and

$$\begin{aligned}
f_{d2}(l, p, g, \mathcal{L}, \varphi) &= \int_{\tilde{y}=0}^{\infty} \tilde{Q} \left(\sqrt{2\gamma_b \tilde{y}} \right) f_{s8}(l, p, g, \mathcal{L}, \tilde{y}) d\tilde{y} \\
&= \mathcal{R}_2 \int_{\tilde{y}=0}^{\infty} \tilde{Q} \left(\sqrt{2\gamma_b \tilde{y}} \right) e^{-\tilde{y} \mathcal{R}_h} \\
&\quad {}_1F_1 \left(\mathcal{R}_{n2}, \frac{3}{2}; \tilde{y} \mathcal{R}_c \right) d\tilde{y} \\
&= \frac{\mathcal{R}_2}{\pi} \int_{\theta=0}^{\varphi} d\theta \int_{\tilde{y}=0}^{\infty} e^{-\tilde{y} \left(\frac{\gamma_b}{\sin^2 \theta} + \mathcal{R}_h \right)} \\
&\quad {}_1F_1 \left(\mathcal{R}_{n2}, \frac{3}{2}; \tilde{y} \mathcal{R}_c \right) d\tilde{y} \quad (94)
\end{aligned}$$

where (variable - definition): $f_{s8}(\cdot, \cdot, \cdot, \cdot)$ - (78), \mathcal{R}_2 - (79), \mathcal{R}_h - (73), \mathcal{R}_{n2} - (80), and \mathcal{R}_c - (75). With $\tilde{y} \mathcal{R}_c = x$,

$$f_{d2}(l, p, g, \mathcal{L}, \varphi) = \frac{\mathcal{R}_2}{\pi \mathcal{R}_c} \int_{\theta=0}^{\varphi} d\theta \int_{x=0}^{\infty} e^{-x \mathcal{S}} {}_1F_1 \left(\mathcal{R}_{n2}, \frac{3}{2}; x \right) dx,$$

\mathcal{S} is defined in (86). $f_{d2}(l, p, g, \mathcal{L}, \varphi)$ can be evaluated using the following infinite series expansion for the Gauss hypergeometric function

$$\begin{aligned}
{}_1F_1(a, b; r) dr &= \sum_{\tilde{m}=0}^{\infty} \frac{a_{\tilde{m}} r^{\tilde{m}}}{b_{\tilde{m}} \tilde{m}!}, \\
a_{\tilde{m}} &= 1, a_1 (a_1 + 1) \cdots (a_1 + \tilde{m} - 1), \\
a_1 &= \mathcal{R}_{n2}, \\
b_{\tilde{m}} &= 1, b_1 (b_1 + 1) \cdots (b_1 + \tilde{m} - 1), \\
b_1 &= \frac{3}{2}.
\end{aligned}$$

With the above series representation, $f_{d2}(l, p, g, \mathcal{L}, \varphi)$ can now be written as

$$\begin{aligned}
f_{d2}(l, p, g, \mathcal{L}, \varphi) &= \frac{\mathcal{R}_2}{\pi \mathcal{R}_c} \sum_{\tilde{m}=0}^{\infty} \frac{a_{\tilde{m}}}{b_{\tilde{m}}} \int_{\theta=0}^{\varphi} d\theta \int_{x=0}^{\infty} \frac{e^{-x \mathcal{S}} x^{\tilde{m}}}{\tilde{m}!} dx, \\
&= \frac{\mathcal{R}_2}{\pi \mathcal{R}_c} \sum_{\tilde{m}=0}^{\infty} \frac{a_{\tilde{m}}}{b_{\tilde{m}}} \int_{\theta=0}^{\varphi} \mathcal{S}^{-(\tilde{m}+1)} d\theta, \\
&= \frac{\mathcal{R}_2}{\pi \mathcal{R}_c} \sum_{\tilde{m}=0}^{\infty} \frac{a_{\tilde{m}} \mathcal{R}_d^{\tilde{m}+1} \mathcal{D}(\varphi, c_1, \tilde{m} + 1)}{b_{\tilde{m}}} \quad (95)
\end{aligned}$$

where $\mathcal{D}(\varphi, c_1, \tilde{m} + 1)$ is defined in (91) and c_1 is defined in (88). Finally $f_{d1}(l, p, g, \mathcal{L}, \varphi)$ is given by

$$f_{d1}(l, p, g, \mathcal{L}, \varphi) = f_{d2}(l, p, g, 1, \varphi).$$

⁴Note that the convergence of the above series is not a problem since $d^{-1} < 1$.

APPENDIX-III

In this section, for $m \geq 3$, with the help of the analytically tractable approximation of $Q(x)$ given in (35), we derive closed-form expressions for $A_p E [Q^m (\sqrt{2\gamma_b z_1})]$ and $(1 - A_p) E [Q^m (\sqrt{2\gamma_b z_2})]$.

$$\begin{aligned} A_p E [Q^m (\sqrt{2\gamma_b z_1})] &= \int_{z_1=0}^{\infty} Q^m (\sqrt{2\gamma_b z_1}) \hat{p}_1(z_1) dz_1 \\ &+ \int_{z_1=0}^{\infty} Q^m (\sqrt{2\gamma_b z_1}) \hat{p}_2(z_1) dz_1 \\ &= U_1(\gamma_b) + U_2(\gamma_b) \end{aligned} \quad (96)$$

where $p_{z_1}(z_1)$, $\hat{p}_1(z_1)$, and $\hat{p}_2(z_1)$ are defined in (67), (70), and (77) respectively and

$$\begin{aligned} U_1(\gamma_b) &= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{e1}(l, p, g, \mathcal{L}) + \\ &\quad \mathcal{K}_{p_2} f_{e2}(l, p, g, \mathcal{L}) \}, \quad (97) \\ f_{e2}(l, p, g, \mathcal{L}) &= \mathcal{R}_1 \int_{z_1=0}^{\infty} Q^m (\sqrt{2\gamma_b z_1}) z_1^{-\frac{1}{2}} e^{-z_1 \mathcal{R}_h} \\ &\quad {}_1F_1 \left(\mathcal{R}_n, \frac{1}{2}; z_1 \mathcal{R}_c \right) dz_1 \\ &= \mathcal{R}_1 \int_{z_1=0}^{\infty} \sum_{k_1=0}^{\infty} c_{k_1} (2\gamma_b)^{\frac{k_1}{2}} z_1^{\frac{k_1-1}{2}} \\ &\quad e^{-z_1(m\gamma_b + \mathcal{R}_h)} {}_1F_1 \left(\mathcal{R}_n, \frac{1}{2}; z_1 \mathcal{R}_c \right) dz_1 \\ &= \mathcal{R}_1 \sum_{k_1=0}^{\infty} c_{k_1} (2\gamma_b)^{\frac{k_1}{2}} \\ &\quad \mathcal{F} \left(\mathcal{R}_n, \frac{1}{2}, k_1 + \frac{1}{2}, \mathcal{R}_c, m\gamma_b + \mathcal{R}_h \right) \end{aligned}$$

where (variable - definition): n - (46), δ - (55), \mathcal{L} - (52), \mathcal{K}_{p_1} - (58), \mathcal{K}_{p_2} - (59), \mathcal{R}_1 - (72), \mathcal{R}_h - (73), \mathcal{R}_n - (74), and \mathcal{R}_c - (75), and

$$\begin{aligned} \mathcal{F}(a, b, \alpha, k, s) &= \int_{x=0}^{\infty} x^{\alpha-1} e^{-sx} {}_1F_1(a, b; kx) dx \\ &= \Gamma(\alpha) s^{-\alpha} {}_2F_1(a, \alpha; b; ks^{-1}), \\ &\quad [\alpha > 0, |s| > |k|]. \quad (98) \end{aligned}$$

To complete the calculation of $U_1(\gamma_b)$ in (97) we still need $f_{e1}(l, p, g, \mathcal{L})$ which is given by

$$f_{e1}(l, p, g, \mathcal{L}) = f_{e2}(l, p, g, 1).$$

We now focus on $U_2(\gamma_b)$ in (96):

$$\begin{aligned} U_2(\gamma_b) &= \sum_{l=0}^n \sum_{p=0}^{2n-l-1} \sum_{g=0}^{\delta} \{ \mathcal{K}_{p_1} f_{e3}(l, p, g, \mathcal{L}) + \\ &\quad \mathcal{K}_{p_2} f_{e4}(l, p, g, \mathcal{L}) \}, \quad (99) \\ f_{e4}(l, p, g, \mathcal{L}) &= \int_{z_1=0}^{\infty} Q^m (\sqrt{2\gamma_b z_1}) \hat{p}_2(z_1) dz_1 \\ &= \mathcal{R}_2 \int_{z_1=0}^{\infty} Q^m (\sqrt{2\gamma_b z_1}) e^{-z_1 \mathcal{R}_h} \\ &\quad {}_1F_1 \left(\mathcal{R}_{n2}, \frac{3}{2}; z_1 \mathcal{R}_c \right) dz_1 \\ &= \mathcal{R}_2 \int_{z_1=0}^{\infty} \sum_{k_1=0}^{\infty} c_{k_1} (2\gamma_b)^{\frac{k_1}{2}} z_1^{\frac{k_1}{2}} \\ &\quad e^{-z_1(m\gamma_b + \mathcal{R}_h)} {}_1F_1 \left(\mathcal{R}_{n2}, \frac{3}{2}; z_1 \mathcal{R}_c \right) dz_1 \\ &= \mathcal{R}_2 \sum_{k_1=0}^{\infty} c_{k_1} (2\gamma_b)^{\frac{k_1}{2}} \\ &\quad \mathcal{F} \left(\mathcal{R}_{n2}, \frac{3}{2}, \frac{k_1}{2} + 1, \mathcal{R}_c, m\gamma_b + \mathcal{R}_h \right) \end{aligned} \quad (100)$$

where (variable - definition): \mathcal{R}_2 - (79), and \mathcal{R}_{n2} - (80). To complete $U_2(\gamma_b)$ in (99), $f_{e3}(l, p, g, \mathcal{L})$ is given by

$$f_{e3}(l, p, g, \mathcal{L}) = f_{e4}(l, p, g, 1).$$

Substituting (97) and (99) in (96) gives the final closed form expression for $A_p E [Q^m (\sqrt{2\gamma_b z_1})]$. It is straightforward to show that

$$(1 - A_p) E [Q^m (\sqrt{2\gamma_b z_2})] = U_1(\gamma_b) - U_2(\gamma_b). \quad (101)$$

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