A Block-Diagonal Structured Model Reduction Scheme for Power Grid Networks

Abstract—**We propose a block-diagonal structured model order reduction (BDSM) scheme for fast power grid analysis. Compared with existing power grid model order reduction (MOR) methods, BDSM has several advantages. First, unlike many power grid reductions that are based on terminal reduction and thus error-prone, BDSM utilizes an exact column-by-column moment matching to provide higher numerical accuracy. Second, with similar accuracy and macromodel size, BDSM generates very sparse block-diagonal reduced-order models (ROMs) for massive-port systems at a lower cost, whereas traditional algorithms such as PRIMA produce full dense models inefficient for the subsequent simulation. Third, different from those MOR schemes based on extended Krylov subspace (EKS) technique, BDSM is input-signal independent, so the resulting ROM is reusable under different excitations. Finally, due to its block-diagonal structure, the obtained ROM can be simulated very fast. The accuracy and efficiency of BDSM are verified by industrial power grid benchmarks.**

I. Introduction

Power grid analysis has been a major topic in modern VLSI design. The challenges for power grid analysis mainly stem from the large problem size and massive port number. A typical power grid model usually has millions of nodes and up to thousands of input sources, rendering it extremely difficult to simulate. During the past decade, numerous efforts have been made to speed up the analysis and/or simulation of power grid networks, such as domain decomposition technique [1], preconditioned Krylov-subspace iterative method [2], random walk algorithm [3], and multi-grid reduction technique [4]. One issue of these methods is that the computation on the large model needs to be repeated for different inputs or time steps.

A viable solution is to approximate the original network by model order reduction (MOR), and then use the much smaller model in the subsequent simulation. Popular MOR algorithms include Krylov-subspace projections [5], [6] and balanced truncations [7], which have been highly successful in interconnect macromodeling. Krylov-subspace projections such as PRIMA [5] have superior efficiency over balanced truncations. Therefore, they have been modified to reduce power grid models [8]–[10]. However, the efficiency significantly degrades as the port number increases. First, the MOR cost increases linearly with the port number, making the computation inefficient. Second, the reduced order model (ROM) size increases linearly with the port number, resulting in a quadratic increase on storage cost. Consequently, the normally large and dense ROMs make the simulation very inefficient.

To address the problems induced by the large port number, extended Krylov subspace (EKS [10]) and extended truncated balance realization (ETBR [11]) treat the product of input vector and input matrix as a new frequency-dependent "single-input matrix", and then reduce a "single-input multi-output" system. Based on a similar idea, triangularization based structure preserving MOR (TBS [9]) generates structured ROMs to further speed up power grid simulation. However, these methods are highly dependent on the input signals, and the obtained ROMs can not be reused for different input patterns. Since there exist some correlations between the input-output pairs, singular value decomposition (SVD) can be used to compress the terminals before MOR [12]. Similarly, Ref. [13] uses frequency-dependent packing to improve the numerical accuracy; Ref. [14] has proposed decentralized MOR (DeMOR) for multi-port system reduction. However, the terminal reduction process is error-prone, because the *true* transfer matrix moments can not be matched.

In this paper, we present a novel method, called block-diagonal structured MOR (BDSM), for power grid reduction subject to the following criteria:

1) The ROMs should be cheap to simulate;
2) The ROMs should be reusable;
3) The MOR accuracy should be comparable to that of PRIMA.

By BDSM, we get ROMs having the same sizes and similar accuracy as those from PRIMA. Even more, the models from BDSM are sparse and block-diagonal, thereby facilitating fast simulation. Since BDSM does not involve terminal reduction, it is more accurate over terminal-reduction based MOR. On the other hand, BDSM is input-independent, so the ROMs can be reused for different input patterns. Due to their block-diagonal structure, the ROMs can be efficiently simulated, which allows for parallel calculations.

II. Background

A. Problem Formulation

We consider the modified nodal analysis (MNA) equation of a power grid network

$$C\frac{dx(t)}{dt} = Gx(t) + Bu(t), \quad y(t) = Lx(t) \tag{1}$$

where $C, G \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $L \in \mathbb{R}^{p\times n}$. The input vector $u(t)$ normally represents time-varying current sources from transistor-level circuit blocks; the output vector $y(t)$ contains the voltage or current variables of interest; the state vector $x(t)$ represents nodal voltages and the branch currents across inductive components. The system matrix C includes the capacitance and inductance terms; G denotes the conductance matrix; B and L are the input and output matrices, respectively.

Provided the matrix pencil (C, G) being regular (i.e., $\exists s \in \mathbb{C}$ such that $(sC - G)$ is nonsingular), in Laplace domain, the $p \times m$ transfer matrix can be written as

$$H(s) = L(sC - G)^{-1}B. \tag{2}$$

In MOR, we attempt to find the left and right projection matrices $W, V \in \mathbb{R}^{n \times q}$ with $q \ll n$, to construct a small size-q linear system $\Sigma_r : (C_r, G_r, B_r, L_r)$

$$C_r \frac{dz(t)}{dt} = G_r z(t) + B_r u(t), \quad y_r(t) = L_r z(t) \tag{3}$$

with $C_r = W^T CV$, $G_r = W^T GV$, $B_r = W^T B$ and $L_r = LV$, such that $H_r(s) = L_r(sC_r - G_r)^{-1}B_r \approx H(s)$ or $y_r(s) = H_r(s)u(s) \approx y(s)$, subject to some accuracy requirements.

If $W = V$, the projection is a congruence transform. For simplicity, we use congruence transform in this paper. The projection matrices can be constructed by (rational) Krylov subspace moment matching [5], [6], [15] or balanced truncations (BT) [7]. Although BT approaches may provide *a priori* error estimation, they become inefficient for such large-scale systems as power grid networks whose problem sizes may be in the millions. Therefore, Krylov-subspace projections are discussed in this paper.

B. Problems with Existing Krylov Subspace Projection

Given a matrix M and (block) vector R with compatible sizes, an l-th order (block) Krylov subspace $\mathcal{K}_l(M, R)$ is the space spanning the range of a set of (block) vectors, i.e.,

$$\mathcal{K}_l(M, R) = \{R, MR, M^2R, \cdots, M^{l-1}R\}.$$

By (block) Arnoldi algorithm [5], the projection matrices are constructed as

$$W = V = \mathcal{K}_l\{(s_0C - G)^{-1}C, (s_0C - G)^{-1}B\} \tag{4}$$

with s_0 being a specific expansion point. Then a size-q ROM with $q = ml$ can be constructed, such that $H_r(s)$ matches the first l moments of $H(s)$ around the expansion point s_0, i.e.,

$$H(s) = H_r(s) + O\left((s - s_0)^l\right). \tag{5}$$

If the input signals are distributed in a wide frequency band, multi-point Krylov-subspace projection may be used to improve the accuracy [15]. We proceed with single-point projection, and the multi-point scheme straightforwardly follows.

The standard projections have some problems when applied to power grid networks. First, the obtained ROMs are not efficient for computer-aided simulation. Since the ROM size q increases linearly with the port number m, it is clear that the ROM size can be very large. Since the ROM's matrices from standard projections are normally dense, storing the ROMs becomes challenging for a general PC, let alone simulating the ROM. Second, the cost of projection matrix construction is high for large many-port systems. To construct the projection matrix in (4), we need to perform $\frac{ml(ml-1)}{2}$ steps of long-vector orthonormalization, whose cost quadratically increases with m. Therefore, standard moment-matching based projection would be inefficient for power grid reduction.

Some modifications have been made for MOR of power grid networks. These approaches are mainly based on terminal reduction [12]–[14] or ideas similar to extended Krylov-subspace projection [9]–[11]. The former captures the moments of a low-rank approximated transfer matrix [12]–[14], rather than the original one. Therefore, essentially the model compactness is obtained at the cost of model accuracy. EKS, ETBR and TBS generate compact ROMs via moment matching of the output response under a predefined input excitation [9]–[11]. However, due to their strong dependency on input signal waveforms, the ROMs need to be rebuilt every time as the excitation vector changes. Since the cost in MOR is much more expensive over simulating a ROM, this kind of approaches may be inefficient for power grid analysis if we need to simulate the response under different excitations.

III. BDSM SCHEME

This section presents the proposed BDSM algorithm to generate block-diagonal structured ROMs for power grid networks. We first decompose the original MIMO (multi-input multi-output) system into m MIMO subsystems (each with a $p \times m$ transfer matrix), via input matrix splitting. Then, we show that the Krylov-subspace projection matrix of each MIMO subsystem is in fact identical to that of a SIMO (single-input multi-output) subsystem. To match l moments, the proposed method generates an $ml \times ml$ ROM as by PRIMA. However, the resulting ROM's system matrices only contain m blocks in the diagonal, with each one being a small $l \times l$ matrix. This structure makes the subsequent simulation highly efficient. For simplicity, we only discuss the projection at a single point, and the multi-point projection follows analogously.

A. Input Matrix Splitting

Denoting the i-th column of $B \in \mathbb{R}^{n \times m}$ by b_i, the input matrix can be splitted to m rank-1 matrices, i.e.,

$$B = \sum_{i=1}^{m} B_i, \quad \text{with } B_i \in \mathbb{R}^{n \times m}, \ B_i(:, j) = \begin{cases} b_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{6}$$

for $i, j = 1, 2, \cdots, m$. Here $B_i(:, j)$ denotes the j-th column vector of B_i. The linear time-invariant (LTI) system (C, G, B_i, L) is called a *splitted system*, denoted by Σ_i. Σ_i's transfer matrix is written as $H_i(s) = L(sC - G)B_i$. Clearly, the original transfer matrix $H(s)$ can be rewritten as

$$H(s) = \sum_{i=1}^{m} H_i(s). \tag{7}$$

Subsequently, the original network can be reformulated as the *parallel connection* of $\Sigma_i{}'s$, and then realized by a size-mn

model $(\mathcal{C},\ \mathcal{G},\ \mathcal{B},\ \mathcal{L})$:

$$\mathcal{C} = \begin{bmatrix} C & & \\ & \ddots & \\ & & C \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix}$$
$$\mathcal{B} = \begin{bmatrix} B_1^T & \cdots & B_m^T \end{bmatrix}^T, \ \mathcal{L} = \begin{bmatrix} L & \cdots & L \end{bmatrix}. \tag{8}$$

This larger-size block-diagonal system is an equivalent model of the original power grid network. Note that $H_i(s)$ is a $p \times m$ matrix with only one column vector (the i-th column) being non-zero, which is identical to the i-th column of $H(s)$.

Generally, if we attempt to match the first l moments of a general m-port size-mn model via standard Krylov subspace projection such as PRIMA [5] at a single expansion point, a size-ml ROM would be generated. During the projection, the block-diagonal structure would be destroyed and a dense ROM would be produced, which makes the ROM-based simulation very inefficient. Additionally, reducing a size-mn linear system is normally much more expensive over reducing a size-n system, since more calculations are needed in the LU decomposition, linear system solution and Gram-Schmidt orthonormalization. In BDSM, we aim to keep the block-diagonal structure of (8) such that the storage and calculations could be much cheaper in the subsequent simulation steps. We also expect the MOR cost to be ***cheaper*** than traditional projection frameworks on (1). More importantly, the resulting ROM is expected to be ***reusable*** for repeated simulation under varying input patterns.

To proceed, we consider the i-th splitted system Σ_i. Excited by the input vector $u(s)$, the output vector is $y_i(s) = H_i(s)u(s)$, and it can be rewritten as

$$y_i(s) = L(sC - G)^{-1}B_i u(s) = L(sC - G)^{-1}b_i u_i(s) \tag{9}$$

since B_i has only one nonzero vector in the i-th column. Here, $u_i(s)$ denotes the i-th input scalar. This reformulation shows that $y_i(s)$ is only dependent on the input u_i, and B_i shields the effects induced by other input signals, although the splitted system Σ_i has m input ports. Since $y(s)$ is the sum of $y_i(s)$ for $i = 1, \cdots, m$, the above input matrix splitting is physically equivalent to decomposing the output response into m independent components, with each excited by a single input signal. This property in fact allows for a block-diagonal structure-preserving reduction for model (8), at a lower computational cost over PRIMA.

B. Block-Diagonal Structured Projection

Unlike traditional projection reduction methods that directly match the moments of $H(s)$, BDSM uses an ***indirect moment matching***. Specifically, the ROM of each splitted model Σ_i, denoted by Σ_{ir}, is built such that its transfer matrix $H_{ir}(s)$ matches the first l moments of $H_i(s)$, and then all reduced models are parallely connected to approximate the original linear network (1).

Let us consider the splitted model $\Sigma_i : (L, C, G, B_i)$. At a single expansion point s_0, a projection matrix spanning the l-th order block Krylov subspace can be constructed:

$$V^{(i)} = \mathcal{K}_l\{(s_0C - G)^{-1}C,\ (s_0C - G)^{-1}B_i\}. \tag{10}$$

Then the ROM of Σ_i, denoted by $\Sigma_{ir} : (C_{ir}, G_{ir}, B_{ir}, L_{ir})$, can be constructed by the congruence transform

$$\begin{aligned} &C_{ir} = (V^{(i)})^T C V^{(i)},\ G_{ir} = (V^{(i)})^T G V^{(i)}, \\ &B_{ir} = (V^{(i)})^T B_i \text{ and } L_{ir} = L V^{(i)}. \end{aligned} \tag{11}$$

It can be proved that the ROM's transfer matrix H_{ir} matches the first l moments of $H_i(s)$, i.e.,

$$H_{ir}(s) = L_{ir}(s_iC_{ir} - G_{ir})^{-1}B_{ir} = H_i(s) - O\left((s - s_0)^l\right) \tag{12}$$

Since $B_i \in \mathbb{R}^{n \times m}$, it seems that $V^{(i)}$ is a $n \times ml$ matrix and the size of the ROM Σ_{ir} would be ml. But it is ***not*** the case. By noting that B_i has only one nonzero vector b_i as its i-th column, it is straightforward to prove

$$V^{(i)} = \mathcal{K}_l\left\{(s_0C - G)^{-1}C,\ (s_0C - G)^{-1}b_i\right\} \in \mathbb{R}^{n \times l} \tag{13}$$

provided that no vectors are deflated in the orthonormalization steps. *Therefore, $V^{(i)}$ is in fact a $n \times l$ projection matrix, and Σ_{ir} is a very small size-l ROM, although Σ_i is an MIMO system.*

After computing the projection matrix for each splitted system Σ_i, a projection matrix can be constructed for model (8). Using the congruence transform $C_r = \mathcal{V}^T\mathcal{C}\mathcal{V}$, $G_r = \mathcal{V}^T\mathcal{G}\mathcal{V}$, $B_r = \mathcal{V}^T\mathcal{B}$ and $L_r = \mathcal{L}\mathcal{V}$, the system matrices of the final ROM of (8) [denoted by Σ_r, which is also the final ROM of (1)], can be decided as

$$\begin{aligned} &C_r = \text{blkdiag}\left(C_{1r}, \cdots, C_{mr}\right), \quad G_r = \text{blkdiag}\left(G_{1r}, \cdots, G_{mr}\right) \\ &B_r = \begin{bmatrix} B_{1r} \\ \vdots \\ B_{mr} \end{bmatrix} = \text{blkdiag}\left((V^{(1)})^T b_1, \cdots, (V^{(m)})^T b_m\right) \\ &\text{and } L_r = \mathcal{L}\mathcal{V}, \quad \text{where } \mathcal{V} = \text{blkdiag}\left(V^{(1)}, \cdots, V^{(m)}\right). \end{aligned} \tag{14}$$

Here "blkdiag" denotes the Matlab function that constructs a block-diagonal matrix from the input arguments. It is clearly shown that the final size-ml ROM is block-diagonal structured. All diagonal blocks of C_r and G_r (i.e., C_{ir} and G_{ir} for $i = 1, \cdots, m$) are small $l \times l$ matrices. The i-th block of B_r (i.e., B_{ir}) contains only one nonzero vector as its i-th column.

From (14) and (12), the transfer matrix of Σ_r can be written as

$$H_r(s) = \sum_{i=1}^{m} H_{ir}(s) = H(s) - O\left((s - s_0)^l\right). \tag{15}$$

Therefore, $H_r(s)$ matches the first l moments of $H(s)$, and BDSM has similar accuracy to PRIMA [5]. In PRIMA, the first l moments of $H(s)$ are matched in a matrix format. However, in BDSM, each $p \times m$ transfer matrix $H_{ir}(s)$ captures the first l moments of $H(s)$'s i-th column. Consequently, their sum, $H_r(s)$, captures $H(s)$'s first l moment matrices in a ***column-by-column*** style, as illustrated in the BDSM flow of Fig 1.

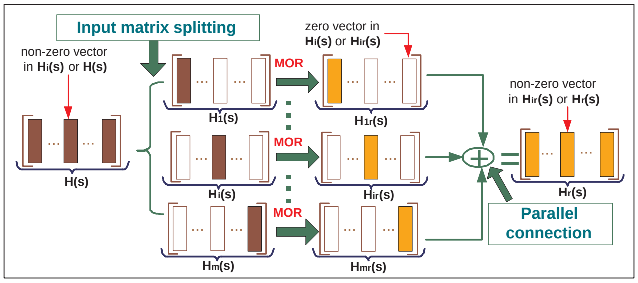

Fig. 1. The BDSM model reduction scheme for a linear network with m input ports, which is based on column-by-column moment matching. After input matrix splitting, the original model is decomposed into m MIMO subsystems. Then using the projection process, $H_{ir}(s)$ captures the first l moments of $H(s)$'s i-th column. Finally, the parallel connection of all ROMs guarantees the preservation of $H(s)$'s first l moments.

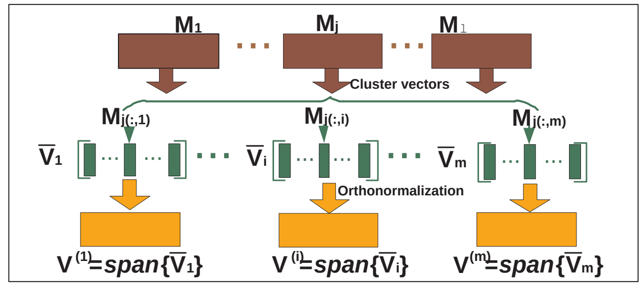

Fig. 2. Projection matrix construction in BDSM . In this figure, $M_j = \left((s_0C-G)^{-1}C\right)^{j-1}(s_0C-G)^{-1}B$, $j = 1, \cdots, l$. The i-th columns of M_j's are grouped to form $\bar{V}_i$ $(i = 1, \cdots, m)$. And then V_i is computed such that $V_i = \bar{V}_i$, for $i = 1,\ \cdots,\ m$. Note that, in PRIMA the projection matrix for (1) is constructed without clustering, such that $V = \text{span}\{M_1, \cdots, M_l\}$ with more computational cost.

The detailed implementation is presented in Algorithm 1. Assume that no vectors are deflated in the Krylov subspace projection. To match l moments for a system with m inputs, BDSM and PRIMA both need one sparse LU factorization, $l-1$ multiplications of sparse matrices and block vectors, and l steps of backward plus forward substitutions. The cost difference comes from the orthonormalization process (cf. Step 4 of Algorithm 1). In PRIMA, all nl column vectors need to be orthonormalized, which costs $\frac{ml(ml-1)}{2}$ long vector-vector production. While in BDSM algorithm, the vectors are clustered into m groups, and then each group of vectors are orthonormalized separately. Consequently, BDSM only requires $\frac{ml(l-1)}{2}$ vector-vector production in the orthonormalization step. For many-terminal large-scale systems, the computational savings of BDSM can be very remarkable. An explanation of the cluster-and-orthonormalization flow is given in Fig 2.

Algorithm 1 Block-diagonal structured MOR (BDSM)

1: **Input:** $C, G \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$,$L \in \mathbb{R}^{p\times n}$, and l
2: Perform LU factorization: $LU = (s_0C - G)$, calculate $X = U^{-1}(L^{-1}B)$, and normalize each column of X
3: Set $V^{(i)} = X(:, i)$ for $i = 1,\ \cdots,\ m$
4: **for** $j = 1, \cdots, l-1$ **do**
 4.1 calculate $X_{\text{temp}} = CX$ and $X = U^{-1}(L^{-1}X_{\text{temp}})$
 4.2 **for** $j = 1, \cdots, l-1$ **do**
  orthonormalize $X(:, i)$ to all columns of $V^{(i)}$ to get $\bar{x}_i$, update $V^{(i)}$: $V^{(i)} = \left[V^{(i)},\ \bar{x}_i\right]$
5: Construct the reduced model for Σ_i as in (11) for $i = 1, \cdots, m$, and then form the reduced model of (1) by (14)
6: **Output:** ROM matrices C_r, G_r, L_r and B_r.

Next, we contrast the resulting ROMs. PRIMA generate dense ROMs with $O(m^2l^2)$ nonzeros, while only ml^2 nonzero entries need to be stored in a BDSM ROM. When m becomes large, the ROMs by BDSM would be very sparse (with $\frac{1}{m}$ sparsity). The resulting sparse and block-diagonal structured ROMs would significantly facilitate numerical simulation. To simulate the ROM from PRIMA, $O(m^3l^3)$ cost is required, whereas only $O(ml^3)$ flops are needed for the BDSM ROM.

TABLE I
COMPARISON OF VARIOUS MULTI-PORT MOR SCHEMES. IN SVDMOR, α REPRESENTS THE PORT COMPRESSION RATIO.

MOR method	ROM size	ROM pattern	Matched moments	ROM reusable?	ROM scalable?
BDSM	ml	block-diagonal	l	yes	yes
PRIMA	ml	full dense	l	yes	no
SVDMOR	αml	full dense	N/A	yes	no
EKS	l	full dense	N/A	no	no

C. Comparison with Existing Power Grid MORs

Table I compares BDSM with some typical massive-port MOR schemes: EKS [10], PRIMA [5], and SVDMOR [12] (a typical MOR based on terminal reduction). In SVDMOR, we assume that the port compression ratio is α (i.e., the ratio of port number after terminal reduction w.r.t. the original port number), and then l moments of the "thin" transfer matrix is matched; in EKS, it is assumed that the first l moments of the response under a *predefined* excitation are captured. In SVD-MOR and EKS, the "true" moments of $H(s)$ are not captured, so they are *not* exact moment matching schemes. Among these approaches, PRIMA and SVDMOR generate full dense matrices, which are expensive for subsequent frequency/time-domain simulation. Although SVDMOR can compress the port size to some extent (at the cost of accuracy sacrifice), the obtained dense-matrix ROMs are still memory- and time-consuming for many-terminal systems. And when the input-output correlation is not strong, large errors may be induced by the terminal reduction process. EKS is capable of generating very small (size-l) macromodels, but the resulting ROMs are *not* reusable. These problems lead to remarkable efficiency degradation in ROM-based simulation. Compared with these existing MORs, BDSM does not have these limitations, thereby allowing for more efficient simulation of massive-port networks. We remark that EKS ROM is very inaccurate under varying input patterns, due to its strong dependency on the predefined input waveforms. To increase its accuracy, more moments of the response should be captured. However, as will be shown in Section IV, EKS is not comparable with PRIMA and BDSM in terms of accuracy, even if the ROM size is increased to ml, at a cost similar to that of PRIMA.

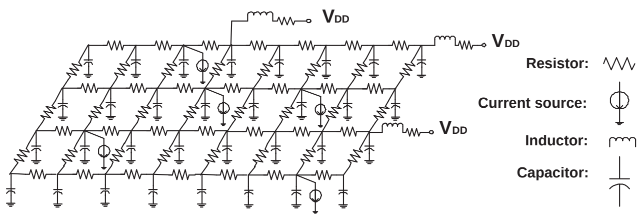

Fig. 3. The RLC model of a power grid network, with consideration of package inductance.

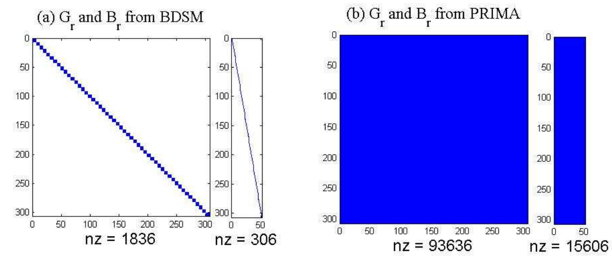

Fig. 4. The matrix structures of ckt1's ROMs, obtained from BDSM and PRIMA, respectively.

D. Application Issues

BDSM can be directly used for fast power grid analysis, or interconnected system-level simulation when $H(s)$ is not the admittance/impedance matrix. Theoretically, the resulting ROM may be non-passive, but fortunately the possible passivity violation is normally very weak due to the high accuracy of BDSM. If $H(s)$ represents the impedance/admittance parameters and the resulting ROM is non-passive, some modifications are needed before system-level simulation (e.g., when the ROM is connected to other networks (e.g. package) for IR-drop or package resonance analysis).

One solution is to incorporate other passive networks (e.g., a package model) with the power grid network (as shown in Figure 3), and then reduce the whole RLC model by BDSM.

Another solution is to perform passivity enforcement [16] after detecting the possible non-passive regions [17] in the frequency band(s) of interest. Due to the block-diagonal structure, passivity verification and enforcement can be finished at a low cost. We assume that Σ_r is obtained by single-point projection thus the size of Σ_{ir} is l, then Σ_{ir} can be transformed to a standard state-space model Σ_{ir}^s: (I, G_{ir}^s, B_{ir}^s, L_{ir}) at the cost of $O(l^3)$. An eigenvalue decomposition can be further performed on G_{ir}^s at a cost of $O(l^3)$

$$G_{ir}^s = X_i \Lambda_i X_i^{-1} \tag{16}$$

where Λ_i is a diagonal matrix. Then Σ_{ir} can be realized by (I, Λ_i, $X_i^{-1}B_{ir}^s$, $L_{ir}X_i$), which is a diagonal-structured LTI system. Finally, the passivity test and enforcement can be simplified via Laguerre's method at the cost of only $O(q^2)$ [18], which is negligible compared to the cost of BDSM.

IV. Numerical Results

We use several industrial power grid benchmarks to verify the proposed scheme. As shown in Fig. 3, the power grid and package are connected and modeled as a whole large-scale linear circuit including resistance, capacitance and inductance. Time-varying current sources are used to describe the behavior of active circuit blocks. The MNA LTI models are extracted from some industrial SPICE netlists. All experiments are performed on a 2.6GHz 4-GB RAM Linux workstation.

We begin by timing different MOR schemes using single-point moment matching on 5 RLC power grid benchmarks (ckt1-ckt5 in Table II). The port numbers range from several tens to over $1k$; and the node numbers are from $6k$ to $1.7M$. For simplicity, all ports are assumed to be excited by unit-impulse signals in EKS [9]; in SVDMOR, α is set around 0.6 for all examples. Specifically, $H(s)$ is first approximated by $U_l^T \mathcal{H}(s) U_r$ with $\mathcal{H}(s) \in \mathbb{C}^{\hat{p}\times\hat{m}}$, $\hat{p} = [\alpha p]$ and $\hat{m} = [\alpha m]$ [12], and then the "thiner" LTI $\mathcal{H}(s)$ is reduced by PRIMA. Since sparse LU may still introduce large amounts of nonzero elements for some cases, this factorization is skipped in ckts3-5 to save memory, at the cost of more simulation time.

The CPU times and resulting ROM sizes are listed in Table II. With the same number of moments matched, BDSM and PRIMA generate ROMs with the same size. Since much fewer long-vector orthonormalizations are needed, BDSM is faster than PRIMA, and this speedup becomes more remarkable as the problem size and port number increase. In SVDMOR, although the terminals can be reduced to some extent, it still needs more orthonormalization steps and thus is slower than BDSM in many-terminal cases (cf. ckts3-4). Even more, PRIMA and SVDMOR may fail in very-large-size many-port cases (cf. ckts4-5). This is because: 1) the resulting full-dense ROMs of PRIMA and SVDMOR can be memory-consuming in many-port cases; 2) the "fat" projection matrix V ($\in \mathbb{R}^{n\times ml}$ or $\mathbb{R}^{n\times[\alpha m]l}$) is also dense and even more CPU-consuming. While in BDSM the projection matrix $V^{(i)}$ for each splitted system is very thin, and the final sparse block-diagonal ROM is cheap to store. To illustrate this, Fig. 4 has compared the ROM matrix structures of ckt1, from BDSM and PRIMA, respectively. Due to the special structure of G_r, C_r and B_r, the subsequent simulation can be very fast. For example, if $m = 1000$, the BDSM ROM is expected to enjoy a $10^6\times$ speedup over PRIMA ROM in the subsequent simulation. EKS is the fastest one among these schemes. However, the EKS ROM need to be rebuilt each time when the input pattern changes, making the simulation very inefficient in practice. Furthermore, it is also difficult to *exactly* predict the input signals of a power grid network, whereas inexactly modeled inputs may make the EKS ROM unreliable. Therefore, a ***reusable*** ROM is preferred for repeated circuit simulation. As shown by Table II, BDSM provides the best numerical efficiency among those reusable power grid MOR schemes.

Fig. 5 has plotted the transfer function of port(1,2) for ckt1. In EKS, all inputs are set as unit impulse signals. For fairness, 6 moments are matched in all MOR schemes. EKS's size-6 ROM has very low accuracy. Then we construct a larger

TABLE II
CPU TIMES (IN SECOND) OF VARIOUS MOR SCHEMES.

ckt	node	port number	PRIMA [5]		SVDMOR [12] ($\alpha = 0.6$)		EKS [10][1]		BDSM		No. of matched moments
			MOR time	ROM size	MOR time	ROM size	MOR time	ROM size	MOR time	ROM size	
ckt1	6k	51	29.37	306	35.60	180	0.30	6	8.18	306	6
ckt2	20k	108	5.0×10^3	1080	1.4×10^3	640	15.4	10	3.7×10^3	1080	10
ckt3	80k	204	1.2×10^4	2040	1.0×10^4	1220	17.7	10	7.1×10^3	2040	10
ckt4	123k	315	break down	N/A	break down	N/A	39.8	8	2.6×10^4	2520	8
ckt5	1.7M	1429	break down	N/A	break down	N/A	610	10	5.9×10^4	14290	10

[1] The EKS ROMs are ***not*** reusable.

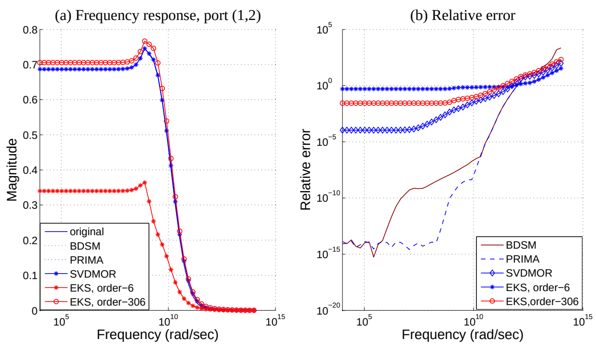

Fig. 5. Accuracy comparison of various MORs for ckt1.

EKS ROM by matching 306 moments of the response, which costs $36sec$ for ckt1. However, the size-306 EKS ROM is still very inaccurate. This is not surprising, because the EKS ROM constructed under a specific excitation is not reusable for new input patterns. Fig. 5 has also plotted the relative errors of these MOR schemes. PRIMA and BDSM have very high accuracy (relative error $< 10^{-6}$ for $\omega < 10^{10}$ rad/s), due to their ***exact*** moment matching properties. The error of SVDMOR is several orders larger than BDSM and PRIMA, due to the error-prone terminal reduction.

V. Conclusion

This paper has proposed a novel MOR scheme, BDSM, highly applicable to multi-port systems such as power grid networks. BDSM has similar accuracy to PRIMA due to the same number of matched moments; yet it is faster and more memory-efficient over PRIMA (and SVDMOR in many-terminal cases) in model generation, since lots of long-vector orthonormalizations are skipped. Unlike EKS and TBS, because BDSM is input-independent, the obtained ROMs are reusable for time/frequency-domain analysis under varying input patterns. More importantly, BDSM ROMs have block-diagonal structures, thereby allowing for very fast subsequent simulation. The efficiency and accuracy of BDSM have been verified by industrial benchmarks.

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