International Journal of Control, Automation, and Systems (2014) 12(1):71-83
DOI 10.1007/s12555-012-0234-7
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

# Canonical Projector Techniques for Analyzing Descriptor Systems

**Zheng Zhang and Ngai Wong***

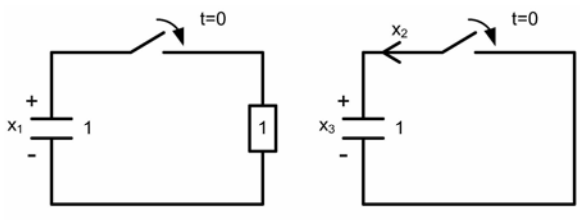
**Abstract:** Physical systems are often naturally formulated as descriptor systems (DSs) which form a superset of the more restrictive standard state spaces. The analysis of a DS, however, is complicated by the algebraic coupling between its proper and improper subsystems. The recently emerging canonical projector technique, stemming from iterative matrix chain construction, provides a theoretically sound and numerically effective way to completely decouple these subsystems and largely facilitates the reuse or adaptation of standard state space techniques for DS analysis. Nonetheless, results concerning canonical projectors are scattered and their potential use is currently less appreciated. The objectives of this paper are twofold: i) It serves as a tutorial that collects distributed results about canonical projectors and presents them in a coherent manner; and more than just a tutorial, it elaborates and provides new/elegant/corrected proofs to some fundamental properties of canonical projectors. An iterative procedure for canonical projector construction, lacking in the literature, is also described. ii) Obvious applications, including some latest development, of projector techniques in practical circuit design problems are succinctly illustrated. By creating a self-contained repository of important canonical projector theories, it is hoped that more interest will be drawn and efficient numerical implementations will follow.

**Keywords:** Canonical projector, descriptor system, passivity test, spectral projector.

## 1. INTRODUCTION

Modeling and simulation tools have become an indispensable part of modern electronic design automation (EDA) due to the ever-increasing size and complexity of systems. Fast and accurate modeling and simulation of, for instance, on-chip circuit components (e.g., wires, vias, pin packages and devices), while preserving important physical properties (e.g., causality, stability and passivity), are critical for verifying signal integrity and ensuring circuit functionality, e.g., [1-6]. These on-chip elements, modeled through discretization of partial differential equations (PDEs) or electrical modeling techniques such as modified nodal analysis (MNA) [6], are naturally cast as differential algebraic equations (DAEs) known also as singular or descriptor systems (DSs), e.g., [7-14]. Specifically, DSs represent a bigger class with much higher modeling capability compared to the more restrictive standard state space systems. For example, algebraic equations like Kirchhoff's laws or controlled voltage/current sources are readily described by DSs (e.g., [6]) but not representable as standard state spaces. Even when a DS is reducible to a standard state space, it is often desirable to work directly in the DS format due to the computation-friendly structural stamps (e.g., sparse or banded matrices) arising from circuit extraction, e.g., [2,6].

The possible presence of both proper and improper parts (also respectively called the slow and fast subsystems [15,16], or non-impulsive and impulsive subsystems) of a general DS constitutes a major source of numerical difficulties in decoupling or analyzing them [1]. To illustrate, Fig. 1 shows an index-2 (see definition in Section 2.1) circuit DS where the system matrices are already in their decoupled form [viz. $E$ and $A$ in (2.1) and (3)]. Simple circuit analysis shows that for $t \geq 0$, with $t = 0^-$ denoting the instance just before time zero, the




$$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

Fig. 1. An illustrative index-2 DS system where $u(t) \equiv 0$.

Manuscript received May 30, 2012; revised July 4, 2013; accepted September 23, 2013. Recommended by Associate Editor Juhoon Back under the direction of Editor Hyungbo Shim.

This work was supported in part by the Hong Kong Research Grants Council under project 718711E, and in part by the University Research Committee of The University of Hong Kong.

Zheng Zhang is with the Computational Prototyping Group, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 U.S.A. (e-mail: z_zhang@mit.edu).

Ngai Wong is with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong (e-mail: nwong@eee.hku.hk).

* Corresponding author.

proper part (top-left subsystem) follows $x_1(t) = e^{-t}x_1(0^-)$ whereas the improper part (top-right subsystem) follows $x_2(t) = -x_3(0^-)\delta(t)$ and $x_3(t) = 0$. In most cases, however, the system matrices are not presented in their decoupled forms (say, in a similarly transformed linear system), and therefore subsystem dynamics are not readily recognized.

Various approaches to decouple these subsystems include Weierstrass decomposition and Drazin inverse etc., mostly based on the generalized Schur/QZ decompositions of a general matrix pencil or by solving successive Sylvester matrix equations [17-19]. However, these methods often call for complicated subspace computation and are either numerically ill-conditioned or approximate in nature, and sometimes can even be inappropriate [15]. The works of März [20-24] on matrix projector chain, marked by their elegant algebras and theory, provide an effective way to constructively generate the *canonical projectors*, whose product then forms the spectral projectors for completely decoupling the proper and improper subsystems of a DS so that straightforward analysis as in the above example can be carried out. Nonetheless, despite the potential applications of canonical projectors, they appear only rarely in some work on model order reduction (MOR) of DSs (where projectors have led to projected generalized Lyapunov and Riccati equations for direct DS-MOR unamenable before) and partial realization [12,15,16,25-28].

On the other hand, there are increasingly widespread use of DAEs or DSs in circuit modeling and simulation. For example, recent attempts have been made to couple linear RLC networks to (nonlinear) transistors for direct system-level co-simulation [29-31]. Specifically, due to different physical natures between wires and solid-state devices, DAEs or DSs are first derived from nonlinear subcircuits like MOSFET or RF circuits [31] which are then interfaced to the interconnect or transmission line systems and co-simulated. Similarly, DSs also find important application in transient noise analysis where standard techniques like MNA are incorporated with stochastic noise models, resulting in stochastic DAEs [32,33]. Projector techniques can then be employed to decouple and reduce general stochastic DAEs into explicit stochastic differential equations (SDEs) solvable by a number of advanced numerical techniques, e.g., [34]. Furthermore, DSs are recently applied to describe second-order systems, such as RLCK circuits, as first-order DS realizations [2] (here 'K' refers to the susceptance, or inverse of inductance, whose matrix can be sparsified much more easily due to its localized nature). Based on this first-order DS, existing first-order reduction methods such as balanced truncation can be adopted for second-order MOR. In all these cases, canonical projector techniques can play a major role to streamline implementations of DS-oriented modeling procedures like DS-MOR, passivity check and enforcement.

While this paper will not present a truly extensive coverage of canonical projector techniques and applications, it contributes uniquely in the following ways:

- It collects previously scattered results on canonical projectors, and presents a relatively complete, self-contained and rigorous treatment of this topic with coherent notations. We are not aware of such an article in the literature to date.
- It provides new/corrected/more compact proofs to some fundamental properties and theorems in [20-22] where the ground work on canonical projectors is laid. Regarding corrected proof, we remark that Theorem 2 of [20] (Theorem 1 of this paper) serves as the founding theorem upon which the canonical projector theory is developed. However, its proof in [20] regarding the singularity of intermediate chain matrices $E_j$'s is flawed due to its presumption of non-zero intermediate projectors $Q_j$'s, i.e., the proof is a self-referential one. We give a new and easier proof in Appendix A.1.
- It describes a systematic and iterative canonical projector construction routine which is lacking in the literature.
- It highlights some latest deployments of projector techniques in DS analysis, with an aim to popularize the projector approach and arouse research interests into its potential applications.

Nonetheless, to limit the scope and length of this paper, the emphasis of this paper falls on the analytical aspects of canonical projectors. Efficient and application-specific numerical implementations of projector-based algorithms are possible: many new circuit modeling examples utilizing canonical projectors can be found in the latest works, by the authors and other researchers, listed in Section 4.

The organization of this paper is as follows. Section 2 reviews the basics of projector theory and outlines some key properties necessary for the rest of the paper. Section 3 details the core routine for systematically constructing canonical projectors and subsequently the spectral projectors for completely decoupling a DS. Some practical application examples are succinctly discussed in Section 4. Finally, Section 5 draws the conclusion.

## 2. PROJECTOR BASICS

### 2.1. Matrix chain and projectors

Throughout this paper, a linear time invariant (LTI) DS in the state space form is assumed:

$$E\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx + Du, \tag{1b}$$

where $E, A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^{n\times m}$. Also, $u, y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ are the input, output and state vectors, respectively. Here $E$ is generally singular with $\text{rank}(E) = r \le n$. We assume a regular matrix pencil $\lambda E - A$, namely, there exists a $\lambda_0 \in \mathbb{C}$ such that $\det(\lambda_0 E - A) \neq 0$. Then, there always exist nonsingular $W, T \in \mathbb{R}^{n\times n}$ that transform $E$ and $A$ into the so-called Weierstrass form [35]:

$$W^{-1}ET^{-1} = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix}, \quad W^{-1}AT^{-1} = \begin{bmatrix} J & 0 \\ 0 & I_{n-q} \end{bmatrix}, \quad (2)$$

where $I_k$ denotes an identity matrix of dimension $k$. The matrix $J \in \mathbb{R}^{q\times q}$ corresponds to the finite eigenvalues of $(\lambda E - A)$ whereas $N \in \mathbb{R}^{(n-q)\times(n-q)}$ is nilpotent and corresponds to infinite eigenvalues. The matrix pencil is stable if all eigenvalues of $J$ are stable (i.e., having negative real parts). The nilpotency index $\mu$ of $N$, viz. $N^{\mu-1} \neq 0$ and $N^{\mu} = 0$, is called the index of the matrix pencil $\lambda E - A$. The left and right spectral projectors, respectively $P_l$ and $P_r$, are defined as

$$P_l = W \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \text{ and } P_r = T^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} T. \quad (3)$$

Obviously, $P_l$ and $P_r$ ($Q_l = I - P_l$ and $Q_r = I - P_r$) project onto the left and right deflating subspaces corresponding to the finite (infinite) eigenvalues.

For a regular index-$\mu$ pencil $\lambda E - A$, setting $E_0 = E$ and $A_0 = A$, an initial projector $Q_0$ onto $\ker E_0$ is constructed, namely, $Q_0^2 = Q_0$ and $\text{im}Q_0 = \ker E_0$. Here $\text{im}(\circ)$ and $\ker(\circ)$ denote the image and kernel of a matrix, respectively. Subsequently, a matrix chain is formed:

$$E_{j+1} = E_j + A_j Q_j \text{ and } A_{j+1} = A_j P_j, \quad (4)$$

for $j = 0,1,\cdots,$ where $Q_j^2 = Q_j$ is the projector onto $\ker E_j$ and $P_j = I - Q_j$. That is, if $z \in \ker E_j$, then $Q_j z = z$ or $P_j z = 0$. For the matrix chain in (13), we have the following important theorem:

**Theorem 1** [20]**:** Assume a regular matrix pencil $\lambda E - A$ of index $\mu$, then $E_0,\cdots,E_{\mu-1}$ are singular while $E_\mu$ is nonsingular.

**Proof:** A more compact proof than that in [20] is given in Appendix A (see also the remark in Section 1).

From (4) and the properties of a projector, we get

$$E_{j+1} Q_j = A_j Q_j, \quad (5a)$$

$$E_{j+1} P_j = E_j, \quad (5b)$$

$$E_j = E_\mu P_{\mu-1} P_{\mu-2} \cdots P_j, \quad (5c)$$

$$A_\mu = A_j P_j P_{j+1} \cdots P_{\mu-1}, \quad (5d)$$

for $j = 0,1,\cdots,\mu-1$. From (14) and (15), we also have $\text{im}(A_{j-1}Q_{j-1}) \subseteq \text{im}E_j \subseteq \text{im}E_{j+1}$ and so on.

2.2. Useful properties

Some important properties of projectors for matrix pencils, especially those necessary for the proofs in this paper, are reviewed. The reader is referred to [22] for further properties. We first define a projector $W_j$ whose kernel is the image of $E_j$, i.e., $W_j^2 = W_j$ and $\ker W_j = \text{im}E_j$, and the subspace $S_j = \ker(W_j A_j) = \{z \mid A_j z \in \text{im}E_j\}$ that appears in some properties below. Moreover, '+' is used to denote union of subspaces and, in particular, '$\oplus$' denotes direct sum of disjoint subspaces (subspaces whose only intersection is the zero vector).

**Useful matrix chain properties:**

i) $S_j = \{z \mid A_j z \in \text{im}E_j\} = \{z \mid A_{j-1} z \in \text{im}E_j\} = \cdots = \{z \mid Az \in \text{im}E_j\}$.

ii) $S_j \subseteq S_{j+1}$ and $\ker E_0 + \cdots + \ker E_j \subseteq S_{j+1}$.

iii) $(\ker E_j \cap \ker A_j) = (\ker E_j \cap \ker E_{j+1}) \subseteq (\ker E_{j+1} \cap \ker E_{j+2})$.

iv) $\dim(\ker E_{j+1}) = \dim((\ker E_j) \cap S_j)$, where $\dim(\circ)$ denotes dimension.

v) For a regular pencil, the projectors $Q_0,\cdots,Q_{\mu-1}$ can be constructed such that $Q_j Q_i = 0$ for $j > i$. Projectors satisfying this property are called *admissible projectors*.

vi) For admissible projectors $Q_0,\cdots,Q_{\mu-1}$, we have $\ker(P_0 P_1 \cdots P_j) = \text{im}Q_0 \oplus \cdots \oplus \text{im}Q_j$.

**Proofs:**

i) From the end of Section 2.1, $A_j z = A_{j-1} P_{j-1} z = E_j w \Rightarrow A_{j-1} z = E_j w + A_{j-1} Q_{j-1} z \in \text{im}E_j \Rightarrow \cdots \Rightarrow Az \in \text{im}E_j$. Similarly, $Az = E_j w \Rightarrow A_1 z = A_0 P_0 z = E_j w - A_0 Q_0 z \in \text{im}E_j$ and so on.

ii) The first one is obvious from (i) and $\text{im}E_j \subseteq \text{im}E_{j+1}$. Next, we have $\ker E_j \subseteq S_{j+1}$, since $A_{j+1} Q_j = A_j P_j Q_j = 0$. Consequently, we have $\ker E_0 \subseteq S_1 \subseteq S_2$ and $\ker E_1 \subseteq S_2$, which implies $\ker E_0 + \ker E_1 \subseteq S_2$. The second part then follows by induction.

iii) For the first part: $z \in (\ker E_j \cap \ker E_{j+1}) \Leftrightarrow Q_j z = z$ and $(E_j + A_j Q_j) z = 0 \Leftrightarrow E_j z = 0$ and $A_j Q_j z = A_j z = 0$. For the second part: $z \in (\ker E_j \cap \ker E_{j+1}) \Leftrightarrow Q_j z = z$ and $Q_{j+1} z = z \Rightarrow Q_{j+1} z = z$ and $E_{j+2} z = A_{j+1} z = A_j P_j z = A_j P_j Q_j z = 0$.

iv) $Q_j$ is in general not unique as only its range is constrained. Suppose $Q'_j$ is another projector onto $\ker E_j$, we have $Q'_j Q_j = Q_j$ and also $Q_j Q'_j = Q'_j$, from which it follows $-Q'_j P_j = Q_j P'_j$. This permits the factorization

$$\begin{aligned} E_{j+1} &= E_j + A_j Q'_j Q_j \\ &= E_j + A_j Q'_j (I - P_j) \\ &= (E_j + A_j Q'_j)(I + Q_j P'_j) \\ &= E'_{j+1}(I + Q_j P'_j), \end{aligned} \quad (6)$$

where $E'_{j+1} = E_j + A_j Q'_j$. In particular, let $E_j^\dagger$ be the pseudo inverse of $E_j$, we may choose the orthogonal projector [22] $Q'_j = I - E_j^\dagger E_j$ which projects onto $\ker E_j$ along $(\ker E_j)^\perp$. Further, define $W'_j = I - E_j E_j^\dagger$ which projects onto $(\text{im}E_j)^\perp$ along $\text{im}E_j$, we arrive at $E_{j+1} = (E_j + W'_j A_j Q'_j)(I + E_j^\dagger A_j Q'_j)(I + Q_j P'_j)$. The second and last brackets on the right of equality are invertible whose inverses are obtained by changing the '+' signs into '–' signs. Consequently, we have $\ker E_{j+1} = (I - Q_j P'_j)(I - E_j^\dagger A_j Q'_j) \ker(E_j + W'_j A_j Q'_j)$.
But

$$\begin{aligned} &(E_j + W'_j A_j Q'_j) z = 0 \\ &\Leftrightarrow E_j z = 0 (\text{so} Q'_j z = z) \text{ and } W'_j A_j Q'_j z = W'_j A_j z = 0 \\ &\Leftrightarrow z \in \ker E_j \text{ and} A_j z \in \text{im}E_j \Leftrightarrow z \in (\ker E_j) \cap S_j, \end{aligned}$$

from which the proof follows. In particular, we have $\dim(\ker E_\mu) = \dim((\ker E_{\mu-1}) \cap S_{\mu-1}) = 0$ for a regular pencil.

v) From Theorem 1 and property (iii), $\ker E_\mu = 0$ so we must have $(\ker E_0 \cap \ker E_1) = \cdots = (\ker E_{\mu-1} \cap \ker E_\mu) = 0$. This renders $(\mathrm{im}Q_0 \cap \mathrm{im}Q_1) = \cdots = (\mathrm{im}Q_{\mu-2} \cap \mathrm{im}Q_{\mu-1}) = 0$ or in other words, $(\mathrm{im}Q_0 \oplus \mathrm{im}Q_1 \oplus \cdots \oplus \mathrm{im}Q_{j-1}) \cap \mathrm{im}Q_j = 0$. This permits a $Q_j$ such that $(\mathrm{im}Q_0 \oplus \mathrm{im}Q_1 \oplus \cdots \oplus \mathrm{im}Q_{j-1}) \subseteq \ker Q_j$. Specifically, if $U$ and $V$ are two block column matrices with linearly independent columns such that $\mathrm{im}U \cap \mathrm{im}V = 0$, a projector $Q$ with $\mathrm{im}Q = \mathrm{im}U$ and $\mathrm{im}V \subseteq \ker Q$ is

$$Q = H\begin{bmatrix} I & \\ & 0 \end{bmatrix}(H^* H)^{-1} H^*, \text{ where } H = [U \quad V].$$

(See also Lemma 2.5 and Proposition 2.6 of [22]).

vi) This is easily checked from the property that $(P_0 \cdots P_j)Q_i = 0$ for $0 \le i \le j$, and that $(P_0 \cdots P_j)z = 0 \Rightarrow z \in (\mathrm{im}Q_0 \oplus \cdots \oplus \mathrm{im}Q_j)$. (See also Proposition 3.1 of [22].)

In addition to the above, many new properties arise from the use of admissible projectors, for example,

$$A_j Q_j = E_{j+1} Q_j = \cdots = E_\mu Q_j, \tag{7a}$$

$$A_{j+1} = A - E_\mu (Q_0 + \cdots + Q_j), \tag{7b}$$

$$P_j P_{j-1} \cdots P_0 = I - (Q_0 + \cdots + Q_j), \tag{7c}$$

for $j = 0, \cdots, \mu-1$. Furthermore, we have the following important decompositions

$$I = (P_0 \cdots P_{\mu-1}) + (Q_0 P_1 \cdots P_{\mu-1}) + (Q_1 P_2 \cdots P_{\mu-1}) + \cdots + (Q_{\mu-2} P_{\mu-1}) + Q_{\mu-1}, \tag{8a}$$

$$I = (P_0 \cdots P_{\mu-1}) + (P_0 \cdots P_{\mu-2} Q_{\mu-1}) + (P_0 \cdots P_{\mu-3} Q_{\mu-2}) + \cdots + (P_0 Q_1) + Q_0, \tag{8b}$$

which play an essential role to completely decouple and solve (1) (cf. Section 4.1). It can be verified that for admissible projectors, $Q_j P_{j+1} \cdots P_{\mu-1} Q_j = Q_j$ and $Q_j P_0 \cdots P_{j-1} Q_j = Q_j$, from which it is readily shown that all terms on the right of (8a) and (8b) are then projectors by themselves.

## 3. CANONICAL AND SPECTRAL PROJECTORS

Canonical projectors are admissible projectors that satisfy

$$Q_j = Q_j P_{j+1} \cdots P_{\mu-1} E_\mu^{-1} A_j = Q_j P_{j+1} \cdots P_{\mu-1} E_\mu^{-1} A, \quad j = 0, \cdots, \mu-2, \tag{9a}$$

$$Q_{\mu-1} = Q_{\mu-1} E_\mu^{-1} A_{\mu-1} = Q_{\mu-1} E_\mu^{-1} A. \tag{9b}$$

The last equalities in (9a) and (9b) can be readily established using (7b). Canonical projectors allow the easy construction of spectral projectors $P_r$ and $P_l$ in (3), as well as the solution of $x$ in (1) through a decoupling approach. These are addressed in the following.

### 3.1. Construction of canonical projectors

We use the superscripts in $E_j^{(n)}$, $A_j^{(n)}$ and $Q_j^{(n)}$ to denote the $n$th matrix chain (here the chain index $n$ should not be confused with the state order $n$ in (1)). Then, with a regular index-$\mu$ pencil, setting $E_0^{(0)} = E$ and $A_0^{(0)} = A$, the initial matrix chain $E_{j+1}^{(0)} = E_j^{(0)} + A_j^{(0)} Q_j^{(0)}$ and $A_{j+1}^{(0)} = A_j^{(0)} P_j^{(0)}$, $j = 0, \cdots, \mu-1$, is formed whereby admissible projectors are assumed, i.e., $Q_j^{(0)} Q_i^{(0)} = 0$ for $j > i$. Generally, $Q_j^{(0)}$, $j = 0, \cdots, \mu-2$, are not canonical but $Q_{\mu-1}^{(0)}$ can be assumed canonical without loss of generality (see Appendix A.2). Using this initial set of projectors $Q_j^{(0)}$'s, an iterative matrix chain formation procedure then allows the generation of all canonical projectors. Specifically, based on the $(n-1)$th matrix chain parameters, $E_j^{(n-1)}$, $A_j^{(n-1)}$ and (admissible) $Q_j^{(n-1)}$, the $n$th chain is formed by setting $E_0^{(n)} = E$ and $A_0^{(n)} = A$ and

$$Q_{\mu-1}^{(n)} = Q_{\mu-1}^{(n-1)} (E_\mu^{(n-1)})^{-1} A_{\mu-1}^{(n-1)}, \tag{10a}$$

$$Q_j^{(n)} = Q_j^{(n-1)} P_{j+1}^{(n-1)} \cdots P_{\mu-1}^{(n-1)} (E_\mu^{(n-1)})^{-1} A_j^{(n-1)}, \tag{10b}$$

for $j = 0, \cdots, \mu-2$. It can be seen that if $Q_j^{(n-1)}$'s are admissible, $A_j^{(n-1)}$ $(j = 0, \cdots, \mu-1)$ in (10) can be replaced with $A$ due to (7b). Of course, it remains to show that the projectors $Q_j^{(n)}$, $j = 0, \cdots, u-1$, in (10) are indeed valid projectors onto $\ker E_j^{(n)}$. To show this, we need to study some projector properties arising from the definition in (10).

#### 3.1.1 Relationship between adjacent chains

In particular, the following hold:

**Further matrix chain properties:**

vii) $Q_j^{(n)} Q_i^{(n-1)} = 0$ for $j > i$.

viii) $Q_j^{(n)} Q_i^{(n)} = 0$ for $j > i$, i.e., $Q_j^{(n)}$'s are also admissible.

ix) $Q_j^{(n)} Q_i^{(n)} = Q_j^{(n-1)} Q_i^{(n)}$ for $j < i$.

x) $Q_j^{(n)} Q_j^{(n-1)} = Q_j^{(n-1)}$, $Q_j^{(n-1)} Q_j^{(n)} = Q_j^{(n)}$, and $(Q_j^{(n)})^2 = Q_j^{(n)}$. This implies $Q_j^{(n)}$ is a valid projector with the same range as $Q_j^{(n-1)}$.

xi) $Q_j^{(n-1)} P_j^{(n)} = -Q_j^{(n)} P_j^{(n-1)}$.

xii) $Q_j^{(n)} P_j^{(n-1)} Q_i^{(n)} = Q_j^{(n)} P_j^{(n-1)} Q_i^{(n-1)} = 0$ for any $0 \le i, j \le \mu-1$.

xiii) $P_0^{(n)} \cdots P_{j-1}^{(n)} Q_j^{(n)} = P_0^{(n-1)} \cdots P_{j-1}^{(n-1)} Q_j^{(n)}$, leading to $A_j^{(n)} Q_j^{(n)} = E_{j+1}^{(n-1)} Q_j^{(n)}$.

xiv) $Q_j^{(n)} P_j^{(n-1)} = Q_j^{(n)} P_j^{(n-1)} P_{j+1}^{(n-1)} = \cdots = Q_j^{(n)} P_j^{(n-1)} \cdots P_{\mu-1}^{(n-1)}$.

**Proofs:** The proofs for (vii), (viii), (x), (xi) are trivial, and also in the case of (xii) for $j \ge i$. To prove (xii) for $j < i$, we note that

$$Q_j^{(n)} P_j^{(n-1)} Q_i^{(n)} = Q_j^{(n)} P_j^{(n-1)} (Q_{j+1}^{(n-1)} + P_{j+1}^{(n-1)}) Q_i^{(n)}$$

$$= Q_j^{(n-1)} P_{j+1}^{(n-1)} \cdots P_{\mu-1}^{(n-1)} (E_\mu^{(n-1)})^{-1} A_{j+1}^{(n-1)} Q_{j+1}^{(n-1)} Q_i^{(n)}$$
$$+ Q_j^{(n)} P_j^{(n-1)} P_{j+1}^{(n-1)} Q_i^{(n)}$$
$$= Q_j^{(n-1)} P_{j+1}^{(n-1)} \cdots P_{\mu-1}^{(n-1)} (E_\mu^{(n-1)})^{-1} E_\mu^{(n-1)} Q_{j+1}^{(n-1)} Q_i^{(n)}$$
$$+ Q_j^{(n)} P_j^{(n-1)} P_{j+1}^{(n-1)} Q_i^{(n)}$$
$$= Q_j^{(n)} P_j^{(n-1)} P_{j+1}^{(n-1)} Q_i^{(n)}$$
$$= Q_j^{(n)} P_j^{(n-1)} P_{j+1}^{(n-1)} (Q_{j+2}^{(n-1)} + P_{j+2}^{(n-1)}) Q_i^{(n)}$$
$$= \cdots = Q_j^{(n)} P_j^{(n-1)} \cdots P_i^{(n-1)} Q_i^{(n)} = 0.$$

The proof for $Q_j^{(n)} P_j^{(n-1)} Q_i^{(n-1)} = 0$ proceeds similarly. Property (ix) then follows from (x) and (xii) by noting that $Q_j^{(n)} Q_i^{(n)} = Q_j^{(n)} (Q_j^{(n-1)} + P_j^{(n-1)}) Q_i^{(n)} = Q_j^{(n-1)} Q_i^{(n)}$. For (xiii), it is recognized that the left hand side of the first equality is $(I - Q_0^{(n)}) \cdots (I - Q_{j-1}^{(n)}) Q_j^{(n)}$. The equal sign can then be verified by applying (ix) repeatedly. This gives

$$A_j^{(n)} Q_j^{(n)} = A P_0^{(n)} \cdots P_{j-1}^{(n)} Q_j^{(n)} = A P_0^{(n-1)} \cdots P_{j-1}^{(n-1)} Q_j^{(n)}$$
$$= A_j^{(n-1)} Q_j^{(n-1)} Q_j^{(n)} = E_{j+1}^{(n-1)} Q_j^{(n)}.$$

Finally, (xiv) can be proven similarly to (xii).

Now we establish an important relationship between $E_j^{(n)}$ and $E_j^{(n-1)}$ and subsequently show that $Q_j^{(n)}$'s defined in (10) are actually projectors onto kernels of $E_j^{(n)}$'s. It follows from (6) that

$$E_1^{(n)} = E_1^{(n-1)} (I + Q_0^{(n)} P_0^{(n-1)}). \tag{11}$$

Also, (x) and (xii) above imply $E_1^{(n)} Q_1^{(n)} = E_1^{(n-1)} (I + Q_0^{(n)} P_0^{(n-1)}) Q_1^{(n)} = E_1^{(n-1)} Q_1^{(n-1)} Q_1^{(n)} = 0$, so $Q_1^{(n)}$ is a valid projector onto $\ker E_1^{(n)}$. Consequently, noting that $E_2^{(n-1)} P_1^{(n-1)} = E_1^{(n-1)}$ [see (5b)] and $P_1^{(n-1)} Q_0^{(n)} = (I - Q_1^{(n-1)})\ Q_0^{(n-1)} Q_0^{(n)} = Q_0^{(n-1)} Q_0^{(n)} = Q_0^{(n)}$, we have

$$\begin{aligned} E_2^{(n)} &= E_1^{(n)} + A_1^{(n)} Q_1^{(n)} \\ &= E_1^{(n-1)} (I + Q_0^{(n)} P_0^{(n-1)}) + A_1^{(n)} Q_1^{(n)} \\ &= E_2^{(n-1)} (P_1^{(n-1)} + Q_0^{(n)} P_0^{(n-1)}) + E_2^{(n-1)} Q_1^{(n)} \text{ [(xiii)]} \\ &= E_2^{(n-1)} (P_1^{(n-1)} + Q_0^{(n)} P_0^{(n-1)} + Q_1^{(n-1)} Q_1^{(n)}) \text{ [(x)]} \\ &= E_2^{(n-1)} (I + Q_0^{(n)} P_0^{(n-1)} + Q_1^{(n)} P_1^{(n-1)}) \text{ [(xi)]}, \end{aligned} \tag{12}$$

which can be used to show $E_2^{(n)} Q_2^{(n)} = 0$. By induction,

$$E_j^{(n)} = E_j^{(n-1)} (I + Q_0^{(n)} P_0^{(n-1)} + \cdots + Q_{j-1}^{(n)} P_{j-1}^{(n-1)}), \tag{13}$$

for $j = 1, \cdots, \mu$, and that $E_j^{(n)} Q_j^{(n)} = 0$. From (viii), if $Q_j^{(n-1)}$'s are admissible projectors, so are $Q_j^{(n)}$'s. Using (xii), it is also seen that the bracket in (13) is invertible and

$$(E_\mu^{(n)})^{-1} = (I - Q_0^{(n)} P_0^{(n-1)} - \cdots - Q_{\mu-1}^{(n)} P_{\mu-1}^{(n-1)}) (E_\mu^{(n-1)})^{-1}. \tag{14}$$

This suggests that $(E_\mu^{(n)})^{-1}$, which is required for computing the $(n+1)$th matrix chain, needs not be formed explicitly.

3.1.2 Proof of canonicity

Till now, it has been shown that given an admissible matrix chain, a new (also admissible) one can be formed by redefining its projectors as in (10). The columns in Table 1 shows the case where $\mu$ chains are formed, starting with the 0th chain with its last projector being canonical (marked by a gray cell). In practice, the last $n$ projector(s) of the $n$th chain are equal to those of the previous chain by definition of (10) and need not be computed again, as marked by the equal signs in the table, and each new chain creates one more canonical projector $Q_{\mu-1-n}^{(n)}$.

To begin with, we show that this is the case for the 1st chain. First, $Q_{\mu-1}^{(1)} = Q_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A_{\mu-1}^{(0)} = Q_{\mu-1}^{(0)}$, where the first equality is by definition and the second is by the canonicity of $Q_{\mu-1}^{(0)}$ (therefore $P_{\mu-1}^{(1)} = P_{\mu-1}^{(0)}$, too). To see $Q_{\mu-1}^{(1)}$ is further canonical in the context of the 1st chain,

$$\begin{aligned} &Q_{\mu-1}^{(1)} (E_\mu^{(1)})^{-1} A_{\mu-1}^{(1)} = Q_{\mu-1}^{(1)} (E_\mu^{(1)})^{-1} A \\ &= Q_{\mu-1}^{(1)} (I - Q_0^{(1)} P_0^{(0)} - \cdots - Q_{\mu-1}^{(1)} P_{\mu-1}^{(0)}) (E_\mu^{(0)})^{-1} A \\ &= Q_{\mu-1}^{(1)} (E_\mu^{(0)})^{-1} A = Q_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A = Q_{\mu-1}^{(1)}, \end{aligned} \tag{15}$$

where in the second line we have made use of the fact that $Q_{\mu-1}^{(1)} P_{\mu-1}^{(0)} = Q_{\mu-1}^{(0)} P_{\mu-1}^{(0)} = 0$. To show $Q_{\mu-2}^{(1)}$ is also canonical, we first get

$$\begin{aligned} &Q_{\mu-2}^{(1)} P_{\mu-1}^{(1)} (E_\mu^{(1)})^{-1} A_{\mu-2}^{(1)} = Q_{\mu-2}^{(1)} P_{\mu-1}^{(1)} (E_\mu^{(1)})^{-1} A \\ &= Q_{\mu-2}^{(1)} P_{\mu-1}^{(1)} (I - Q_0^{(1)} P_0^{(0)} - \cdots - Q_{\mu-2}^{(1)} P_{\mu-2}^{(0)}) (E_\mu^{(0)})^{-1} A \\ &= Q_{\mu-2}^{(1)} P_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A - Q_{\mu-2}^{(1)} P_{\mu-2}^{(0)} (E_\mu^{(0)})^{-1} A. \end{aligned} \tag{16}$$

The first term in the last line of (16) is re-expressed as

$$\begin{aligned} &Q_{\mu-2}^{(1)} P_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A \\ &= Q_{\mu-2}^{(1)} (Q_{\mu-2}^{(0)} + P_{\mu-2}^{(0)}) P_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A \\ &= Q_{\mu-2}^{(0)} P_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A + Q_{\mu-2}^{(1)} P_{\mu-2}^{(0)} P_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A \\ &= Q_{\mu-2}^{(0)} P_{\mu-1}^{(0)} (E_\mu^{(0)})^{-1} A + Q_{\mu-2}^{(1)} P_{\mu-2}^{(0)} (E_\mu^{(0)})^{-1} A. \text{ [by(xiv)]} \end{aligned} \tag{17}$$

Combining (16) and (17) renders

Table 1. Iterative generation of canonical projector chain (gray cells denote canonical projectors).

| 0 th | $\rightarrow$ iterations $\rightarrow$ | | $\cdots$ | $(\mu-1)$ th |
|---|---|---|---|---|
| $Q_0^{(0)}$ | $Q_0^{(1)}$ | $Q_0^{(2)}$ | $\cdots$ | $Q_0^{(\mu-1)}$ |
| $Q_1^{(0)}$ | $Q_1^{(1)}$ | $Q_1^{(2)}$ | $\cdots$ | $= Q_1^{(\mu-1)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $Q_{\mu-3}^{(0)}$ | $Q_{\mu-3}^{(1)}$ | $Q_{\mu-3}^{(2)} =$ | $\cdots$ | $= Q_{\mu-3}^{(\mu-1)}$ |
| $Q_{\mu-2}^{(0)}$ | $Q_{\mu-2}^{(1)} =$ | $Q_{\mu-2}^{(2)} =$ | $\cdots$ | $= Q_{\mu-2}^{(\mu-1)}$ |
| $Q_{\mu-1}^{(0)} =$ | $Q_{\mu-1}^{(1)} =$ | $Q_{\mu-1}^{(2)} =$ | $\cdots$ | $= Q_{\mu-1}^{(\mu-1)}$ |

$$
\begin{aligned}
&Q_{\mu-2}^{(1)}P_{\mu-1}^{(1)}(E_{\mu}^{(1)})^{-1}A_{\mu-2}^{(1)} = Q_{\mu-2}^{(0)}P_{\mu-1}^{(0)}(E_{\mu}^{(0)})^{-1}A \\
&= Q_{\mu-2}^{(0)}P_{\mu-1}^{(0)}(E_{\mu}^{(0)})^{-1}A_{\mu-2}^{(0)} = Q_{\mu-2}^{(1)},
\end{aligned}
\tag{18}
$$

where the last equality is again by definition of $Q_{\mu-2}^{(1)}$. This verifies $Q_{\mu-2}^{(1)}$ is also canonical.

In the general case, when $Q_j^{(n)} = Q_j^{(n-1)}$ (so that $P_j^{(n)} = P_j^{(n-1)}$), $j = \mu-1, \mu-2, \cdots, k,$ are canonical, we have

$$
\begin{aligned}
&Q_{k-1}^{(n)}P_k^{(n)}\cdots P_{\mu-1}^{(n)}(E_{\mu}^{(n)})^{-1}A_{k-1}^{(n)} \\
&= Q_{k-1}^{(n)}P_k^{(n)}\cdots P_{\mu-1}^{(n)}(E_{\mu}^{(n)})^{-1}A \\
&= Q_{k-1}^{(n)}P_k^{(n)}\cdots P_{\mu-1}^{(n)}(E_{\mu}^{(n-1)})^{-1}A - Q_{k-1}^{(n)}P_{k-1}^{(n-1)}(E_{\mu}^{(n-1)})^{-1}A \\
&= Q_{k-1}^{(n)}(Q_{k-1}^{(n-1)} + P_{k-1}^{(n-1)})P_k^{(n-1)}\cdots P_{\mu-1}^{(n-1)}(E_{\mu}^{(n-1)})^{-1}A \\
&\quad - Q_{k-1}^{(n)}P_{k-1}^{(n-1)}(E_{\mu}^{(n-1)})^{-1}A \\
&= Q_{k-1}^{(n-1)}P_k^{(n-1)}\cdots P_{\mu-1}^{(n-1)}(E_{\mu}^{(n-1)})^{-1}A\,[\text{(xiv)}] \\
&= Q_{k-1}^{(n-1)}P_k^{(n-1)}\cdots P_{\mu-1}^{(n-1)}(E_{\mu}^{(n-1)})^{-1}A_{k-1}^{(n-1)} \\
&= Q_{k-1}^{(n)}. \ [\text{by definition}]
\end{aligned}
$$

By induction, in the $n$th chain generated by the projectors in (10), the last $n$ entries are identical to those of the previous chain and that $Q_{\mu-1-n}^{(n)}$ is made canonical, too. This confirms the results in Table 1, whereas Fig. 2 summarizes the flow for canonical projector construction which we consider to be a much clearer exposition than that in [21]. In other words, the iterative matrix chain formation algorithm via (10) always converges in $\mu$ steps leading to all canonical projectors $(Q_0^{\mu-1}, Q_1^{\mu-1}, \cdots, Q_{\mu-1}^{\mu-1})$ in the $(\mu-1)$th chain.

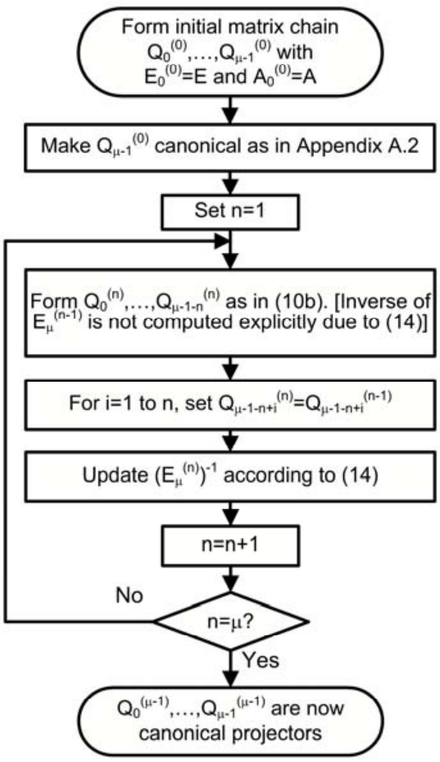



Fig. 2. The flow for constructing canonical projectors.

3.2. Construction of spectral projectors

Suppose now canonical projectors $Q_0, Q_1, \cdots, Q_{\mu-1}$ are produced (with superscripts omitted for notational simplicity) by the iterative chain formation procedure in Section 3.1. The right spectral projector $P_r$ in (3) is then readily obtained by

$$
P_r = P_0P_1\cdots P_{\mu-1} = T^{-1}\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}T. \tag{19}
$$

The proof in [21], nonetheless, is based on an indirect comparison with spectral projection and Drazin inverse. A constructive and more compact proof is given in Appendix A.3.

To obtain the left projector, canonical projectors are constructed from the matrix chain starting instead with $E_0^{(0)} = E^T$ and $A_0^{(0)} = A^T$. It is then easily checked that with the canonical projectors $\hat{Q}_0, \hat{Q}_1, \cdots, \hat{Q}_{\mu-1}$ computed for that new $\lambda E^T - A^T$ pencil,

$$
P_l = (\hat{P}_0\,\hat{P}_1\cdots\,\hat{P}_{\mu-1})^T = W\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}W^{-1}, \tag{20}
$$

where $\hat{P}_0 = I - \hat{Q}_0$ etc.

## 4. APPLICATIONS OF PROJECTOR TECHNIQUES

We enumerate, among others, some obvious and useful applications of canonical projectors. First, we present how to use canonical projector technique to decouple a DS arising in circuit modeling and simulation, which is then illustrated by index-1 and index-2 real-world examples. Based on this system decomposition, the recent application of projector technique to passivity test and enforcement of circuit models are described. Finally, we show that with projector technique, we can also perform passive MOR directly on a DS-form circuit model, which could also preserve the possible polynomial parts that are normally missed with conventional moment-matching MOR algorithms. In line with our scope and not to overwhelm the length of this paper, we give only concise description while the reader is referred to the references for efficient numerical implementations.

4.1. Decoupling and solution of (1)

The projector framework allows a simple interpretation and solution to the system equation (1) [21]. First, premultiplying (1a) by $E_{\mu}^{-1}$ and noting (5c), (1a) is re-casted as

$$
(P_{\mu-1}\cdots P_0)\dot{x} = E_{\mu}^{-1}Ax + E_{\mu}^{-1}Bu. \tag{21}
$$

Assuming canonical projectors, by respectively premultiplying (21) with the $\mu+1$ projectors in (8a) and by recognizing $I = (P_{j+1}\cdots P_{\mu-2}P_{\mu-1}) + (P_{j+1}\cdots P_{\mu-2}Q_{\mu-1}) + \cdots + (P_{j+1}Q_{j+2}) + Q_{j+1},$ we obtain the decoupling of (1a), or equivalently (21), in (22). The decoupled form in (22) then allows the solution of $(P_0 \cdots P_{\mu-1})x$, viz. the state variable, in (22a). Specifically, $Q_{\mu-1}x$ is solved through (22c), whose derivative then allows the solution of $Q_{u-2}x$ in (22b) and so on. If we consider a homogeneous system with $u = 0$, it is easily checked from (22b) and (22c) that $x$ is confined in the set (23).

$$(P_0 \cdots P_{\mu-1})\dot{x} = (P_0 \cdots P_{\mu-1})E_\mu^{-1}A(P_0 \cdots P_{\mu-1})x + (P_0 \cdots P_{\mu-1})E_\mu^{-1}Bu, \tag{22a}$$

$$\begin{bmatrix} Q_0 & & & & \\ & Q_1 & & & \\ & & \ddots & & \\ & & & Q_{\mu-3} & \\ & & & & Q_{\mu-2} \end{bmatrix} \begin{bmatrix} I & P_1 & P_1P_2 & \cdots & (P_1\cdots P_{\mu-2}) \\ & I & P_2 & \cdots & (P_2\cdots P_{\mu-2}) \\ & & \ddots & & \vdots \\ & & & I & P_{\mu-2} \\ & & & & I \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mu-2} \\ Q_{\mu-1} \end{bmatrix} \dot{x} = -\begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{\mu-3} \\ Q_{\mu-2} \end{bmatrix} x - \begin{bmatrix} Q_0P_1\cdots P_{\mu-1} \\ Q_1P_2\cdots P_{\mu-1} \\ \vdots \\ Q_{\mu-3}P_{\mu-2}P_{\mu-1} \\ Q_{\mu-2}P_{\mu-1} \end{bmatrix} E_\mu^{-1}Bu, \tag{22b}$$

$$0 = Q_{\mu-1}x + Q_{\mu-1}E_\mu^{-1}Bu. \tag{22c}$$

$$\{x \mid Q_{\mu-1}x = 0, \cdots, Q_0x = 0\} \Leftrightarrow \{x \mid x = P_0 \cdots P_{\mu-1}x\}. \tag{23}$$

And subsequently, (22) reduces to

$$\dot{x} = P_0 \cdots P_{\mu-1}E_\mu^{-1}Ax = P_rE_\mu^{-1}Ax, \tag{24a}$$

$$x = P_0 \cdots P_{\mu-1}x = P_rx. \tag{24b}$$

### 4.2. Index-1 and index-2 examples

DSs in practical problems often contain structured matrices. In the following, we demonstrate how canonical projector techniques may be utilized for real-world index-1 and index-2 applications. The examples mainly arise from discretization of PDEs from thermal simulations [26], but similar structures may also be found in circuit equations. For example, in electrical circuit modeling, a passive system can at most be of index 2 (i.e., $\mu = 2$, see also Section 4.3) [1,35] whose MNA equations may assume the index-2 pencil below.

We start with an index-1 case stemming from the discretization of the Euler equation, namely,

$$E = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{25}$$

where $E_{11}$ and $\Omega = A_{22} - A_{21}E_{11}^{-1}E_{12}$ are nonsingular. Choosing an arbitrary initial projector $Q_0$ (in this case not canonical) onto ker $E$, we get

$$\begin{bmatrix} E_{11} & E_{12}+A_{12}-A_{11}E_{11}^{-1}E_{12} \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 & -E_{11}^{-1}E_{12} \\ 0 & I \end{bmatrix}. \tag{26}$$

This verifies $\mu = 1$ by Theorem 1. The canonical projector is then obtained by computing $Q_0^{(0)} = Q_0E_1^{-1}A$ (cf. Appendix A.2) which equals

$$Q_0^{(0)} = \begin{bmatrix} -E_{11}^{-1}E_{12}\Omega^{-1}A_{21} & -E_{11}^{-1}E_{12}\Omega^{-1}A_{22} \\ \Omega^{-1}A_{21} & \Omega^{-1}A_{22} \end{bmatrix} = \begin{bmatrix} -E_{11}^{-1}E_{12} \\ I \end{bmatrix} \Omega^{-1}[A_{21} \quad A_{22}],$$

from which the canonical projector is $P_r = I - Q_0^{(0)}$.

The next DS structure, an index-2 example, arises from discretizing the convection equation, namely,

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}, \tag{27}$$

where $E_{11}$ and $A_{21}E_{11}^{-1}A_{12}$ are nonsingular. Subsequently, the initial matrix chain can be built as

$$\begin{bmatrix} E_{11} & A_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} E_{11} & A_{12}-A_{11}E_{11}^{-1}A_{12} \\ 0 & -A_{21}E_{11}^{-1}A_{12} \end{bmatrix} = \begin{bmatrix} E_{11} & A_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} 0 & -E_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}. \tag{28}$$

Using the procedures in Section 3, it can be shown that

$$P_r = \begin{bmatrix} \Pi_r & 0 \\ -(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}E_{11}^{-1}A_{11}\Pi_r & 0 \end{bmatrix},$$

where $\Pi_r = I - E_{11}^{-1}A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1}A_{21}$ is a projector itself. Higher index examples can also be found in [26]. These closed-form spectral projectors, once available, then facilitate significant acceleration in computation, say, for DS-MOR [26].

### 4.3. Passivity test

Passivity of a linear system is an important property to guarantee electrical synthesizability and stable global simulation when multiple systems are interconnected and simulated [1,3] (see also, e.g., the early work of Cauer [36] based on available transfer function where subsystem decomposition is a much easier task than the otherwise DS formulations here). The numerical passivity check for a DS, however, has only been investigated rigorously in the recent work [35] which proposes a linear matrix inequality (LMI)-based test. Nonetheless, such LMI test incurs $O(n^6)$ complexity and is simply prohibitive for higher order systems. The direct Weierstrass-form transformation (e.g., by the GUPTRI algorithm [17,19]) has a worst-case $O(n^4)$ work and is numerically unstable. Though Hamiltonian transformation techniques have been proposed to overcome part of the computational bottleneck [1], they still involve frequent singular value decompositions, Gram-Schmidt orthogonalizations and the highly expensive ($O((2n)^3)$) solution of an algebraic Riccati equation plus a Lyapunov equation.

To this end, the canonical projector techniques provide a natural way to decouple the proper (impulse-free) and improper (impulsive) parts of a DS, which translates into a highly efficient $O(n^3)$ passivity test, recently reported in [4,5]. Recalling (2), the transfer function of (1) can be expressed as

$$\begin{aligned} G(s) &= D + C(sE-A)^{-1}B \\ &= D + C\left( sW\begin{bmatrix} I_q & \\ & N \end{bmatrix}T - W\begin{bmatrix} J & \\ & I_{n-q} \end{bmatrix}T \right)^{-1} B \\ &= D + CT^{-1}\begin{bmatrix} (sI_q - J)^{-1} & \\ & (sN - I_{n-q})^{-1} \end{bmatrix} W^{-1}B \\ &= D + \begin{bmatrix} C_p & C_\infty \end{bmatrix}\begin{bmatrix} (sI_q - J)^{-1} & \\ & -(I_{n-q} - sN)^{-1} \end{bmatrix}\begin{bmatrix} B_p \\ B_\infty \end{bmatrix} \\ &= \underbrace{D - C_\infty B_\infty + C_p(sI_q - J)^{-1}B_p}_{G_p(s)} \\ &\quad \underbrace{-sC_\infty N B_\infty - s^2 C_\infty N^2 B_\infty - s^3 C_\infty N^3 B_\infty - \cdots}_{G_\infty(s)}, \end{aligned} \tag{29}$$

which consists of the proper part $G_p(s)$ (viz. bounded as $s\to\infty$) and improper part $G_\infty(s)$ (viz. unbounded as $s\to\infty$). The definitions $[C_p \quad C_\infty] = CT^{-1}$ and $\begin{bmatrix} B_p \\ B_\infty \end{bmatrix} = W^{-1}B$ are conformal to the partitions in the Weierstrass form. Then, $G(s)$ is positive real if and only if $G_p(s)$ is positive real and $-C_\infty N B_\infty \geq 0$, while $C_\infty N^i B_\infty = 0,\ i = 2,3,\cdots$ [4,5,35]. Subsequently, canonical projectors can be utilized for DS passivity test. Without going into details, it can be checked that $G_p(s)$ results from the pencil $\lambda E P_r - A$ which can be reduced to a standard state space system due to its impulse-free nature, while $C_\infty N B_\infty = CA^{-1}(E(I-P_r))A^{-1}B$ (the invertibility of $A$ is guaranteed when a DS is passive since its finite eigenvalues must be stable [1]). The high-level DS passivity test flow is captured in Fig. 3. Standard deployments on some electrical circuits from MNA extraction, with conventional passivity test for $G_p(s)$, have shown promising speedups as in Table 2.

Recently, a new passivity test technique called generalized Hamiltonian method (GHM [37]) and its variants [38-40] have been developed by the authors to directly check the validity of DS-form circuit models. Compared with traditional Hamiltonian-based passivity assessments, GHM does not require the system or the $D$ term in the state-space equations to be nonsingular. Furthermore, by a purely algebraic generalized eigenvalue computation, GHM can effectively find all possible passivity violation regions. However, GHM assumes that the DS improper part is passive, which is generally not true. To make GHM applicable to general DS models, spectral projectors can be constructed via canonical projector techniques. After that, the proper and improper subsystems can be decomposed and checked separately [4,5]. For large-scale systems (such as MNA models or the DS models from parasitic field solvers), spectral projectors can be efficiently computed through a sparse LU-based algorithm, by exploiting matrix sparsity and structures [5]. For large-scale proper subsystem, [5] also gives an efficient GHM implementation, making the projector-based flow feasible for problems with orders in the $10^4$.

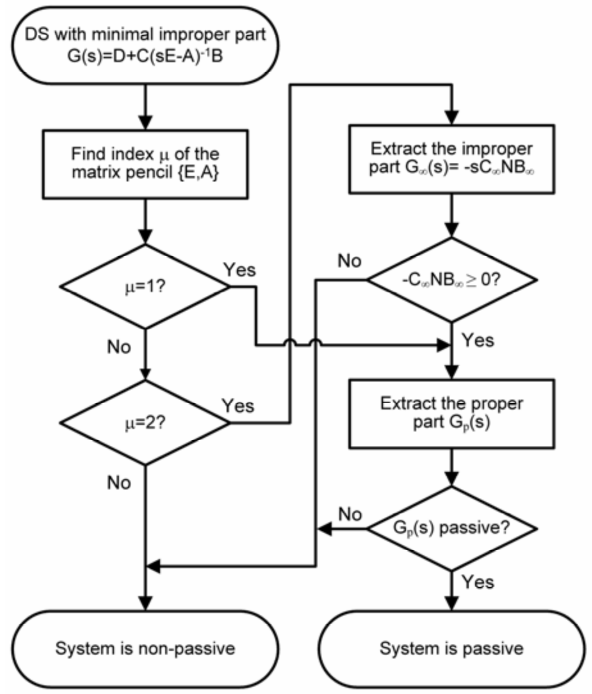



Fig. 3. Canonical projector based passivity test for DSs.

Table 2. CPU times in seconds of various passivity tests on a 3GHz PC.

| order of circuit | projector | SHH [1] | GUPTRI [19] |
|---|---|---|---|
| 100 | 0.22 | 1.47 | 0.43 |
| 200 | 1.64 | 20.17 | 26.45 |
| 300 | 5.61 | 77.25 | 73.55 |
| 400 | 13.09 | 207.7 | 417.75 |

4.4. Passivity enforcement

After passivity test, the nonpassive DS models need to be enforced with passivity to ensure stable global or system-level simulation (namely, when various DS models are interconnected and simulated). Previous enforcement techniques apply mainly to standard state-space models, such as those rebuilt from proper rational transfer functions. Regarding the DS-form circuit models, little work has been reported. The recent work [41,42] of the authors shows that based on the projector-based system decomposition, nonpassive DS models can be perturbed to passive ones. The DS passivity enforcement flow consists of three steps:

i) Use spectral projector to extract the possible improper subsystem;
ii) Check if the improper part is passive. If nonpassive, a small-size optimization problem is solved for improper-part passivation;
iii) Use GHM [37-40] to find all possible nonpassive regions of the proper part, followed by recursive perturbations of generalized eigenvalues to enforce passivity subject to an accuracy requirement.

In all cases, spectral projector plays a key role in extracting the improper subsystems.

4.5. DS-MOR

Besides the application in passivity test and enforcement, we also find spectral projectors crucial in the MOR of passive circuit models [6,27,28]. Because MNA formulation of RLC networks and numerical discretization of Maxwell equations normally yield DS models, conventional positive-real balanced truncation is not feasible. Although the Krylov-subspace projection methods can be used to reduce DS models, they have two drawbacks:

- Krylov-subspace projections can only preserve system passivity for such semi-positive definite (SPD) models as linear RLC circuits formed by MNA. For non-SPD models, such as those from electromagnetic parasitic extraction with nonsymmetric formulations or approximate fast matrix-vector products, system passivity cannot be guaranteed.
- If $D = 0$ in the DS state-space equation, a strictly proper reduced model would be generated by Krylov subspace projections. However, there normally exists a polynomial part in the transfer function when $E$ is singular. The polynomial part can be anon-zero constant term. In some cases, the polynomial part may contain an improper term (e.g., the admittance matrix of high-speed interconnects with strong cross-talk effects), which cannot be preserved by Krylov-subspace projection.

To address these issues, [28] uses spectral projectors to construct two projected generalized algebraic Riccati equations (GAREs). By solution of the positive-real gramians, the large circuit models can be balanced and reduced without loss of passivity. The improper subsystem is also reduced, by solving two discrete-time projected Lyapunov equations, again constructed with spectral projectors resulting from canonical projectors. In [43], it is further discovered that a non-PSD structured DS can also be reduced by a more efficient passivity-preserving moment-matching scheme. In this scheme, spectral projectors are required to preserve the possible polynomial part; the non-PSD structured proper subsystem is then reduced by a moment-matching method.

## 5. CONCLUSION

This paper has systematically put together canonical projector theorems and procedures, formerly scattered through the literature, for forming spectral projectors that decompose a DS into its proper and improper parts. New or more elegant proofs have also been presented for some fundamental projector properties. Such techniques are readily applicable to the analysis and manipulation of DSs commonly encountered in system and circuit macromodeling and simulation. Several examples of applying projectors to DS analysis are highlighted. An apparent product of DS decoupling is that the proper part, which is impulse-free and therefore representable as a standard state space, is conveniently extracted so that all procedures for standard state spaces (e.g., MOR, passivity check and enforcement) can be reused for DSs with little modification.

## APPENDIX A

A.1. Proof of Theorem 1

We assume a regular matrix pencil $\lambda E - A$ of index $\mu$. Due to the regularity assumption and without loss of generality, we assume the matrix chain in (4) begins with the Weierstrass form of $E_0$ and $A_0$ in (2), where $N^{\mu-1} \neq 0$ and $N^{\mu} = 0$. This is possible via pre- and post-multiplication of $W^{-1}$ and $T^{-1}$ to (4), under $j = 0$, as follows (dimensional subscripts of identity matrices are omitted but should be clear from context):

$$\overbrace{\begin{bmatrix} I & 0 \\ \tilde{M}_0 & N+\tilde{Q}_0 \end{bmatrix}}^{W^{-1}E_1T^{-1}} = \overbrace{\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}}^{W^{-1}ET^{-1}} + \overbrace{\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}}^{W^{-1}AT^{-1}} \overbrace{\begin{bmatrix} 0 & 0 \\ \tilde{M}_0 & \tilde{Q}_0 \end{bmatrix}}^{TQ_0T^{-1}},$$

$$\overbrace{\begin{bmatrix} J & 0 \\ -\tilde{M}_0 & \tilde{P}_0 \end{bmatrix}}^{W^{-1}A_1T^{-1}} = \overbrace{\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}}^{W^{-1}AT^{-1}} \overbrace{\begin{bmatrix} I & 0 \\ -\tilde{M}_0 & \tilde{P}_0 \end{bmatrix}}^{TP_0T^{-1}},$$

where $\tilde{Q}_0$ is a projector onto $\ker N$ and $\tilde{P}_0 = I - \tilde{Q}_0$, while $\tilde{M}_0 \in \mathbb{R}^{(n-q)\times q}$ is an arbitrary matrix with $\tilde{Q}_0\tilde{M}_0 = \tilde{M}_0$. Carrying on with the above for $j = 1, \cdots, \mu-1$ we get

$$\overbrace{\begin{bmatrix} I & 0 \\ \Phi_{j+1} & N_{j+1} \end{bmatrix}}^{W^{-1}E_{j+1}T^{-1}} = \overbrace{\begin{bmatrix} I & 0 \\ \Phi_j & N_j \end{bmatrix}}^{W^{-1}E_jT^{-1}} + \overbrace{\begin{bmatrix} J & 0 \\ -\Phi_j & \tilde{P}_0\cdots\tilde{P}_{j-1} \end{bmatrix}}^{W^{-1}A_jT^{-1}} \overbrace{\begin{bmatrix} 0 & 0 \\ \tilde{M}_j & \tilde{Q}_j \end{bmatrix}}^{TQ_jT^{-1}}, \quad \text{(A.1a)}$$

$$\overbrace{\begin{bmatrix} J & 0 \\ -\Phi_{j+1} & \tilde{P}_0\cdots\tilde{P}_j \end{bmatrix}}^{W^{-1}A_{j+1}T^{-1}} = \overbrace{\begin{bmatrix} J & 0 \\ -\Phi_j & \tilde{P}_0\cdots\tilde{P}_{j-1} \end{bmatrix}}^{W^{-1}A_jT^{-1}} \overbrace{\begin{bmatrix} I & 0 \\ -\tilde{M}_j & \tilde{P}_j \end{bmatrix}}^{TP_jT^{-1}}, \quad \text{(A.1b)}$$

where $\Phi_j = \tilde{M}_0 + \tilde{P}_0\tilde{M}_1 + \cdots + (\tilde{P}_0\cdots\tilde{P}_{j-2})\tilde{M}_{j-1}$ and $N_j = N_{j-1} + (\tilde{P}_0\cdots\tilde{P}_{j-2})\tilde{Q}_{j-1} = N + \tilde{Q}_0 + \tilde{P}_0\tilde{Q}_1 + \cdots + (\tilde{P}_0\cdots\tilde{P}_{j-2})$

Table 3. Table of $N^i\tilde{Q}_j$.

| | $i$=1 | $i$=2 | $i$=3 | $\cdots$ |
|---|---|---|---|---|
| $N^i\tilde{Q}_0$ | 0 | 0 | 0 | $\cdots$ |
| $N^i\tilde{Q}_1$ | $-\tilde{Q}_0\tilde{Q}_1$ | 0 | 0 | $\cdots$ |
| $N^i\tilde{Q}_2$ | $-\tilde{Q}_2$ $+\tilde{P}_0\tilde{P}_1\tilde{Q}_2$ | $-N\tilde{Q}_1\tilde{Q}_2$ | 0 | $\cdots$ |
| $N^i\tilde{Q}_3$ | $-\tilde{Q}_3$ $+\tilde{P}_0\tilde{P}_1\tilde{P}_2\tilde{Q}_3$ | $-N\tilde{Q}_3$ $+N\tilde{P}_1\tilde{P}_2\tilde{Q}_3$ | $-N^2\tilde{Q}_2\tilde{Q}_3$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

$\tilde{Q}_{j-1}$. Similarly, $\tilde{Q}_j$ projects onto $\ker N_j$ and $\tilde{Q}_j\tilde{M}_j = \tilde{M}_j$.

Obviously, $E_\mu$ is nonsingular if and only if $N_\mu$ is nonsingular. To prove so, we see that $N_j$ can be re-expressed as

$$N_j = N + I - (\tilde{P}_0 \cdots \tilde{P}_{j-1}). \tag{A.2}$$

By post-multiplying $\tilde{Q}_j$ to (A.2) (thus nulling the left hand side of the equality) and pre-multiplying with different powers of $N$, Table 3 is constructed from which it is easy to deduce $N^i\tilde{Q}_j = 0$ (or equivalently $N^i\tilde{P}_j = N^i$) whenever $i > j$. First, we show that $\tilde{Q}_j$ (and therefore $Q_j$), $j = 0,\cdots,\mu-1$, are nonzero or in other words, $N_j$ (and therefore $E_j$) are singular for $j = 0,\cdots,\mu-1$. Multiplying $N^{\mu-1}$ to the left of (A.2) under $j = \mu-1$ yields $N^{\mu-1}N_{\mu-1} = N^{\mu-1} - N^{\mu-1}\tilde{P}_0\cdots\tilde{P}_{\mu-2} = 0$ (since $N^{\mu-1}\tilde{P}_0\cdots\tilde{P}_{\mu-2} = N^{\mu-1}$ from above). If $N_{\mu-1}$ is nonsingular, then $N^{\mu-1} = 0$ which contradicts our assumption, so $\tilde{Q}_{\mu-1} \neq 0$. Then, by property (iv) in Section 2.2, the kernels of $N_j$ have monotonically non-increasing ranks as $j$ increases, so $\tilde{Q}_j \neq 0$ for $j = 0,\cdots,\mu-1$. Next, we show the product $(\tilde{Q}_0\cdots\tilde{Q}_{\mu-1})$ is also nonzero. Suppose $(\tilde{Q}_0\cdots\tilde{Q}_{\mu-1}) = 0$, because $\tilde{Q}_0 = N_1 - N$, we have $(N_1 - N)(\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) = 0$. Since $N_1\tilde{Q}_1 = 0$ by definition, we get $(N\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) = 0$ so $(\tilde{Q}_1\cdots\tilde{Q}_{\mu-1})$ is in the range of $\tilde{Q}_0$. This gives rise to $(\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) = (\tilde{Q}_0\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) = 0$. Consequently, $(\tilde{P}_0\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) = (N_2 - N_1)(\tilde{Q}_2\cdots\tilde{Q}_{\mu-1}) = 0$, from which we derive $(\tilde{Q}_2\cdots\tilde{Q}_{\mu-1}) = (\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) = 0$. Continuing this way, we have $\tilde{Q}_{\mu-1} = 0$ which is a contradiction, so $(\tilde{Q}_0\cdots\tilde{Q}_{\mu-1}) \neq 0$. Now we show the nonsingularity of $N_\mu$. From the subdiagonal entries of Table 3, we have

$$N^i\tilde{Q}_i = (-1)^i(\tilde{Q}_0\tilde{Q}_1\cdots\tilde{Q}_i), \quad i = 1,\cdots,\mu-1. \tag{A.3}$$

Setting $j = \mu$ in (A.2) and pre-multiply by $N^{\mu-1}$ gives

$$\begin{aligned} N^{\mu-1}N_\mu &= N^{\mu-1} - N^{\mu-1}\tilde{P}_{\mu-1} = N^{\mu-1}\tilde{Q}_{\mu-1} \\ &= (-1)^{\mu-1}(\tilde{Q}_0\tilde{Q}_1\cdots\tilde{Q}_{\mu-1}) \neq 0. \end{aligned} \tag{A.4}$$

If $N_\mu$ is singular, there exists a $z \neq 0$ such that $N_\mu z = 0$ which implies $(\tilde{Q}_0\tilde{Q}_1\cdots\tilde{Q}_{\mu-1})z = 0$. Using similar arguments as before, this leads to $\tilde{Q}_{\mu-1}z = 0$, which in turn implies $\tilde{P}_0\cdots\tilde{P}_{\mu-2}\tilde{Q}_{\mu-1}z = (N_\mu - N_{\mu-1})z = -N_{\mu-1}z = 0$ so that $z = \tilde{Q}_{\mu-1}z = 0$. This contradicts the nonzero assumption of $z$ and therefore $N_\mu$ must be nonsingular.

A.2. Assumption of a canonical $Q_{\mu-1}$

Suppose a matrix chain is constructed where $Q_j$, $j = 0,\cdots,\mu-1$, are admissible. It can be assumed without loss of generality that $Q_{\mu-1}$ in $E_\mu = E_{\mu-1} + A_{\mu-1}Q_{\mu-1}$ is canonical. Otherwise, we replace $Q_{\mu-1}$ by $Q'_{\mu-1} = Q_{\mu-1}E_\mu^{-1}A_{\mu-1}$ and update the last matrix chain equation to $E'_\mu = E_{\mu-1} + A_{\mu-1}Q'_{\mu-1}$. To see $Q'_{\mu-1}$ is a valid projector onto $\ker E_{\mu-1}$, we note that $Q_{\mu-1}Q'_{\mu-1} = Q'_{\mu-1}$ and $Q'_{\mu-1}Q_{\mu-1} = Q_{\mu-1}E_\mu^{-1}A_{\mu-1}Q_{\mu-1} = Q_{\mu-1}E_\mu^{-1}E_\mu Q_{\mu-1} = Q_{\mu-1}$ [cf. (7a)], which implies $Q'^2_{\mu-1} = Q'_{\mu-1}$ is a projector onto the same range as $Q_{\mu-1}$. Moreover, for $i = 0,\cdots,\mu-2$, $Q'_{\mu-1}Q_i = Q_{\mu-1}E_\mu^{-1}A_0P_0\cdots P_{\mu-2}Q_i = 0$ by property (vi) of Section 2.2, thus $Q_0,\cdots,Q_{\mu-2},Q'_{\mu-1}$ are still admissible. Finally, to prove $Q'_{\mu-1}$ is indeed canonical, we make use of the property in (6), which entails $E'_\mu = E_\mu(I + Q'_{\mu-1}P_{\mu-1})$, and obtain $Q'_{\mu-1}E'^{-1}_\mu A_{\mu-1} = Q'_{\mu-1}(I - Q'_{\mu-1}P_{\mu-1})E_\mu^{-1}A_{\mu-1} = Q'_{\mu-1}Q_{\mu-1}E_\mu^{-1}A_{\mu-1} = Q_{\mu-1}E_\mu^{-1}A_{\mu-1} = Q'_{\mu-1}$.

A.3. Constructing right projector $P_r$

Given canonical projectors $Q_0,Q_1,\cdots,Q_{\mu-1}$, we show that the right projector is given by $P_r = P_0P_1\cdots P_{\mu-1}$. First, we assume general (not necessarily canonical) projectors $Q_0$ to $Q_{\mu-1}$ and make use of the notations and results in Appendix A.1 Referring to (A.1) and by virtue of the fact that $E_\mu$ is nonsingular, $N_\mu$ is nonsingular. With some care, it can be shown that

$$P_0P_1\cdots P_{\mu-1} = T^{-1}\begin{bmatrix} I & 0 \\ -\Phi_\mu & \tilde{P}_0\tilde{P}_1\cdots\tilde{P}_{\mu-1} \end{bmatrix}T, \tag{A.5a}$$

$$E_\mu^{-1} = T^{-1}\begin{bmatrix} I & 0 \\ -N_\mu^{-1}\Phi_\mu & N_\mu^{-1} \end{bmatrix}W^{-1}. \tag{A.5b}$$

(Again the dimensional subscripts of identity matrices are omitted for simplicity.) Now because the projectors are canonical, we must have

$$Q_{\mu-1} = Q_{\mu-1}E_\mu^{-1}A, \tag{A.6a}$$

$$Q_j = Q_jP_{j+1}\cdots P_{\mu-1}E_\mu^{-1}A, \tag{A.6b}$$

for $j$ equals $\mu-2$ down to 0. Substituting the notions of $Q_j$, $P_j$ and $E_\mu^{-1}$ in (A.1) and (A.5) into (A.6) and equating both sides, we get

$$\tilde{M}_{\mu-1} = \tilde{M}_{\mu-2} = \cdots = \tilde{M}_0 = 0, \tag{A.7a}$$

$$\tilde{Q}_{\mu-1} = \tilde{Q}_{\mu-1}N_\mu^{-1}, \tag{A.7b}$$

$$\tilde{Q}_j = \tilde{Q}_j\tilde{P}_{j+1}\cdots\tilde{P}_{\mu-1}N_\mu^{-1}, \tag{A.7c}$$

for $j$ equals $\mu - 2$ down to 0. Subsequently, $\Phi_\mu = 0$ and

$$P_0 P_1 \cdots P_{\mu-1} = T^{-1} \begin{bmatrix} I & 0 \\ 0 & \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} \end{bmatrix} T.$$

Now it remains to show $\tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} = 0.$ From (A.2),

$$\begin{aligned} N_\mu &= N + \tilde{Q}_0 + \tilde{P}_0 \tilde{Q}_1 + \cdots + (\tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-2}) \tilde{Q}_{\mu-1} \\ &= N + I - \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1}. \end{aligned} \tag{A.8}$$

Multiplying $\tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1}$ (a projector itself) onto both sides of (A.8) yields

$$N \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} = N_\mu \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1}. \tag{A.9}$$

On the other hand, (A.7b), (A.7c), (8) and the first equation in (A.8) imply

$$\begin{aligned} N_\mu &= N + (\tilde{Q}_0 + \cdots + \tilde{Q}_{\mu-1}) N_\mu \\ &= N + \left( I - (\tilde{P}_{\mu-1} \tilde{P}_{\mu-2} \cdots \tilde{P}_0) \right) N_\mu \\ &\Rightarrow (\tilde{P}_{\mu-1} \tilde{P}_{\mu-2} \cdots \tilde{P}_0) N_\mu = N. \end{aligned} \tag{A.10}$$

But from (5c), $N_\mu (\tilde{P}_{\mu-1} \tilde{P}_{\mu-2} \cdots \tilde{P}_0) = N,$ which together with (A.10) and the nonsingularity of $N_\mu$ gives

$$N_\mu N = N N_\mu. \tag{A.11}$$

Now multiply $N$ to the left of (A.9) results in

$$\begin{aligned} N^2 \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} &= N N_\mu \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} \\ &= N_\mu (N \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1}) \quad \text{[by (A.11)]} \\ &= N_\mu^2 \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1}. \quad \text{[by (A.9)]} \end{aligned}$$

Keep multiplying $N$ to the left of the (A.9) as above and recursively using (A.11) finally gives

$$N^\mu \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} = N_\mu^\mu \tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1}. \tag{A.12}$$

Since $N^\mu = 0$ but $N_\mu^\mu$ is nonsingular, we must then have $\tilde{P}_0 \tilde{P}_1 \cdots \tilde{P}_{\mu-1} = N_\mu^{-\mu} 0 = 0$ and consequently

$$P_0 P_1 \cdots P_{\mu-1} = T^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T = P_r.$$

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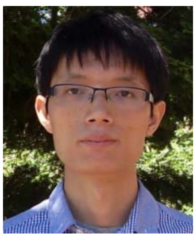

**Zheng Zhang** received his B.Eng. degree from Huazhong University of Science and Technology, China, in 2008, and his M.Phil. degree from the University of Hong Kong, Hong Kong, in 2010. He is a Ph.D. student in Electrical Engineering and Computer Science at the Massachusetts Institute of Technology (MIT), Cambridge, MA. His research interests include numerical methods for uncertainty quantification, computer-aided design (CAD) of integrated circuits and microelectromechanical systems (MEMS), and model order reduction. In 2009, Mr. Zhang was a visiting scholar with the University of California, San Diego (UCSD), La Jolla, CA. In 2011, he collaborated with Coventor Inc., working on CAD tools for MEMS design. In the summer of 2013, he was a visiting scholar at the Applied Math Division of Brown University, Providence, RI. He was recipient of the Li Ka Shing Prize (university best M.Phil/Ph.D thesis award) from the University of Hong Kong, in 2011, and the Mathworks Fellowship from MIT, in 2010.

**Ngai Wong** received his B.E. (with first class honors) and Ph.D. degrees in Electrical and Electronic Engineering from the University of Hong Kong, Pokfulam, Hong Kong, in 1999 and 2003, respectively. He was an Intern with Motorola, Inc., Kowloon, Hong Kong, from 1997 to 1998, specializing in product testing. He was a Visiting Scholar with Purdue University, West Lafayette, IN, in 2003. Currently, he is an Associate Professor with the University of Hong Kong. His current research interests include linear and nonlinear circuit modeling and simulation, model order reduction, passivity test and enforcement, and numerical algorithms in electronic design automation.